Mean Value, Taylor, and all that

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November 2009
Careful: Not proofread!
Derivative

Recall the definition of the derivative of a function $f$ at a point $p$:

$$f'(p) = \lim_{w \to p} \frac{f(w) - f(p)}{w - p}$$ (1)
Thus, to say that
\[ f'(p) = 3 \]
means that if we take any neighborhood \( U \) of 3, say the interval \((1, 5)\), then the ratio
\[ \frac{f(w) - f(p)}{w - p} \]
falls inside \( U \) when \( w \) is close enough to \( p \), i.e. in some neighborhood of \( p \). (Of course, we can’t let \( w \) be equal to \( p \), because of the \( w - p \) in the denominator.)
So if
\[ f'(p) = 3 \]

then the ratio
\[ \frac{f(w) - f(p)}{w - p} \]

lies in (1, 5) when \( w \) is close enough to \( p \), i.e. in some neighborhood of \( p \), but not equal to \( p \).
So if \( f'(p) = 3 \)

then the ratio

\[
\frac{f(w) - f(p)}{w - p}
\]

lies in \((1, 5)\) when \( w \) is close enough to \( p \), i.e. in some neighborhood of \( p \), but not equal to \( p \).

In particular,

\[
\frac{f(w) - f(p)}{w - p} > 0 \quad \text{if } w \text{ is close enough to } p, \text{ but } \neq p.
\]
From $f'(p) = 3$ we found that

$$\frac{f(w) - f(p)}{w - p} > 0$$

if $w$ is close enough to $p$, but $\neq p$.

Looking at this you see that:

• when $w > p$, but near $p$, the value $f(w)$ is $> f(p)$.

• when $w < p$, but near $p$, the value $f(w)$ is $< f(p)$. 
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Looking at this you see that:

- when \( w > p \), but near \( p \), the value \( f(w) \) is > \( f(p) \).

- when \( w < p \), but near \( p \), the value \( f(w) \) is < \( f(p) \).
Looking back at the argument, we see that the only thing about the value 3 for $f'(p)$ which made it all work is that it is $> 0$. 

Thus:

**Theorem**

If $f'(p) > 0$ then:

- the values of $f$ to the right of $p$, but close to $p$, are $> f(p)$,
- and the values of $f$ to the left of $p$, but close to $p$, are $< f(p)$. 

Looking back at the argument, we see that the only thing about the value 3 for \( f'(p) \) which made it all work is that it is \( > 0 \). Thus:

**Theorem**

If \( f'(p) > 0 \) then:

The values of \( f \) to the right of \( p \), but close to \( p \), are \( > f(p) \),

and the values of \( f \) to the left of \( p \), but close to \( p \), are \( < f(p) \).
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Positive Derivative and Increasing behavior

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- the values of $f$ to the left of $p$, but close to $p$, are $< f(p)$. 
Negative Derivative and Decreasing behavior

Similarly,

**Theorem**

*If \( f'(p) < 0 \) then:*

- The values of \( f \) to the right of \( p \), but close to \( p \), are \(< f(p)\),
- The values of \( f \) to the left of \( p \), but close to \( p \), are \( > f(p) \).
Negative Derivative and Decreasing behavior

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**Theorem**

*If* \( f'(p) < 0 \) *then:*

*the values of* \( f \) *to the right of* \( p \), *but close to* \( p \), *are* \( < f(p) \),
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**Theorem**

If $f'(p) < 0$ then:

the values of $f$ to the right of $p$, but close to $p$, are $< f(p)$,

and

the values of $f$ to the left of $p$, but close to $p$, are $> f(p)$. 
Local Maxima and Minima

A function $f$ is said to have a *local maximum* at a point $p$ if there is a neighborhood $U$ of $p$ such that for all $x \in U$ in the domain of $f$, the value $f(x)$ is $\geq f(p)$.

A function $f$ is said to have a *local minimum* at a point $p$ if there is a neighborhood $U$ of $p$ such that for all $x \in U$ in the domain of $f$, the value $f(x)$ is $\leq f(p)$. 
Local Maxima and Minima

Figure: Local Maxima and Minima
The local Maxima/Minima theorem

Theorem
Suppose $f$ is defined in a neighborhood of a point $p \in \mathbb{R}$, and
The local Maxima/Minima theorem

**Theorem**

Suppose $f$ is defined in a neighborhood of a point $p \in \mathbb{R}$, and $f(p) \geq f(x)$ for all $x$ in a neighborhood of $p$. Then $f'(p)$ must be $0$. If $f(p) \leq f(x)$ for all $x$ in a neighborhood of $p$, and $f'(p)$ exists, then $f'(p)$ is $0$.

Note that we are requiring that $f$ be defined in a neighborhood of $p$, and so on both sides of $p$. 
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Suppose $f$ is defined in a neighborhood of a point $p \in \mathbb{R}$, and $f(p) \geq f(x)$ for all $x$ in a neighborhood of $p$. Suppose also that $f'(p)$ exists.

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If \( f(p) \leq f(x) \) for all \( x \) in a neighborhood of \( p \), and \( f'(p) \) exists, then \( f'(p) \) is 0.

Note that we are requiring that \( f \) be defined in a neighborhood of \( p \), and so on both sides of \( p \).
Proof of the Local Max/Min Theorem

Proof Suppose $f'(p)$ exists but is not 0. Then $f'(p)$ is either $> 0$ or $< 0$.
If $f'(p) > 0$ then we know that to the right of $p$, but close to $p$, the values of $f$ are $> \text{ than } f(p)$,
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Proof Suppose $f'(p)$ exists but is not 0. Then $f'(p)$ is either $> 0$ or $< 0$. If $f'(p) > 0$ then we know that to the right of $p$, but close to $p$, the values of $f$ are $> f(p)$, and to the left of $p$, but close to $p$, the values are $< f(p)$. Therefore, $f'(p)$ must be 0.
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But this would mean that $p$ is neither a local maximum nor a local minimum for $f$. 
Proof Suppose $f'(p)$ exists but is not 0. Then $f'(p)$ is either $> 0$ or $< 0$.
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Similarly, $f'(p) < 0$ is also not possible.
Proof of the Local Max/Min Theorem

Proof Suppose \( f'(p) \) exists but is not 0. Then \( f'(p) \) is either \( > 0 \) or \( < 0 \).
If \( f'(p) > 0 \) then we know that to the right of \( p \), but close to \( p \), the values of \( f \) are \( > \) than \( f(p) \), and to the left of \( p \), but close to \( p \), the values are \( < f(p) \).

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Similarly, \( f'(p) < 0 \) is also not possible.

Thus, \( f'(p) \) must be 0.
Rolle’s Theorem

Theorem
Consider a function

\[ f : [a, b] \to \mathbb{R} \]

where \( a, b \in \mathbb{R} \) with \( a < b \). Suppose

- \( f \) is continuous function
- \( f \) is differentiable on \((a, b)\)
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- \( f \) is continuous function
- \( f \) is differentiable on \( (a, b) \)
- \( f(a) = f(b) \).

Then there is a point \( c \) strictly between \( a \) and \( b \) where the derivative of \( f \) is 0:

\[ f'(c) = 0 \text{ for some } c \in (a, b). \]
Proof of Rolle’s Theorem

A fundamental theorem about continuous functions on compact intervals says that \( h \) reaches a maximum value and a minimum value in the interval \([a, b]\).
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A fundamental theorem about continuous functions on compact intervals says that \( h \) reaches a maximum value and a minimum value in the interval \([a, b]\).

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The only other possibility is that both the max and the min value occur at the end points \( a \) and \( b \). But \( h \) has the same value at \( a \) and at \( b \). So then the max and the min value must be the same. Thus in this case \( h \) is constant and so its derivative is 0 everywhere.
Useful consequence Rolle’s Theorem

Suppose now that $f$ and $g$ are functions on a compact interval $[a, b]$, and are differentiable in $(a, b)$.

Next suppose also that $f$ and $g$ have the same value at $a$, and also the same value at $b$:

\[ f(a) = g(a), \quad \text{and} \quad f(b) = g(b). \]
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Then $f'$ and $g'$ agree at some point $c$ between $a$ and $b$:

$$f'(c) = g'(c) \quad \text{for some } c \in (a, b).$$
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\]

Then \( f' \) and \( g' \) agree at some point \( c \) between \( a \) and \( b \):

\[
f'(c) = g'(c) \quad \text{for some} \ c \in (a, b).
\]

To see this simply apply Rolle’s theorem to the function \( h = f - g \).
Mean Value Theorem

Theorem

Suppose $f$ is continuous on a compact interval $[a, b]$ and differentiable in $(a, b)$. Then there is a point $c$ in $(a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
Proof of Mean Value Theorem

Proof Compare $f$ with the straight line function $L$ which agrees with $f$ at the points $a$ and $b$:

$$L(a) = f(a), \quad L(b) = f(b),$$

and the slope of $L$ is constant given by

$$\frac{L(b) - L(a)}{b - a} = \frac{f(b) - f(a)}{b - a}$$
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As consequence of Rolle’s theorem we see that there is a point $c \in (a, b)$ where the derivatives of $f$ and $L$ agree. But the derivative of $L$ at any point is the constant value given above.
Proof of Mean Value Theorem

**Proof** Compare $f$ with the straight line function $L$ which agrees with $f$ at the points $a$ and $b$:

\[ L(a) = f(a), \quad L(b) = f(b), \]

and the slope of $L$ is constant given by

\[ \frac{L(b) - L(a)}{b - a} = \frac{f(b) - f(a)}{b - a} \]

As consequence of Rolle’s theorem we see that there is a point $c \in (a, b)$ where the derivatives of $f$ and $L$ agree. But the derivative of $L$ at any point is the constant value given above. Hence:

\[ f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a} \]
Polynomials: coefficients and derivatives at 0

Consider a polynomial

\[ P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \]

Observe that

\[ P'(x) = a_1 + 2a_2 x + 3a_3 x^2 \]
\[ P^{(2)}(x) = 2a_1 + 3 \times 2a_3 x \]
\[ P^{(3)}(x) = 3 \times 2 \times 1 a_3 \]

Of course, \( P^{(3)}(x) \) is constant for all \( x \).
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Observe now that if we put in $x = 0$ we can recover the values of $a_0, a_1, a_2, a_3$:

$$a_0 = P(0)$$
$$a_1 = P'(0)$$
$$a_2 = \frac{1}{2!} P^{(2)}(0)$$
$$a_3 = \frac{1}{3!} P^{(3)}(0)$$

Of course, $P^{(3)}(x)$ is constant for all $x$. 
In general, we have for any polynomial of degree $n$:

$$P(x) = P(0) + P'(0)x + \frac{P^{(2)}(0)}{2!}x^2 + \ldots + \frac{P^{(n)}(0)}{n!}x^n$$  \hspace{1cm} (2)
Polynomials with specified derivatives derivatives at 0

In general, we have for any polynomial of degree $n$:

$$P(x) = P(0) + P'(0)x + \frac{P''(0)}{2!}x^2 + \ldots + \frac{P^{(n)}(0)}{n!}x^n$$  \hspace{1cm} (2)

Moreover, the $n$-th derivative of this polynomial is a constant.
Polynomials with specified derivatives derivatives at 0

Exercise. Find a polynomial function $P$ for which

\[ P(0) = 1, \quad P'(0) = 1, \quad P''(0) = -2, \quad P^{(3)}(0) = 12 \]

Solution: The simplest choice is

\[ 1 + 1.x + \frac{-2}{2!}x^2 + \frac{12}{3!}x^3 \]
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We could also take, for instance,

$$1 + 1 \cdot x + \frac{-2}{2!} x^2 + \frac{12}{3!} x^3 + \frac{K}{4!} x^4,$$

where $K$ is any constant.
Polynomials with specifications

Exercise. Find a polynomial function $P$ for which

$$P(0) = -4, \quad P'(0) = 3, \quad P''(0) = -4, \quad P^{(3)}(0) = 6$$

and also

$$P(1) = 5$$

Solution: To satisfy the conditions at 0 we can take the polynomial

$$P(x) = -4 + 3x + \frac{-4}{2!}x^2 + \frac{6}{3!}x^3 + \frac{K}{4!}x^4,$$

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where $K$ is any constant.

Now tune the constant $K$ to the requirement that $P(1)$ be 5,
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$$P(x) = -4 + 3x + \frac{-4}{2!} x^2 + \frac{6}{3!} x^3 + \frac{K}{4!} x^4,$$

where $K$ is any constant.

Now tune the constant $K$ to the requirement that $P(1)$ be 5, i.e. choose $K$ in such a way that

$$5 = -4 + 3 \times 1 + \frac{-4}{2!} 1^2 + \frac{6}{3!} 1^3 + \frac{K}{4!} 1^4,$$

which we can solve for $K$. 
Taylor Polynomial of a Function

Consider a function $f$ defined in a neighborhood of 0, and differentiable 15 times.

We know that we can choose a polynomial function $P$ whose value and derivatives at 0 up to order the 14th order match those for $f$:

$P(0) = f(0), \quad P'(0) = f'(0), \ldots, \quad P^{(14)}(0) = f^{(14)}(0)$

For instance, we can take

$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(14)}(0)}{14!}x^{14} + K\frac{1}{15!}x^{15}$

where $K$ is any constant (could be 0 too in the simplest case).
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For instance, we can take

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(14)}(0)}{14!}x^{14} + \frac{K}{15!}x^{15} \quad (3)$$

where $K$ is any constant (could be 0 too in the simplest case).
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We could put in an additional requirement, say that

\[ P(4) = f(4) \]

This would let us pin down the constant \( K \).
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We can also get a description of the constant \( K \) by repeatedly applying Rolle's theorem:

Since \( f(x) \) and \( P(x) \) agree at \( x = 0 \) and \( x = 4 \), their derivatives agree at some point \( c_1 \) strictly between 0 and 4:

\[ f'(c_1) = P'(c_1) \]
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But then... \( f' \) and \( P' \) agree at both 0 and \( c_1 \),
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\[ f'(c_1) = P'(c_1) \]

But then... \( f' \) and \( P' \) agree at both 0 and \( c_1 \), hence their derivatives agree at a point \( c_2 \) in between:

\[ f^{(2)}(c_2) = P^{(2)}(c_2) \]
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and on and on .... until ...
Taylor Polynomial of a Function

we have a point $c$, of course still between 0 and 4, where $f^{(15)}$ and $P^{(15)}$ agree:

$$f^{(15)}(c) = P^{(15)}(c)$$

Now if you look back at (3) to see what $P(x)$ was, you can see that the 15th-derivative of $P$ is the constant $K$:

$$P^{(15)}(x) = \frac{K}{15!} 15! x^0 = K$$

Hence,

$$K = f^{(15)}(c)$$

Thus, the constant $K$ happens to be the 15-th derivative of $f$ at some point $c$ between 0 and 4.
There is nothing special about 15. The general result is:

\[
\text{Taylor’s Theorem} \\
\text{Suppose } f \text{ is a function defined in a neighborhood of } 0 \\
\text{and is } n \text{ times differentiable on this neighborhood, where } n \text{ is some positive integer (i.e. } n \in \{1, 2, 3, \ldots \}).
\]

Then for any } x \text{ in this neighborhood there is a point } c \text{ lying between } 0 \text{ and } x \text{ such that}

\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + f^{(n)}(c)\frac{x^n}{n!}
\]

The main point here is the remainder or error term \( R_n \) when } f \text{ is approximated by the Taylor polynomial}

\[
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\]
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The main point here is the *remainder* or *error* term

$$R_n = \frac{f^{(n)}(c)}{n!}x^n$$

when $f$ is approximated by the *Taylor polynomial*

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Thus for such functions \( f \) we have, for \( x \) in some neighborhood of 0,

\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k
\]

The function \( f \) for which this holds for all \( x \) in a neighborhood \( U \) of 0 is said to be *analytic* on \( U \).
Analytic Functions

In class, we proved that the functions $e^x$ and $\sin x$ are analytic, by showing that the Taylor remainder goes to 0 in each case. Polynomials are, of course, analytic, because the remainder term becomes 0 for them eventually.