Chapter 1

Kolomogorov's Theorem

This chapter will describe Kolmogorov's theorem and present its proof at a very leisurely pace.

1.1 Statement of the Basic Result

Here we shall consider the most basic form of Kolmogorov's theorem. It says that if we are given a family of probability measures μ_n on $\mathbf{R}^n$, for all $n \geq 1$, which satisfy a natural consistency condition then these measures fit together to form a measure μ on the infinite product $\mathbf{R}^\infty$. The infinite product $\mathbf{R}^\infty$ consists of all sequences $(x_1, x_2, ...)$ of real numbers x_n, and has the *product sigma–algebra*, which is the smallest sigma–algebra containing all sets of the form $E \times \mathbf{R} \times \mathbf{R} \times \cdots$ with E running over all Borel sets in $\mathbf{R}^n$ and n running over $\{1, 2, 3, ...\}$.

Theorem 1 *Suppose that for each $n \geq 1$, μ_n is a Borel probability measure on $\mathbf{R}^n$ such that the Kolmogorov consistency condition*

$$\mu_{n+k}(E \times \mathbf{R}^k) = \mu_n(E) \text{ for every } n, k \geq 1 \text{ and every Borel set } E \subset \mathbf{R}^n \tag{1.1}$$

holds. Then there is a unique probability measure μ on the product sigma–algebra of $\mathbf{R}^\infty$ such that for any $n \geq 1$ and any Borel subset E of $\mathbf{R}^n$, the measure $\mu(E \times \mathbf{R} \times \mathbf{R} \times \cdots)$ equals $\mu_n(E)$.

Notice that if μ does exist satisfying the conclusion of the theorem then the consistency condition must necessarily hold.

1.2 A key topological result

A key ingredient in Kolmogorov's proof is an intricate fact which guarantees that the intersection of a certain family of sets is non–empty.

Theorem 2 *Suppose that for each positive integer n, we have a non–empty compact set $C_n \subset \mathbf{R}^n$. Assume that these sets satisfy the following condition: for each n, if $(x_1, ..., x_{n+1}) \in C_{n+1}$ then $(x_1, ..., x_n) \in C_n$. Then there exists a sequence $(x_1, x_2, ...) \in \mathbf{R}^\infty$ such that $(x_1, ..., x_n) \in C_n$ for every n.*

Proof. For $1 \leq m \leq n$, let $p_{mn} : \mathbf{R}^n \to \mathbf{R}^m$ be the projection on the first m factors:

$$p_{mn} : \mathbf{R}^n \to \mathbf{R}^m : (x_1, ..., x_n) \mapsto (x_1, ..., x_m)$$

These are continuous maps. Thus each projection $p_{mn}(C_n)$ is a non–empty compact subset of $\mathbf{R}^m$, for any $n \geq m \geq 1$.

We will check that

$$p_{mm}(C_m) \supset p_{m\,m+1}(C_{m+1}) \supset \cdots,$$

i.e. that the sets $p_{mk}(C_k)$ form a decreasing sequence of compact subsets of $\mathbf{R}^m$ as k runs over the integers $m, m+1, ...$ To see this, fix positive integers $n \geq m$, and consider any point $x = (x_1, ..., x_{n+1}) \in C_{n+1}$. According to the hypotheses, it follows that $x' = (x_1, ..., x_n) \in C_n$. Now $p_{m\,n+1}(x)$ is the same as $p_{mn}(x')$, both being $(x_1, ..., x_m)$. Thus every point of $p_{m\,n+1}(C_{n+1})$ is in $p_{mn}(C_n)$. This establishes the claimed property.

The decreasing sequence of non–empty compact sets

$$p_{11}(C_1) = C_1 \supset p_{12}(C_2) \supset \cdots$$

must have non–empty intersection (this fact about compact sets is proven below in Lemma 3). Pick any point x_1 in this intersection. Thus x_1 belongs to the projection $p_{1n}(C_n)$ for each $n \geq 1$. This means that for each $n \geq 1$, there is a point $y \in \mathbf{R}^{n-1}$ such that $(x_1, y) \in C_n$.

Having found x_1, let us consider, for $n \geq 1$, the sets

$$C_n(x_1) = \{y \in \mathbf{R}^{n-1} : (x_1, y) \in C_n\}$$

Thus $C_n(x_1)$ is the projection onto $\mathbf{R}^{n-1}$ of the non–empty compact set $p_{1n}^{-1}(x_1) \cap C_n$. Moreover, the sets $C_2(x_1), C_3(x_1), C_4(x_1), ...$ satisfy the same conditions as the original sequence $C_1, C_2, ...$, for if $(y_2, ..., y_{n+1}) \in C_{n+2}(x_1)$ then $(x_1, y_2, ..., y_{n+1}) \in C_{n+2}$ and so $(x_1, y_2, ..., y_n) \in C_n$ which means $(y_1, ..., y_n) \in C_{n+1}(x_1)$. Thus, by the argument with which we obtained the point x_1, there is a point x_2 which is such that for every $n \geq 2$ there is some $y \in \mathbf{R}^{n-1}$ such that $(x_2, y) \in C_n(x_1)$. Thus, for each $n \geq 2$, $(x_1, x_2) \in p_{2n}(C_n)$.

Running the argument above by induction, we obtain a sequence of real numbers $x_1, x_2, x_3, ...$, such that for any $n \geq m \geq 1$, the point $(x_1, ..., x_m)$ lies in the projection $p_{mn}(C_n)$. In particular, $(x_1, ..., x_n) \in C_n$ for each $n \geq 1$. QED

Recall that a subset K of $\mathbf{R}^n$ is said to be *compact* if every open covering of K has a finite subcovering. Taking complements of this definition shows that if a collection of compact sets is such that every finite subcollection has non-empty intersection then the intersection of the entire family is also non-empty. Here is a formal statement:

Lemma 3 *Let $C_1 \supset C_2 \supset \cdots$ be non-empty compact subsets of $\mathbf{R}^n$. Then $\cap_{n\geq 1} C_n \neq \emptyset$.*

Proof. Suppose $\cap_{n\geq 1} C_n = \emptyset$. Then taking complements we have $\cup_{n\geq 1} C_n^c = \mathbf{R}^n$. In particular, every point of C_1 must be in some C_n^c; of course, this n would have to be ≥ 2. Thus C_1 is covered by the open sets $C_2^c, C_3^c, ...$ Since C_1 is compact it follows that finitely many of these cover C_1. Now $C_2^c \subset C_3^c \subset \cdots$. So C_1 is a subset of C_n^c for some n. But we also know that $C_1 \supset C_n$. This would mean that C_n is a subset of C_n^c, which would imply that C_n is empty, a contradiction. QED

1.3 Regularity of measures on R

Consider a measure μ on the Borel σ–algebra $\mathcal{B}$ of a topological space X. The measure μ is said to be inner–regular on a Borel set $A \subset X$ if this set can be approximated arbitrarily well from the inside by compact sets, i.e. if

$$\mu(A) = \sup\{\mu(K) : K \text{ is a compact subset of } A\}$$

The measure μ is *outer–regular* on A if A can be approximated arbitrarily well from the outside by open sets:

$$\mu(A) = \inf\{\mu(U) : U \text{ is an open set and } U \supset A\}$$

We say that A is *regular* for μ if μ is both inner–regular and outer–regular on A. If the measure of A is finite then regularity is equivalent to the statement that for any $\epsilon > 0$ there exists compact K and open U with

$$K \subset A \subset U$$

and

$$\mu(U - K) < \epsilon$$

On a space like $\mathbf{R}^n$, it turns out that any Borel measure which assigns finite measure to compact sets is regular on every Borel set. All we shall need is summarized in the following:

Theorem 4 *Suppose μ is a finite measure on the Borel sigma–algebra of $\mathbf{R}^n$. Then every Borel set is regular for μ.*

Proof. Let $\mathcal{A}$ be the collection of all Borel subsets of $\mathbf{R}^n$ which are regular for μ. We shall prove that $\mathcal{A}$ is a sigma–algebra and contains all open subsets of $\mathbf{R}^n$.

Clearly, $\emptyset \in \mathcal{A}$.

Next, suppose $A_1, A_2, ... \in \mathcal{A}$. We will show that $A = \cup_{n\geq 1} A_n$ is also regular. Fix $\epsilon > 0$. Then there exist compact sets K_n and open U_n with $K_n \subset A_n \subset U_n$ with

$$\mu(U_n - K_n) < \epsilon/2^n$$

The sets $\cup_{n=1}^m A_n$ increase with m and have union A. So

$$\lim_{m\to\infty} \mu(\cup_{n=1}^m A_n) = \mu(A)$$

Since $\mu(A)$ is finite, there is an $N \geq 1$ such that

$$\mu(A) - \mu(A_1 \cup \cdots \cup A_N) < \epsilon$$

Let

$$K = K_1 \cup \cdots \cup K_N$$

This is a compact subset of A and

$$\mu\left(\cup_{n=1}^N A_n - \cup_{n=1}^N K_n\right) \leq \mu\left(\cup_{n=1}^N (A_n - K_n)\right) \leq \sum_{n=1}^N \epsilon/2^n < \epsilon$$

and so

$$\mu(A) - \mu(K) = \mu(A) - \mu(A_1 \cup \cdots \cup A_N) + \mu\left(\cup_{n=1}^N A_n - \cup_{n=1}^N K_n\right) \leq 2\epsilon$$

Thus A is inner–regular. Next, let $U = \cup_{n=1}^\infty U_n$. This is an open set containing A as a subset and

$$\mu(U) - \mu(A) \leq \sum_{n=1}^{\infty} \mu(U_n - A_n) < \epsilon$$

which shows that A is also outer–regular.

Finally, we check that $\mathcal{A}$ is closed under complementation. Let $A \in \mathcal{A}$, and $\epsilon > 0$. Then there exist compact K and open U with $K \subset A \subset U$ and $\mu(U - K) < \epsilon$. Taking complements we have

$$U^c \subset A^c \subset K^c$$

Moreover, since $K^c - U^c = U - K$ we have

$$\mu(K^c - U^c) = \mu(U - K) < \epsilon$$

Now K^c is open and so we see that A^c is outer–regular. However, U^c, though certainly closed, may not be compact. To fix this problem, note that $\mathbf{R}^n$ is the union of an increasing sequence of compact sets $D_1, D_2, \ldots$ (for example, D_j can be taken to be the closed ball centered 0 of radius j). So $\mu(D_j) \to \mu(\mathbf{R}^n)$ as $j \uparrow \infty$. Choose j large enough so that $\mu(\mathbf{R}^n) - \mu(D_j) < \epsilon$. Now consider the set

$$K' = U^c \cap D_j$$

Being the intersection of the closed set U^c and the compact set D_j it is compact; moreover, it is contained as a subset in U^c and hence also in A^c. The complement of D_j has measure $< \epsilon$. So certainly, the part of U^c not in D_j has measure $< \epsilon$. This means

$$\mu(U^c - K') < \epsilon$$

Therefore,

$$\mu(A^c - K') = \mu(A^c - U^c) + \mu(U^c - K') < 2\epsilon$$

So A^c is also inner–regular.

Thus $\mathcal{A}$ is a σ–algebra. It remains to show that $\mathcal{A}$ contains all open sets. Let U be any open set. It is automatically outer–regular. To prove inner–regularity, recall that in $\mathbf{R}^n$, every open set is a countable union of compact sets. Thus there are compact sets $K_1, K_2, ... \subset U$ whose union is all of U. Let $C_n = K_1 \cup \cdots \cup K_n$. Then each C_n is a compact subset of U, the sets C_n form an increasing sequence of sets with union U. So $\mu(C_n) \to \mu(U)$, as $n \to \infty$. Thus U is inner–regular. QED

1.4 Proof of Kolmogorov's theorem

Let us recall the framework and hypotheses. For each $n \geq 1$ we are given a Borel probability measure μ_n on $\mathbf{R}^n$ satisfying the consistency condition that

$$\mu_{n+1}(E \times \mathbf{R}) = \mu_n(E)$$

for every Borel set $E \subset \mathbf{R}^n$. Our task is to construct a measure μ on the product sigma–algebra in $\mathbf{R}^\infty$ such that for any Borel $E \subset \mathbf{R}^n$ the measure $\mu(E \times \mathbf{R} \times \mathbf{R} \times \cdots)$ equals $\mu_n(E)$.

Let $\mathcal{F}$ be the set of all measurable *cylinder sets* in $\mathbf{R}^\infty$, i.e. sets of the form $E \times \mathbf{R} \times \mathbf{R} \times \cdots$ with E a Borel set in some $\mathbf{R}^n$. It will be convenient to denote by $\mathcal{F}_n$ the collection of all sets of the form $E \times \mathbf{R} \times \mathbf{R} \times \cdots$ with E a Borel set in $\mathbf{R}^n$. Clearly $\mathcal{F}_n$ is a σ–algebra, and

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset ...$$

The fact that $\mathcal{F}$ is the union of an increasing family of σ–algebras implies that $\mathcal{F}$ is an algebra. For, it contains $\emptyset$ which is in every $\mathcal{F}_n$, it is closed under complementation since each $\mathcal{F}_n$ is, and the union of $A, B \in \mathcal{F}$ is in $\mathcal{F}$ because if $A \in \mathcal{F}_n$ and $B \in \mathcal{F}_m$ then $A, B \in \mathcal{F}_{\max\{m,n\}}$ and so $A \cup B \in \mathcal{F}_{\max\{m,n\}} \subset \mathcal{F}$.

$\mathcal{F}$ is not a sigma–algebra since it is not closed under countably infinite unions. Any sigma–algebra which contains $\mathcal{F}$ contains all the cylinder sets of the form $\mathbf{R} \times \mathbf{R} \times \cdots \times A \times \mathbf{R} \times \cdots$ for all Borel $A \subset \mathbf{R}$. Therefore, such a sigma–algebra would contain the product sigma–algebra of $\mathbf{R}^\infty$. Thus $\sigma(\mathcal{F})$ is the product sigma–algebra of $\mathbf{R}^\infty$.

Define a set function μ' on $\mathcal{F}$ by

$$\mu'(E \times \mathbf{R} \times \mathbf{R} \times \cdots) = \mu_n(E)$$

for every Borel $E \subset \mathbf{R}^n$. The consistency hypothesis on the system of measures μ_n guarantees that μ' is well–defined.

We shall prove that μ' is a probability measure on $\mathcal{F}$. The Caratheodory extension theorem will then imply that there is a probability measure μ on $\sigma(\mathcal{F})$ which agrees with μ' on $\mathcal{F}$.

It is clear that $\mu' \geq 0$, that $\mu'(\emptyset) = 0$, $\mu'(\mathbf{R}^\infty) = 1$. Moreover, μ' is finitely additive because if $A, B \in \mathcal{F}$ then $A, B \in \mathcal{F}_k$ for some k, and so A, B are of the form

$$A = A' \times \mathbf{R} \times \mathbf{R} \times \cdots, \quad B = B' \times \mathbf{R} \times \mathbf{R} \times \cdots$$

for some Borel sets $A', B' \subset \mathbf{R}^k$. If A, B are disjoint then A', B' must also be disjoint and so

$$\mu'(A \cup B) = \mu_k(A' \cup B') = \mu_k(A') + \mu_k(B') = \mu'(A) + \mu'(B)$$

Thus μ' has all the properties of a measure except possibly countable additivity. We shall now prove that μ' is in fact also countably additive.

Since μ' is a finitely–additive measure, to prove countable additivity it will suffice to show that if $A_1, A_2, ...$ are a decreasing sequence of sets in $\mathcal{F}$ with $\inf_n \mu'(A_n) > 0$ then $\cap_{n\geq 1} A_n \neq \emptyset$.

Let

$$\alpha = \inf_{n\geq 1} \mu'(A_n)$$

We shall show later that it will suffice to assume that the A_n's are arranged in such a way that each A_n belongs to $\mathcal{F}_n$. This means that

$$A_n = A'_n \times \mathbf{R} \times \mathbf{R} \times \cdots$$

for a Borel $A'_n \subset \mathbf{R}$. The condition $A_n \supset A_{n+1}$ means then that for every $(y_1, ..., y_{n+1}) \in A'_{n+1}$ we have $(y_1, ..., y_n) \in A'_n$. Now

$$\mu'(A_n) = \mu_n(A'_n) > \alpha$$

So, by regularity of A'_n, there is a compact set $D'_n \subset A'_n$ such that

$$\mu_n(A'_n - D'_n) < \alpha/2^{n+1}$$

Let

$$D_n = D'_n \times \mathbf{R} \times \mathbf{R} \times \cdots$$

and let

$$C_n = D_1 \cap D_2 \cap \cdots \cap D_n$$

Then

$$C_1 \supset C_2 \supset ...$$

and

$$\mu'(A_n - C_n) = \mu'\left(\cap_{k=1}^n A_k - \cap_{k=1}^n D_k\right) \leq \sum_{k=1}^n \mu_k'(A_k - D_k) < \alpha/2$$

Since $\mu'(A_n)$ is $> \alpha$ it follows that $\mu'(C_n) > \alpha/2$.

Now C_n is in $\mathcal{F}_n$ and so is of the form $C_n' \times \mathbf{R} \times \cdots$. Moreover,

$$C_n' = (D_1' \times \mathbf{R}^{n-1}) \cap (D_2' \times \mathbf{R}^{n-2}) \cap \cdots \cap D_n'$$

is compact and is non–empty since $\mu_n(C_n') = \mu'(C_n) > \alpha/2 > 0$. Furthermore, since $C_1 \supset C_2 \supset \cdots$ it follows that for each $(y_1, ..., y_{n+1}) \in C_{n+1}'$ we have $(y_1, ..., y_n) \in C_n'$. Finally, note also that $C_n' \subset A_n'$, since $C_n' \subset D_n'$.

The result proved in Section 1.2 now implies that there is a sequence $x = (x_1, x_2, ...) \in \mathbf{R}^\infty$, with $(x_1, ..., x_n) \in C_n'$ for each $n \geq 1$. It follows that the point x lies in all the sets A_n. Thus

$$\cap_{n=1}^\infty A_n \neq \emptyset$$

All that remains now is to justify the assumption that the sets A_n can be arranged in such a way that each A_n belongs to $\mathcal{F}_n$. Recall that each A_n is in $\mathcal{F}$. Therefore each A_n is in some $\mathcal{F}_{j(n)}$; since the $\mathcal{F}_j$'s form an increasing family, it follows that A_n belongs to all $\mathcal{F}_k$ with $k \geq j(n)$. Let

$$k(1) = j(1),\ k(2) = \max\{j(1), j(2)\} + k(1), k(3) = \max\{j(1), j(2), j(3)\} + k(2) ...$$

Then

$$A_1 \in \mathcal{F}_{k(1)}, \quad A_2 \in \mathcal{F}_{k(2)}, \cdots$$

and

$$k(1) < k(2) < \cdots$$

Now we reorganize the list of sets A_k. Let $\tilde{A}_1 = A_1$, and set $\tilde{A}_r = A_1$ for $1 \leq r \leq k(1)$. Then it follows automatically that

$$\tilde{A}_1 \in \mathcal{F}_1, ..., \tilde{A}_{k(1)} \in \mathcal{F}_{k(1)}$$

Next let $\tilde{A}_{k(1)+1} = A_2$, and continue with

$$\tilde{A}_r = A_2 \quad \text{for } k(1) < r \leq k(2)$$

This ensures that

$$\tilde{A}_r \in \mathcal{F}_r$$

holds for $r \in \{1, ..., k(2)\}$. Continuing in this way we obtain a sequence of sets $\tilde{A}_1, \tilde{A}_2, ...$, which, as a collection of sets is simply the original collection $\{A_1, A_2, ...\}$, but now satisfies the condition $\tilde{A}_n \in \mathcal{F}_n$ for all $n \geq 1$. This is all we needed. QED

1.5 Kolmogorov's theorem for Uncountable Products

Kolmogorov's theorem holds not only for countable products but also for uncountable products. To this end, let us first introduce some notation.

Let J be any non–empty set, possibly uncountable. We shall denote non–empty finite subsets of J by greek letters $\alpha, \beta, ...$

The product $\mathbf{R}^J$ is the set of all maps $x : J \to \mathbf{R}$; an element $x \in \mathbf{R}^J$ may be also denoted $(x_j)_{j \in J}$. We have the coordinate projection maps

$$\pi_j : \mathbf{R}^J \to \mathbf{R} : x \mapsto x_j$$

A *cylinder set* is a subset of $\mathbf{R}^J$ of the form

$$\pi_{j_1}^{-1}(A_{j_1}) \cap \cdots \cap \pi_{j_n}^{-1}(A_{j_n})$$

for $j_1, ..., j_n \in J$ and $A_{j_1}, ..., A_{j_n} \subset \mathbf{R}$. This is the set of all $(x_j)_{j \in J}$ for which the coordinates $x_{j_1}, ..., x_{j_n}$ lie in the sets $A_{j_1}, ..., A_{j_n}$, respectively. We will call this a *measurable* cylinder set if $A_{j_1}, ..., A_{j_n}$ are Borel sets.

If I and K are any non–empty subsets of J with $I \subset K$ then we have a natural projection map

$$p_{IK} : \mathbf{R}^K \to \mathbf{R}^I : x = (x_j)_{j \in K} \mapsto (x_j)_{j \in I} = x|I$$

where $x|I$ is just the restriction of x to I.

The projection map $p_{IJ} : \mathbf{R}^J \to \mathbf{R}^I$ will be denoted simply by p_I.

Thus, a measurable cylinder set is a subset of $\mathbf{R}^J$ of the form

$$p_\alpha^{-1}(A)$$

for some non–empty finite subset α of J and a Borel set $A \subset \mathbf{R}^\alpha$.

The collection of all measurable cylinder sets forms an algebra $\mathcal{F}$, which is not a sigma–algebra unless J is finite.

The *product sigma–algebra* on $\mathbf{R}^J$ is the sigma–algebra generated by all the projections π_j. Equivalently, it is the sigma–algebra $\sigma(\mathcal{F})$ generated by all measurable cylinder sets.

Theorem 5 *Suppose that for every non–empty finite subset $\alpha \subset J$ we are given a Borel probability measure μ_α on $\mathbf{R}^\alpha$. Suppose also that Kolmogorov's consistency condition*

$$\mu_\beta(p_{\alpha\beta}^{-1}(A)) = \mu_\alpha(A) \tag{1.2}$$

holds for every non–empty finite subsets $\alpha \subset \beta$ of J and every Borel set $A \subset \mathbf{R}^\alpha$. Then there is a unique probability measure μ on $\sigma(\mathcal{F})$ such that

$$\mu\left(p_\alpha^{-1}(A)\right) = \mu_\alpha(A)$$

for every non–empty finite subset α of J and every Borel set $A \subset \mathbf{R}^\alpha$.

Proof. Much of the proof is basically the same as it was for the countable case. The essential new element comes in at the end where we prove that the measure on the algebra of cylinder sets is "continuous from above at $\emptyset$." However, we will go through a slow and detailed account of the whole argument.

Let F be the set of all non–empty finite subsets of J.

For any $\alpha \in F$, let $\mathcal{F}_\alpha$ be the collection of all sets of the form $p_\alpha^{-1}(A)$ for some Borel set $A \subset \mathbf{R}^\alpha$. This is readily seen to be a σ–algebra. Indeed, it is the smallest sigma–algebra with respect to which $p_\alpha : \mathbf{R}^J \to \mathbf{R}$ is measurable.

We shall list some obervations about the maps p_α and $p_{\alpha\beta}$:

- It is useful to observe that each set of $\mathcal{F}_\alpha$ is *uniquely* of the form $p_\alpha^{-1}(A)$; for if $p_\alpha^{-1}(A) = p_\alpha^{-1}(B)$ then $A = B$, as can be verified using the fact that the projection map $p_\alpha : \mathbf{R}^J \to \mathbf{R}^\alpha$ is *surjective.*
- If $\alpha \subset \beta$ then
$$p_{\alpha\beta} \circ p_\beta = p_\alpha$$
- For any $\alpha, \beta \in F$, we have $\alpha \cup \beta \in F$.

We have the collection of all measurable cylinder sets in $\mathbf{R}^J$:

$$\mathcal{F} = \cup_{\alpha \in F} \mathcal{F}_\alpha$$

The collection $\mathcal{F}$ is an algebra: (i) the empty set is in $\mathcal{F}$ because it is in any $\mathcal{F}_\alpha$; (ii) if $E \in \mathcal{F}$ then $E \in \mathcal{F}_\alpha$ for some α and then $E^c \in \mathcal{F}_\alpha$ and hence $E^c \in \mathcal{F}$; (iii) if $E_1, E_2 \in \mathcal{F}$ then $E_1 \in \mathcal{F}_{\alpha_1}$ and $E_2 \in \mathcal{F}_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in F$, and then $E_1 \cup E_2 \in \mathcal{F}_{\alpha_1 \cup \alpha_2}$. Note that the last step used the fact that for any two $\alpha_1, \alpha_2 \in F$ the union $\alpha_1 \cup \alpha_2$ is also in F.

Now define μ'_α on $\mathcal{F}_\alpha$ by

$$\mu'_\alpha(p_\alpha^{-1}(A)) = \mu_\alpha(A)$$

for every Borel set $A \subset \mathbf{R}^\alpha$. Because each set in $\mathcal{F}_\alpha$ is of the form $p_\alpha^{-1}(A)$ in a unique way, μ'_α is well-defined.

Now suppose $\alpha, \beta \in F$ with $\alpha \subset \beta$. Then it is readily seen that

$$\mathcal{F}_\alpha \subset \mathcal{F}_\beta$$

Moreover, the Kolmogorov consistency condition (1.2) implies that

$$\mu'_\beta | \mathcal{F}_\alpha = \mu'_\alpha$$

for every $\alpha, \beta \in F$ with $\alpha \subset \beta$. For, suppose $E \in \mathcal{F}_\alpha$; then $E = p_\alpha^{-1}(A)$ for a Borel $A \subset \mathbf{R}^\alpha$ and so

$$\begin{aligned}
\mu'_\beta(E) &= \mu'_\beta\left(p_\alpha^{-1}(A)\right) \\
&= \mu'_\beta\left((p_{\alpha\beta} \circ p_\beta)^{-1}(A)\right) \\
&= \mu'_\beta\left(p_\beta^{-1} p_{\alpha\beta}^{-1}(A)\right) \\
&= \mu_\beta\left(p_{\alpha\beta}^{-1}(A)\right) \\
&= \mu_\alpha(A) \quad \text{by the consistency condition} \\
&= \mu'_\alpha\left(p_\alpha^{-1}(A)\right) \\
&= \mu'_\alpha(A)
\end{aligned}$$

So the μ_α's fit together in a well-defined way to yield a function

$$\mu' : \mathcal{F} \to [0,1]$$

given by

$$\mu'(E) = \mu_\alpha(E) \text{ for every } E \in \mathcal{F}_\alpha$$

Thus μ' is a set function on the algebra of cylinder sets.

It is readily checked that μ' is a *finitely*–additive measure on $\mathcal{F}$. For, if $A, B \in \mathcal{F}$ then $A \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$ and so $A, B \in \mathcal{F}_\gamma \subset \mathcal{F}$, where $\gamma = \alpha \cup \beta$; so, if such A and B are disjoint then

$$\begin{aligned} \mu'(A \cup B) &= \mu_\gamma(A \cup B) \\ &= \mu_\gamma(A) + \mu_\gamma(B) \\ &= \mu'(A) + \mu'(B) \end{aligned}$$

By the Caratheodory theorem, it will suffice now to prove that μ is actually countably additive on $\mathcal{F}$. As we now this is equivalent to proving "continuity from above."

To this end, let $E_1, E_2, ...$ be measurable cylinder sets with

$$E_1 \supset E_2 \supset \cdots$$

and with

$$\inf_{n \geq 1} \mu'(E_n) > 0$$

Our objective is to show that $\cap_{n \geq 1} E_n$ is non–empty.

At this point we use a key fact (see Lemma 6 below) about sigma–algebras: for any measurable set $E \in \sigma(\mathcal{F})$ there is a *countable* subset $I \subset J$ such that $E \in \sigma(\pi_i : i \in I)$.

Thus each E_i is in the sigma–algebra generated by a countable collection of projections π_k for k running over a countable collection $I_i \subset J$. The union $\cup_i I_i$, being a countable union of countable sets is countable. So we may denote this union by

$$\mathbf{P} = \{1, 2, 3, ...\}$$

(If the union $\cup_i I_i$ is a finite set, the proof is much easier.)

Then we have essentially the situation that we had in the proof of Kolmogorov's theorem in the case of countable J. The argument there shows that $\cap_{n \geq 1} E_n \neq \emptyset$. $\boxed{\text{QED}}$

We have used:

Lemma 6 *Let X be a set, $\mathcal{A}$ a non–empty collection of subsets of X, and $\sigma(\mathcal{A})$ the sigma–algebra generated by $\mathcal{A}$. Then for any $A \in \sigma(\mathcal{A})$ there is a countable subset $\mathcal{A}'$ of $\mathcal{A}$ such that $A \in \sigma(\mathcal{A}')$.*

If $\{f_\alpha\}_{\alpha \in J}$ is a non–empty family of functions on X then for each set $A \in \sigma(\{f_\alpha\}_{\alpha \in J})$ there is a countable set $I \subset J$ such that $A \in \sigma(\{f_\alpha\}_{\alpha \in I})$.

Proof. Let $\mathcal{B}$ be the set of all subsets $A \subset X$ such that $A \in \sigma(\mathcal{A}')$ for some countable $\mathcal{A}' \subset \mathcal{A}$. It is readily checked that $\mathcal{B}$ is a sigma–algebra. Of course, $\mathcal{B}$ contains each set of $\mathcal{A}$. Therefore, $\mathcal{B} \supset \sigma(\mathcal{A})$, i.e. every set in $\sigma(\mathcal{A})$ satisfies the desired condition.

The argument for the function version of the result is essentially the same. QED

1.6 Further Generalizations

Kolmogorov's product measure construction does not work for arbitrary probability spaces. However, glancing over the construction we see that if we try to go through it for a general product $\prod_{j\in J}(\Omega_j, \mathcal{B}_j)$, then the probability measures μ_α on the products $\prod_{j\in\alpha}(\Omega_j, \mathcal{B}_j)$, for finite $\alpha \in \mathbf{R}$, need to be "regular" in some sense.

A class of spaces for which the Kolmogorov theorem works and which is closed under countable products, are the spaces of *complete separable metric spaces* which are also called *Polish* spaces.

Exercise

1. Let F be a non–empty set with a partial ordering denoted by $\leq$. Assume that for every $\alpha, \beta \in F$ there is a $\gamma \in F$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

   Suppose that for each $\alpha \in F$, we have a probability space $(\Omega_\alpha, \mathcal{F}_\alpha, \mu_\alpha)$, where Ω_α is a Hausdorff topological space, and $\mathcal{F}_\alpha$ is a σ–algebra of subsets of Ω_α containing all Borel sets.

   Assume that μ_α is *regular* in the sense that for any $A \in \mathcal{F}_\alpha$ and $\epsilon > 0$ there is a compact set $K \subset A$ such that $\mu_\alpha(A - K) < \epsilon$.

   The case to keep in mind as motivation is with F being the set of all non–empty finite subsets of a given non–empty set J, and $\Omega_\alpha = \mathbf{R}^\alpha$.

   Assume that for each $\alpha \leq \beta$ there is given a mapping

   $$p_{\alpha\beta} : \Omega_\beta \to \Omega_\alpha$$

   such that:

   P1. $p_{\alpha\alpha}$ is the identity map on Ω_α

   P2. $p_{\alpha\beta} \circ p_{\beta\gamma} = p_{\alpha\gamma}$ for any $\alpha \leq \beta \leq \gamma$ in F

P3. each map $p_{\alpha\beta}$ is continuous and measurable, in the sense that $p_{\alpha\beta}^{-1}(\mathcal{F}_\alpha) \subset \mathcal{F}_\beta$ for all $\alpha \leq \beta$ in F

P4. each map $p_{\alpha\beta}$ is surjective

KC. the Kolmogorov consistency condition

$$\boxed{\mu_\beta\left(p_{\alpha\beta}^{-1}(E)\right) = \mu_\alpha(E)} \tag{1.3}$$

holds for all $\alpha, \beta \in F$ with $\alpha \leq \beta$ and $E \in \mathcal{F}_\alpha$.

Let $\Omega' = \prod_{\alpha \in F} \Omega_\alpha$ and let Ω be the subset of Ω' consisting of all $(y_\alpha)_{\alpha \in F}$ for which $p_{\alpha\beta}(y_\beta) = y_\alpha$ for each $\alpha \leq \beta$. Let

$$p_\alpha : \Omega \to \Omega_\alpha : y \mapsto y_\alpha$$

be the projection onto the α–th component. We assume that $\Omega \neq \emptyset$ and, moreover, p_α is surjective. Assume, furthermore, the following condition:

P5. if F' is any countable non–empty subset of F and if for each $\alpha \in F'$ we have an element $y_\alpha \in \Omega_\alpha$ such that $p_{\beta\alpha}(y_\alpha) = y_\beta$ for all $\alpha, \beta \in F'$ with $\beta \leq \alpha$, then there is an element $y \in \Omega$ such that $p_\alpha(y) = y_\alpha$ for every $\alpha \in F$.

Let $\mathcal{F}'_\alpha$ be the collection of all subsets of Ω of the form $p_\alpha^{-1}(E')$ with E' running over $\mathcal{F}_\alpha$. Thus $\mathcal{F}'_\alpha$ is the sigma–algebra generated by the projection map p_α. Let

$$\mathcal{F}' = \cup_{\alpha \in F} \mathcal{F}'_\alpha$$

a. Show that

$$p_{\alpha\beta} \circ p_\beta = p_\alpha$$

for all $\alpha, \beta \in F$ with $\alpha \leq \beta$.

b. Show that

$$\mathcal{F}'_\alpha \subset \mathcal{F}'_\beta \text{ if } \alpha \leq \beta \text{ in } F$$

Next show that $\mathcal{F}'$ is an algebra of subsets of Ω.

c. Show that if $E \in \mathcal{F}'_\alpha$ then there is a *unique* $E' \in \mathcal{F}_\alpha$ such that $E = p_\alpha^{-1}(E')$. Show that this E' is in fact $p_\alpha(E)$. [Hint: p_α is surjective.]

d. Define μ'_α on $\mathcal{F}'_\alpha$ by

$$\mu'_\alpha\left(p_\alpha^{-1}(E')\right) = \mu_\alpha(E')$$

Clearly μ'_α is a probability measure on $\mathcal{F}'_\alpha$. Verify that

$$\mu'_\beta|\mathcal{F}'_\alpha = \mu'_\alpha$$

for all $\alpha, \beta \in F$ with $\alpha \leq \beta$.

e. By part d, there is a well-defined function $\mu' : \mathcal{F}' \to [0, 1]$ specified by

$$\mu'(E) = \mu'_\alpha(E) \text{ if } E \in \mathcal{F}'_\alpha$$

Verify that μ' is a *finitely–additive* probability measure (it is clear that $\mu'(\emptyset) = 0$, $\mu'(E) \geq 0$ for all $E \in \mathcal{F}'$, and $\mu'(\Omega) = 1$; so what you have to show is that $\mu'(E \cup F) = \mu'(E) + \mu'(F)$ for all disjoint $E, F \in \mathcal{F}' = \cup_{\alpha \in F} \mathcal{F}'_\alpha$.)

f. Suppose I is a countable subset of F, consisting of distinct elements $\beta_1, \beta_2, ...$ Show that there exist $\alpha_1, \alpha_2, ... \in F$ satisfying: (i) $\beta_n \leq \alpha_n$ for all $n \geq 1$, and (ii) $\alpha_1 \leq \alpha_2 \leq \cdots$.

g. If $E_1, E_2, ...$ is a countable collection of sets in $\mathcal{F}'$, show that there exist $\alpha_1, \alpha_2, ... \in F$ with $\alpha_1 \leq \alpha_2 \leq \cdots$ and $E_n \in \mathcal{F}'_{\alpha_n}$ for each $n \geq 1$.

h. Let $E_1 \supset E_2 \supset \cdots$ be a countable collection of sets in $\mathcal{F}'$ and $\alpha_1 \leq \alpha_2 \leq \cdots$ elements of F such that $E_n \in \mathcal{F}'_{\alpha_n}$ for each $n \geq 1$. Thus each E_n is of the form $p_{\alpha_n}^{-1}(E'_n)$ for a unique $E'_n \in \mathcal{F}_{\alpha_n}$ and, because $p_{\alpha_n} : \Omega \to \Omega_{\alpha_n}$ is surjective, we have

$$E'_n = p_{\alpha_n}(E_n)$$

Show that for $n \leq m$, the condition $E_n \supset E_m$ is equivalent to

$$E'_n \supset p_{\alpha_n \alpha_m}(E'_m)$$

i. Notation and hypotheses as in part h. Fix $\epsilon > 0$ and, by the regularity hypothesis, choose compact $D_n \subset E'_n$ with

$$\mu_{\alpha_n}(E'_n - D_n) < \epsilon/2^{n+1}$$

Let

$$C_n = p^{-1}_{\alpha_1\alpha_n}(D_1) \cap p^{-1}_{\alpha_2\alpha_n}(D_2) \cap \cdots \cap p^{-1}_{\alpha_n\alpha_n}(D_n),$$

where the last term is of course just D_n. Show that C_n is a compact subset of E'_n.

j. With notation and hypotheses as in part h, show that the sequence of sets C_n has the property that for any $n \leq m$,

$$p_{\alpha_n\alpha_m}(C_m) \subset C_n$$

k. With notation and hypotheses as in part h, show that

$$\mu_{\alpha_n}(E'_n - C_n) < \epsilon/2$$

[Hint: Use $E'_n - C_n \subset \cup_{k=1}^n p^{-1}_{\alpha_k\alpha_n}(E'_k - D_k)$, and the consistency condition $\mu_{\alpha_n} \circ p^{-1}_{\alpha_k\alpha_n} = \mu_{\alpha_k}$.]

l. Notation and hypotheses as in h, but assume now also that

$$\inf_{n\geq 1} \mu_{\alpha_n}(E'_n) = \epsilon > 0$$

Show that each $\mu_{\alpha_n}(C_n)$ is $\geq \epsilon/2$.

m. Now suppose that for each $n \in \{1, 2, 3, ...\}$ we have a non–empty set C_n (such as the one in part l above) which is a compact subset of Ω_{α_n} and such that for any $n \geq m \geq 1$, $p_{\alpha_m\alpha_n}(C_n) \subset C_m$. Prove that there is a point $y \in \Omega$ such that $p_{\alpha_n}(y) \in C_n$ for every $n \geq 1$.

n. Suppose that $E_1 \supset E_2 \supset \cdots$ are a countable collection of sets in $\mathcal{F}'$ such that $\inf_{n\geq 1} \mu'(E_n) > 0$. Prove that $\cap_{n\geq 1} E_n \neq \emptyset$.

o. Prove that there is a probability measure μ on the sigma–algebra $\mathcal{F} = \sigma(\mathcal{F}')$ such that

$$\mu\left(p^{-1}_{\alpha}(E')\right) = \mu_\alpha(E')$$

for every $\alpha \in F$ and every $E' \in \mathcal{F}_\alpha$.