Lecture 6. The Dynkin $\pi - \lambda$ Theorem.

It is often the case that two measures which agree on a certain class of sets actually agree on all sets in the relevant σ-algebra. There are a couple of standard tools to prove that the measures are the same: the Monotone Class lemma and the Dynkin $\pi-\lambda$ theorem. They are essentially equivalent devices and it is largely a matter of taste which one to take as standard equipment. We shall do the $\pi - \lambda$ theorem and use it in the case of Lebesgue measure.

Suppose that μ and μ' are translation invariant measures on the Borel $\sigma-$algebra of $\mathbf{R}^d$ both assigning the same (finite) measure to the unit box $[0,1]^d$. We will show that then $\mu = \mu'$. Let L denote the set of all Borel sets $A \subset R^n$ for which $\mu(A) = \mu'(A)$. By hypothesis, $[0,1]^d \in L$. It seems reasonable to conclude from this that the set P of all boxes $[a_1, b_1] \times \cdots \times [a_d, b_d]$, with rational a_i and b_i, would belong to L. Let us accept this for now; i.e. suppose $P \subset L$. Now if we can show from this that L contains the $\sigma-$algebra generated by P then we would be done, because the $\sigma-$algebra generated by π is the Borel $\sigma-$algebra. (This follows from two observations : (i) each box in P is the intersection of open sets :

$$[a_1, b_1] \times \cdots \times [a_d, b_d] = \bigcap_{k \geq 1} \left(a_1 - \frac{1}{k}, b_1 + \frac{1}{k} \right) \times \cdots \times \left(a_d - \frac{1}{k}, b_d + \frac{1}{k} \right)$$

and (ii) every open U subset of $\mathbf{R}^d$ is the union of small boxes $[a_1, b_1] \times \cdots \times [a_d, b_d]$ with rational endpoints and centered at the rational points in U.) Thus L would in fact be the whole Borel $\sigma-$algebra. That is, $\mu(A) = \mu'(A)$ for every Borel set A. Thus the key tool would be the result that L contains the $\sigma-$algebra generated by P. This will, essentially, be proved by the $\pi - \lambda$ theorem. There are some technical problems involved which will be settled later.

6.0. Definition. Let P and L be collections of subsets of a set X. The collection P is called a $\pi-$system if it is closed under finite intersections; i.e. if $A, B \in P$ then $A \cap B \in P$:

P is a π-system if $A \cap B \in P$ for all $A, B \in P$

The collection L is called a $\lambda-$system if the following hold :

(L1) $\emptyset \in L$;

(L2) if $A \in L$ then $A^c \in L$;

(L3) L is closed under countable *disjoint* unions; i.e. if $A_1, A_2, ... \in L$ and if $A_i \cap A_j = \emptyset$ for every $i \neq j$, then $\cup_{j=1}^{\infty} A_j \in L$.

6.1. **Dynkin's $\pi - \lambda$ Theorem**. *Let P be a π-system of subsets of X, and L a λ-system of subsets of X. Suppose also that $P \subset L$. Then :*

$$\sigma(P) \subset L,$$

i.e. *L contains the σ-algebra $\sigma(P)$ generated by P.*

We will do the proof later but let us apply it to prove the uniqueness of Lebesgue measure.

6.2. Proposition. *Every translation-invariant Borel measure on* $\mathbf{R}$ *which assigns finite measure to the unit interval is a constant multiple of Lebesgue measure.*

Proof. Let μ be a translation-invariant Borel measure on $\mathbf{R}$ which assigns finite measure to the unit interval. Let m be Lebesgue measure on $\mathbf{R}$. Our first objective will be to check that

$$\mu\big([a,b)\big) = km\big([a,b)\big)$$

for every rational a, b, where k is the finite constant

$$k = \mu\big([0,1)\big)$$

Then we shall show by a $\pi - \lambda$ argument that $\mu(A) = km(A)$ holds for all sets A in the sigma-algebra generated by the intervals $[a,b)$, i.e. it holds for all Borel sets A.

For any positive integer p, the interval $[0,p)$ is the union of p disjoint translates of $[0,1)$, and so by translation-invariance of μ, we have

$$\mu\big([0,p)\big) = p\mu\big([0,1)\big) = pk$$

By the same argument, for any positive integer p, we also have

$$\mu\big([0,p)\big) = q\mu\big([0,p/q)\big)$$

Combining these two relations we have

$$\mu\big([0,p/q)\big) = k\frac{p}{q} = km\big([0,p/q)\big)$$

Then by translation-invariance it follows that

$$\mu\big([a,b)\big) = km\big([a,b)\big)$$

for all intervals $[a,b)$ for which $b-a$ is rational.

Thus μ is a constant multiple of Lebesgue measure m on intervals $[a,b)$ with rational endpoints. Now we use the $\pi - \lambda$ theorem to jazz this up to all Borel sets. The first idea would be take P to be the collection of all intervals $[a,b)$ with rational endpoints, and L to be the class of all Borel sets A for which $\mu(A) = km(A)$ holds. But there is a problem with this: the collection L satisfies all properties of being a λ system except that we cannot establish closure under complements, as the argument

$$\mu(A^c) = \mu(\mathbf{R}) - \mu(A) = \infty - km(A) = km(A^c)$$

is non-sense. The way to get around this problem with infinite measure is to focus down to a finite interval $[-N,N)$ and then let $N \uparrow \infty$ at the end.

So, fix any positive integer N, and let

$$L_N = \{\text{all Borel sets } A \text{ fotr which } \mu\big(A \cap [-N,N)\big) = km\big(A \cap [-N,N)\big)\}$$

and

$$P = \{\text{all intervals } [a, b) \text{ with } a, b \text{ rational}\}$$

What we have proven before shows that

$$P \subset L_N$$

It is clear that P is a π–system. It is also clear that L contains the empty set and is closed under countable disjoint unions. To check closure under complements, consider any $A \in L_N$. Then

$$\mu\big(A^c \cap [-N, N)\big) = \mu\big([-N, N)\big) - \mu(A) = km\big([-N, N)\big) - km(A) = km\big(A^c \cap [-N, N)\big)$$

(The subtraction works because $m([-N, N)) = 2N$ is finite.) This shows that $A^c \in L_N$. Thus L_N is a λ-system.

By Dynkin's theorem we conclude then that $L_N \supset \sigma(P)$. But $\sigma(P)$ is the entire Borel sigma-algebra. So, in fact, L_N is the entire Borel sigma-algebra, and this means that for any Borel set A we have the relation

$$\mu\big(A^c \cap [-N, N)\big) = km\big(A^c \cap [-N, N)\big)$$

Now let $N \uparrow \infty$. Since

$$\cup_{N\geq 1}\big(A^c \cap [-N, N)\big) = A$$

we conclude that

$$\mu(A) = km(A)$$

for every Borel set A. ■

We can now move this result up to higher dimensions.

6.3. Proposition. *Suppose μ is a translation-invariant measure on the Borel subsets of $\mathbf{R}^d$, for which $k \stackrel{\text{def}}{=} \mu([0, 1]^d) < \infty$. Then*

$$\mu = k\, m$$

Proof. We will use essentially the same argument as in the one-dimensional case. Let $p_1, ..., p_d$ be positive integers. Note that $[0, p_j)$ is the union of p_j translates $[n, n+1)$, with $n \in \{0, 1, ..., p_j - 1\}$, of $[0, 1)$. Taking products of these intervals we see that

$$[0, p_1) \times \cdots \times [0, p_d)$$

is the union of of the boxes

$$[n_1, n_1 + 1) \times \cdots \times [n_d, n_d + 1)$$

where $n_1, ..., n_d \in \{0, 1, ..., p - 1\}$. Any pair of distinct boxes in this collection have at least one 'side' disjoint, and so these boxes are disjoint. There are

$$p_1 \cdots p_d$$

of these boxes. So, by translation-invariance of the measure μ, we have

$$\mu\big([0,p_1)\times\cdots\times[0,p_d)\big)=p_1\cdots p_d\mu\big([0,1)^d\big)=kp_1\cdots p_d$$

Again, by the same reasoning, for any positive integers $q_1,...,q_d$, we have

$$\mu\big([0,p_1)\times\cdots\times[0,p_d)\big)=q_1\cdots q_d\mu\big([0,p_1/q_1)\times\cdots[0,p_d/q_d)\big)$$

Combining the preceding relations we have

$$\mu(B)=km(B)$$

for the box $B=[0,p_1/q_1)\times\cdots\times[0,p_d/q_d)$. Translation invariance then shows that the above equality holds for every box B with rational sides.

Let P be the collection of all boxes with rational corners. This is a π-system which generates the Borel sigma-algebra of $\mathbf{R}^d$. Fix any positive integer N and let L_N be the collection of all Borel sets $A\subset\mathbf{R}^d$ for which

$$\mu(A\cap B_N)=km(A\cap B_N)$$

where

$$B_N=[-N,N)^d$$

Then L_N is a λ-system and, by what we have proven above, $L_N\supset P$. Therefore, by the $\pi-\lambda$ theorem, $L_N\supset\sigma(P)$. Since $\sigma(P)$ is the Borel sigma-algebra, it follows that

$$\mu(A\cap B_N)=km(A\cap B_N)$$

for every Borel set A. Now let $N\uparrow\infty$ to conclude that

$$\mu(A)=km(A)$$

for every Borel set $A\subset\mathbf{R}^d$. ■

Finally, we turn to the proof of the $\pi-\lambda$ theorem. There are a couple of simple observations we will need :

6.4. Lemma. *Let X be a set, and consider λ and π systems of subsets of X.*

(i) *A λ-system which also a π-system* (i.e. is closed under finite intersections) *is a σ-algebra.*

(ii) *A λ system is closed under proper differences: if L is a $\lambda-$system and $A,B\in L$ with $B\subset A$, then $A-B\in L$.*

Proof. (i) Suppose that the $\lambda-$system L is also a π system; i.e. if $A,B\in L$ then $A\cap B\in L$. The only point we have to check is that L is closed under countable unions. So let $E_1,E_2,...\in L$, and set $E=\cup_j E_j$. Since we know that L is closed under countable disjoint unions we need to write E as a disjoint union of sets in L. This is achieved as follows. Let H_j be the set of all points in E_j which do not already belong to any of the 'previous' sets $E_1,...,E_{j-1}$. Then clearly the sets H_j are disjoint and $E=\cup_j H_j$. To see that H_j is in L we note that $H_j = E_j \cap E_{j-1}^c \cap \ldots \cap E_1^c$ (this makes sense for $j > 1$; for $j = 1$, $H_1 = E_1$ is already in L.) Since L is closed under complements (being a λ system) and since we have also assumed that L is closed under finite intersections, we have $H_j \in L$. Hence $E \in L$, as required.

(ii) If $A, B \in L$, and $A \subset B$, then $A - B = A \cap B^c$ can be expressed in the following way :

$$A \cap B^c = (A^c \cup B)^c.$$

Notice that the sets A^c and B are disjoint because $A \subset B$. So, since L is closed under complements and finite disjoint unions, we see that $A - B \in L$. ■

For the next step towards the π-λ theorem, observe that the intersection of any family of λ systems is again a λ system. If X is a set and $P \subset \mathcal{P})(X)$, define $l(P)$ to be the intersection of all λ systems which contain P as subset (for example, the power set $\mathcal{P}(X)$ is a λ system $\supset P$). Thus:

$$l(P) \stackrel{\text{def}}{=} \cap\{\Lambda : \Lambda \text{ is a } \lambda \text{ system and } \Lambda \supset P\}$$

Thus $l(P)$ *is the smallest* λ *system containing* P *as a subset.*

6.4. <u>Lemma</u>. *Let* P *be a* π*-system of subsets of a set* X*. Then* $l(P)$ *is a* σ*-algebra.*

<u>Proof.</u> We have seen that a collection of subsets of X which is both a π system and a λ system is actually a σ-algebra. Thus, it will suffice to prove that $l(P)$ is a π-system.

Let $A, B \in l(P)$. Our goal is to prove that $A \cap B$ is also in $l(P)$. We will do this by establishing the following:

(1) for any fixed $A \in l(P)$, the collection of all $B \subset X$ for which $A \cap B \in l(P)$ is a λ system;
(2) $A \cap B \in l(P)$ whenever $A \in P$ and $B \in l(P)$;
(3) if $A \in l(P)$ and $B \in l(P)$ then $A \cap B \in l(P)$. This would show that $l(P)$ is a π system.

For (1), fix any $A \in l(P)$, and let

$$l_A = \{B \subset X : A \cap B \in l(P)\}$$

It is clear that l_A contains $\emptyset$ and is closed under countable disjoint unions. Next if $B \in l_A$ then

$$A \cap B^c = A - (A \cap B)$$

is a proper difference of sets in the λ-system $l(P)$, and so is in $l(P)$. Thus, l_A is a λ-system. This establishes (1).

Now suppose $A \in P$. Then $A \cap B \in P \subset l(P)$ for every B in the π system P. So $B \in l_A$. Thus, $l_A \supset P$. Then, by definition of l_P as the smallest λ system containing P, we have

$$l_A \supset l(P)$$

Looking back at the definition of l_A, we see that this means simply that $A \cap B \in l(P)$ for every $B \in l(P)$. This establishes (2).

Now we'll bootstrap ourselves up one level: using what has been established in (2) and applying again essentially the same argument we will reach our goal (3). To this end, now fix $B \in l(P)$, and think of

$$l_B = \{A \subset X : A \cap B \in l(P)\}$$

This is a λ-system, as we have already seen. In (2) we proved essentially that

$$l_B \supset P$$

Then, by definition of $l(P)$ as the smallest λ-system containing P, we have

$$l_B \supset l(P)$$

Glancing at the definition of l_B we see that this means

$$A \cap B \in l(P) \text{ for every } B \in l(P)$$

Since A is any element of $l(P)$ we have thus established our goal (3), i.e. $l(P)$ is a π system.

Proof of the $\pi - \lambda$ theorem Let X be a set, and

$$P \subset L \subset \mathcal{P}(X)$$

with P a π system and L a λ system. Then, by definition of $l(P)$ as the smallest λ system containing P, we have

$$l(P) \subset L$$

But $l(P)$ is a σ–algebra, and $l(P) \supset P$. Therefore,

$$\sigma(P) \subset l(P)$$

This completes the argument, since $l(P) \subset L$. Note that since every sigma-algebra is also a λ system it follows that $l(P) \subset \sigma(P)$. Thus, in fact,

$$l(P) = \sigma(P) \blacksquare$$

Problem Set

1. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, and let $A \in \mathcal{F}$. Show that the set of all $B \in \mathcal{F}$ which satisfy
$$\mu(A \cap B) = \mu(A)\mu(B)$$
is a λ-system.
2. Find an example of a λ system which is not a σ-algebra.
3. Let $k \in \mathbf{R}$ and E a Borel subset of $\mathbf{R}^d$. Prove that
$$m(kE) = |k|^d m(E),$$