Math 7330: Functional Analysis Fall 2005

Homework 1 A. Sengupta

In the following, V is a *finite-dimensional* complex vector space with a Hermitian inner-product $(\cdot,\cdot)$, and $A : V \to V$ a linear map.

1. Let $e_1, ..., e_n$ be an *orthonormal* basis of V.
   (i) Show that the matrix for A relative to the basis $e_1, ..., e_n$ has $A_{ij} = (Ae_j, e_i)$ as the entry at the i–th row and j–th column.

   (ii) Show that for the matrix of A^*,

$$(A^*)_{ij} = \overline{A}_{ji}$$

2. Suppose that A is a *normal* operator, i.e. it commutes with its adjoint:

$$AA^* = A^*A$$

Show that

$$|Ax| = |A^*x|$$

for all $x \in V$.

3. Show that for a complex number $\lambda \in \mathbf{C}$ the following are equivalent:
   - $A - \lambda I$ is not invertible
   - there is a *non-zero* vector $x \in V$ for which $Ax = \lambda x$
   - $\det(A - \lambda I) = 0$

If $k \in \mathbf{C}$ and non-zero $y \in V$ satisfy $Ay = ky$ then k is an *eigenvalue* of A and y is an *eigenvector* corresponding to the eigenvalue k. In general, we shall use the notation

$$M_k = \{v \in V : Av = kv\} = \ker(A - kI)$$

The set of all $\lambda \in \mathbf{C}$ for which $A - \lambda I$ is not invertible is called the *spectrum* of A.

4. Determine the spectrum of A if its matrix $[A_{ij}]$ is diagonal

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

5. Prove that the spectrum $\sigma(A)$ of A is non-empty and contains at most n elements, where $n = \dim V$.

6. Suppose A is normal. Show that

$$Ax = \lambda x \quad \Leftrightarrow \quad A^* x = \overline{\lambda} x$$

7. Suppose A is normal. Show that M_λ and M_μ are orthogonal if $\lambda \neq \mu$. (Hint: Let $x \in M_\lambda$ and $y \in M_\mu$, and consider $(x, Ay) = (A^* x, y)$.)

8. Suppose $X \subset V$ a subspace such that $A(X) \subset X$. Show that

$$A^*(X^\perp) \subset X^\perp$$

where $X^\perp$ is the orthogonal complement of X in V.

9. Suppose A is normal, and let X be the subspace spanned by all the subspaces M_λ:

$$X = \sum_{\lambda \in \sigma(A)} M_\lambda$$

Show that $X = V$. [Hint: Use several Problems 8,5 and 6.]

10. **Spectral Theorem** in finite dimensions: Suppose that the operator $A : V \to V$ is normal. Let $P_\lambda : V \to V$ be the *orthogonal projection* onto M_λ. This is the linear operator which satisfies $P_\lambda x = x$ if $x \in M_\lambda$ and $P_\lambda x = 0$ if $x \in M_\lambda^\perp$. Show that

$$A = \sum_{\lambda \in \mathbf{C}} \lambda P_\lambda$$

Let $e_1, ..., e_n$ be any orthonormal basis of V made up of bases of the subspaces M_λ (for $\lambda \in \mathbf{C}$). Show that the matrix of A relative to such a basis is diagonal. Conversely, show that if there is an orthonormal basis relative to which the matrix of a certain operator is diagonal then that operator is normal.

Math 7330: Functional Analysis Fall 2005

Homework 2 A. Sengupta

In the following, H is a complex Hilbert space with a Hermitian inner-product $(\cdot, \cdot)$. All operators are operators on H.

1. Suppose P and Q are orthogonal projections.

   (i) Show that if $PQ = QP$ then PQ is an orthogonal projection.

   (ii) Show that, conversely, if PQ is an orthogonal projection then $PQ = QP$.

2. Let P and Q be orthogonal projections.

(i) Show that if $PQ = P$ then $PQ = QP$ and $\mathrm{Im}(P) \subset \mathrm{Im}(Q)$. Show that the same conclusions hold if $QP = P$.

(ii) Show that if $\mathrm{Im}(P) \subset \mathrm{Im}(Q)$ then $QP = P$.

3. Suppose A, B, C are mutually orthogonal closed subspaces of H, and let P_A, P_B, P_C be the orthogonal projections onto A, B, C, respectively. Let $X = A+B$ and $Y = C+B$, and let P_X and P_Y be the orthogonal projections onto X and Y, respectively.
(i) Show that $P_X P_Y = P_Y P_X$.

(ii) Express P_X and P_Y in terms of P_A, P_B and P_C.

(iii) Express P_A, P_B and P_C in terms of P_X and P_Y.

4. Suppose P and Q are orthogonal projections which commute, i.e. $PQ = QP$. The goal is to show that then the geometric situation of the preceding problem holds, i.e. there are mutually orthogonal closes subspaces A, B, C such that P is the orthogonal projection onto $A + B$ and Q is the orthogonal projection onto $C + B$. Let

$$R = PQ, \qquad S = P(I - Q), \qquad T = Q(I - P)$$

Observe that

$$P = S + R \qquad \text{and} \qquad Q = T + R$$

(i) Show that R, S, and T are orthogonal projections. [Note that if A is an orthogonal projection then so is $I - A$, and B commutes with A then it also commutes with $I - A$.]

(ii) Show that $RS = SR = 0$, $RT = TR = 0$, and $ST = TS = 0$.

(iii) Show that $\mathrm{Im}(R)$, $\mathrm{Im}(S)$, and $\mathrm{Im}(T)$ are mutually orthogonal. Thus R, S, T are orthogonal projections onto mutually orthogonal closed subspaces.

5. Let $x_1, x_2, x_3, ...$ be a sequence of mutually orthogonal vectors in the Hilbert space H. Let $S_n = x_1 + \cdots + x_n$. Let $S'_n = |x_1|^2 + \cdots + |x_n|^2$.

(i) Show that for any integers $m \geq n$,

$$|S_m - S_n|^2 = S'_m - S'_n$$

(ii) Show that the series $\sum_{n=1}^{\infty} x_n$ to converge in H if and only if the series $\sum_n |x_n|^2$ converges.

Spectral Measures

In the following, Ω is a non-empty set, $\mathcal{B}$ is a σ–algebra of subsets of Ω. A *spectral measure* is a mapping E from $\mathcal{B}$ to the set of all orthogonal projections on H satisfying the following conditions:

(i) $E(\emptyset) = 0$

(ii) $E(\Omega) = I$

(iii) if $A_1, A_2, ... \in \mathcal{B}$ are mutually disjoint and their union is the set A then

$$(E(A)x, y) = \sum_{n=1} \big(E(A_n)x, y\big) \tag{1}$$

for every $x, y \in H$

(iv) if $A, B \in \mathcal{B}$ then

$$E(A)E(B) = E(B)E(A) = E(A \cap B)$$

For $x, y \in H$ define $E_{x,y} : \mathcal{B} \to \mathbf{C}$ by

$$E_{x,y}(A) \overset{\text{def}}{=} (E(A)x, y)$$

Conditions (i) and (iii) say that $E_{x,y}$ is a complex measure. If $x = y$ we have

$$E_{x,x}(A) = (E(A)x, x) = |E(A)x|^2 \geq 0 \tag{2}$$

where we used the fact if P is any orthogonal projection then any $x \in H$ decomposes as $Px + x - Px$ with Px being perpendicular to $x - Px$ and so

$$(Px, x) = (Px, Px + x - Px) = (Px, Px) + 0 = |Px|^2 \tag{3}$$

The non-negativity in (2) shows that

$E_{x,x}$ *is an (ordinary) measure on* $(\Omega, \mathcal{B})$

Recall that on the complex Hilbert space H any bounded linear operator A is determined uniquely by the "diagonal values" (Ax, x). It follows that if E and E' are spectral measures for which $E_{x,x} = E'_{x,x}$ for all $x \in H$ then $E = E'$.

6. Let E be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space H. By a "measurable subset of Ω" we mean, of course, a subset of Ω which belongs to the σ–algebra $\mathcal{B}$.

   (i) Show that if A and B are disjoint measurable subsets of Ω then $E(A)$ and $E(B)$ are projections onto orthogonal subspaces, i.e. $\mathrm{Im}\big(E(A)\big)$ and $\mathrm{Im}\big(E(B)\big)$ are orthogonal to each other.

   (ii) Let $A_1, A_2, ...$ be a sequence of disjoint measurable subsets of Ω (i.e. each A_j is in $\mathcal{B}$). Let $A = \cup_{j=1}^{\infty} A_j$. Show that for every $x \in H$ the series

$$\sum_{n=1}^{\infty} E(A_n)x$$

   is convergent in H.

(iii) With notation and hypotheses as before, show that

$$E(A)x = \sum_{n=1}^{\infty} E(A_n)x$$

for every $x \in H$. [Hint: Take inner-product with any $y \in H$]

(iv) Suppose $A_1, A_2, ...$ are as above but assume now also that infinitely many of the projections $E(A_n)$ are non-zero. Prove that the series $\sum_{n=1}^{\infty} E(A_n)$ does not converge in operator norm. [Hint: Let $s_n = E(A_1) + \cdots + E(A_n)$, and suppose $s = \lim_{n\to\infty} s_n$ exists. Then $\lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$. What is $s_n - s_{n-1}$ and what is the norm of a non-zero projection?]

Measure Theory and Integration

We recall a few facts from measure theory and integration. In the following, Ω is a non-empty set, $\mathcal{B}$ is a σ–algebra of subsets of Ω, and μ a measure on $\mathcal{B}$.

(a) A function $f : \Omega \to \mathbf{C}$ is said to be *measurable* if $f^{-1}(U)$ is in $\mathcal{B}$ for every open set $U \subset \mathbf{C}$. Write $f = u + iv$, where u and v are real-valued. Then f is measurable if and only if u and v are measurable. Write u as $u^+ - u^-$, where $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. Then u is measurable if and only if u^+ and u^- are measurable.

(b) A function $s : \Omega \to \mathbf{C}$ is a *simple function* if it has only finitely many values, i.e. $s(\Omega)$ is a finite subset of Ω. If $c_1, ..., c_n$ are all the distinct values of s and $A_i = s^{-1}(c_i)$ the set on which s has value c_i, then

$$s = \sum_{j=1}^{n} c_j 1_{A_j}$$

Here 1_B denotes the *indicator function* of B, equal to 1 on B and 0 outside B. The simple function s is measurable if and only if each of the sets A_i is measurable.

(c) Let $F : \Omega \to [0, \infty]$ be a non-negative function. For each positive integer n, divide $[0, \infty]$ into intervals of length $1/2^n$, i.e. into the intervals $[(k-1)2^{-n}, k2^{-n})$. Define a function s_n which is equal to the lower value $(k-1)2^{-n}$ on the set $A_{nk} = F^{-1}[(k-1)2^{-n}, k2^{-n})$, for $k = 1, ..., n2^n$, but cut off the value of s_n at the maximum value n at all points in the set A'_n where $F > n$. The construction ensures that $0 \le s_n \le F$, $s_n \le n$, and that $|F - s_n| \le 2^{-n}$ at all points where $F \le n$. Thus if the function F is *bounded* then $|F - s_n| < 2^{-n}$ holds for all n large enough and so, in particular, $s_n(x) \to F(x)$ *uniformly* in $x \in \Omega$. If F is measurable so is each of the sets A_{nk} and A'_n and so the function s_n is then also measurable. Now consider a function $f : \Omega \to \mathbf{C}$. Writing $f = u + iv$, with u and v real–valued, and then splitting $u = u^+ - u^-$ and $v = v^+ - v^-$, it follows that we can construct a sequence of simple functions s_n such that $|s_n(x)| \le |f(x)|$ for all $x \in \Omega$, $s_n(x) \to f(x)$ *uniformly* if f is *bounded*, and each s_n is measurable if f is measurable.

(d) If s is a measurable simple function and $c_1, ..., c_n$ are all the distinct values of s then

$$\int s \, d\mu \overset{\text{def}}{=} \sum_{j=1}^{n} c_j \mu([s = c_j])$$

where $[s = c_j]$ is the set $s^{-1}(c_j)$ of all points where s has value c_j.

(e) If s and t are measurable simple functions then considering the number of ways $s+t$ can take a particular value, it follows that $\int (s+t) \, d\mu = \int s \, d\mu + \int t \, d\mu$. Also, $\int \alpha s \, d\mu = \alpha \int s \, d\mu$ for every $\alpha \in \mathbf{C}$. The additivity property has the following consequence: if $s = a_1 1_{A_1} + \cdots + a_m 1_{A_m}$, where $A_1, ..., A_m$ are measurable but *may overlap* then $\int s \, d\mu = \sum_{j=1}^{m} a_j \mu(A_j)$ still holds.

7. Let E be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space H. Let $\mathcal{N}$ be the set of all sets $A \in \mathcal{B}$ for which $E(A) = 0$. Thus $\mathcal{N}$ consists of sets of E–measure 0.

   (i) Show that if A and B are measurable sets and $A \subset B$ and $E(B) = 0$ then $E(A) = 0$.

   (ii) Show that $\mathcal{N}$ is closed under countable unions.

   (ii) Let $f : \Omega \to \mathbf{C}$ be a measurable function. Show that there is a largest open subset U of $\mathbf{C}$ such that $f^{-1}(U)$ is in $\mathcal{N}$.

(iii) The *essential range* σ_f of f is the closed set given by the complement of the open set U of (ii). The *essential supremum* of f, denoted $|f|_\infty$, is the radius of the smallest closed ball (center 0) containing σ_f. Thus

$$|f|_\infty = \inf\{r \geq 0 : E[|f| > r] = 0\}$$

Suppose f and g are measurable functions which are *essentially bounded*, i.e. $|f|_\infty$ and $|g|_\infty$ are finite. Then show

$$|f + g|_\infty \leq |f|_\infty + |g|_\infty$$

and for every complex number α:

$$|\alpha f|_\infty = |\alpha||f|_\infty$$

8. Let E be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space H.

(i) Let $A_1, ..., A_n, B_1, ..., B_m \in \mathcal{B}$ and $a_1, ..., a_n, b_1, ..., b_m \in \mathbf{C}$, and suppose

$$\sum_{j=1}^{n} a_j 1_{A_j} = \sum_{j=m}^{n} b_j 1_{B_j}$$

Show that

$$\sum_{j=1}^{n} a_j E(A_j) = \sum_{j=m}^{n} b_j E(B_j) \tag{4}$$

[Hint: Let $s = \sum_{j=1}^{n} a_j 1_{A_j} = \sum_{j=m}^{n} b_j 1_{B_j}$, and consider the operators $T = \sum_{j=1}^{n} a_j E(A_j)$ and $R = \sum_{j=m}^{n} b_j E(B_j)$. Take any $x \in H$ and show that both (Tx, x) and (Rx, x) equal $\int s\, dE_{x,x}$.] The common value in (4) will be denote

$$\int s\, dE$$

(ii) Check that for any measurable simple function s on Ω:

$$\left((\int s\, dE) x, x \right) = \int s\, dE_{x,x}$$

holds for every $x \in H$.

(iii) Let s, t be measurable simple functions on Ω and $\alpha, \beta \in \mathbf{C}$. Show that

$$\int (\alpha s + \beta t)\, dE = \alpha \int s\, dE + \beta \int t\, dE$$

(iv) Let s, t be measurable simple functions on Ω. Show that

$$\left(\int s\, dE\right)\left(\int t\, dE\right) = \int st\, dE$$

[Hint: Write out s and t in the usual forms $\sum_j a_j 1_{A_j}$ and $\sum_k b_j 1_{B_k}$ and then work out st and write out both sides of the above equation.]

(v) Let s be a measurable simple function on Ω. Show that

$$\left(\int s\, dE\right)^* = \int \overline{s}\, dE$$

(vi) Let s be a measurable simple function on Ω. Show that

$$\left|\int s\, dE\right| \leq |s|_\infty$$

[Hint: Let T be the operator $\int s\, dE$. Then $|T| = \sup_{|x|\leq 1} |Tx|$. Now $|Tx|^2 = (Tx, Tx) = (T^*Tx, x)$. Show that (T^*Tx, x) equals $\int |s|^2\, dE_{x,x}$. Next use $|s| \leq |s|_\infty$ almost-everywhere for the measure E_x.]

(vii) Let $f : \Omega \to \mathbf{C}$ be a bounded measurable function. We know that there exists a sequence of measurable simple functions s_n on Ω such that $s_n(x) \to f(x)$, as $n \to \infty$, *uniformly* for $x \in \Omega$ and $|s_n(x)| \leq |f(x)|$ for all $x \in \Omega$. Part (vi) above shows then that the sequence of operators $\int s_n\, dE$ is Cauchy in operator norm and therefore *converges in operator norm* to a limit which we denote by $\int f\, dE$:

$$\int f\, dE \stackrel{\text{def}}{=} \lim_{n\to\infty} \int s_n\, dE$$

where the limit is in operator norm. Now suppose s'_n is another sequence of measurable functions on Ω which converge to f in the sense that $|s'_n - f|_\infty \to 0$ as $n \to \infty$. Show that $\int s'_n\, dE$ also converges to $\int f\, dE$ as $n \to \infty$. [Hint: Use (vi) for $s_n - s'_n$.] Thus the definition of $\int f\, dE$ does not depend on the choice of the sequence s_n converging to f.

(viii) Show that

$$\Big((\int f\, dE)x, x \Big) = \int f\, dE_{x,x}$$

for every bounded measurable function f and every $x \in H$.

(ix) Prove the analogs of (iii)-(vi) for bounded measurable functions.

9. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. For any measurable functions f and g on Ω let $M_f g$ denote the function fg. If f is bounded and $g \in L^2(\mu)$ then clearly $M_f g$ is also in $L^2(\mu)$ and indeed $M_f : L^2(\mu) \to L^2(\mu)$ is a bounded linear operator with norm $|M_f| \leq |f|_\infty$ (in all practical cases $|M_f|$ is actually equal to $|f|_\infty$). It is clear that $f \mapsto M_f$ is linear and, moreover, $M_{fh} = M_f M_h$.

(i) Show that $M_f^* = M_{\overline{f}}$. (Hint: Let $g, h \in L^2(\mu)$ and work out $(M_f g, h)_{L^2}$.)

(ii) Show that for any measurable set A, the operator M_{1_A} is an orthogonal projection operator.

(iii) Show that $E : A \mapsto M_{1_A}$ is a spectral measure. [Hint: The only non-trivial thing to check is that for any $g \in L^2(\mu)$ and disjoint measurable sets A_n whose union is A we have $\sum_n E(A_n)g = E(A)g$ with the sum $\sum_n$ being L^2–convergent. To this end, let $G_n = \sum_{j=1}^n E(A_j)g$ and look at what happens to $\int |G_n - 1_A g|^2 \, d\mu$ a $n \to \infty$.]

(iv) For any measurable simple function s show that $\int s\,dE = M_s$, where E is as in (iii).

(v) For any bounded measurable function f show that $\int f\,dE = M_f$, where E is as in (iii). [Hint: Choose measurable simple s_n converging uniformly to f, and with $|s_n(x)| \leq |f(x)|$ for all $x \in \Omega$. Consider the norms of $\int f\,dE - \int s_n\,dE$ and $M_f - M_{s_n}$.]

Math 7330: Functional Analysis Fall 2005

Notes/Homework 3: Banach Algebras A. Sengupta

A *complex algebra* is a complex vector space B on which there is a bilinear multiplication map

$$B \times B \to B : (x, y) \mapsto xy$$

which is associative. Bilinearity of multiplication means the distributive law

$$x(y + z) = xy + xz, \qquad (y + z)x = yz + zx$$

for all $x, y, z \in B$, and

$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$

for all $a, b \in B$ and $\lambda \in \mathbf{C}$. In particular, a complex algebra is automatically a ring. An element $e \in B$ is a multiplicative identity (or *unit element*) if

$$xe = ex = x$$

for all $x \in B$. If e' is also a multiplicative identity then

$$e = ee' = e'$$

Thus the multiplicative identity, if it exists, is unique.

Suppose B is a complex algebra with unit e. An element $x \in B$ is *invertible* if there exists an element $y \in B$, called an *inverse* of x, such that

$$yx = xy = e$$

If y' is another element for which both xy' and $y'x$ equal e then

$$y = ey = (y'x)y = y'(xy) = y'e = y'$$

Thus if x is invertible then it has a unique inverse, which is denoted x^{-1}.

The set of all invertible elements in B will be denoted $G(B)$. It is clearly a group.

Assume, moreover, that there is a norm on the complex algebra B which makes it a Banach space, the identity e has norm 1:

$$|e| = 1,$$

and that

$$|xy| \leq |x||y|$$

for all $x, y \in B$. Then B is called a *complex Banach algebra.*

In all that follows B is a complex Banach algebra.

1. Let B be a complex Banach algebra. Let $x \in B$, and let

$$s_N = \sum_{n=0}^{N} x^n = e + x + x^2 + \cdots + x^N$$

(i) Show that

$$(e - x)s_N = s_N(e - x) = e - x^{N+1}$$

(ii) Show that if $|x| \neq 1$ then for any integers $N \geq M \geq 0$,

$$|x^M + x^{M+1} + \cdots + x^N| \leq \frac{|x|^M - |x|^{N+1}}{1 - |x|}$$

(iii) Show that if $|x| < 1$ then the limit

$$s = \sum_{n=0}^{\infty} x^n \stackrel{\text{def}}{=} \lim_{N\to\infty} s_N$$

exists.

(iv) Show that if $|x| < 1$ then

$$s = (e - x)^{-1}$$

Thus for any $x \in B$ with $|x| < 1$ the element $e - x$ is invertible. Note that this conclusion is an *algebraic* property.

The *spectrum* $\sigma(x)$ of an element x in a complex Banach algebra B is the set of all complex numbers $\lambda \in \mathbf{C}$ for which $\lambda e - x$ *does not have an inverse.*

2. Show that for any $x \in B$, the spectrum $\sigma(x)$ is contained in the closed ball $\{\lambda \in \mathbf{C} : |\lambda| \leq |x|\}$:

$$\sigma(x) \subset \{\lambda \in \mathbf{C} : |\lambda| \leq |x|\}$$

3. Let $G(B)$ be the set of all invertible elements of B. Show that $G(B)$ is open by going through the following argument. Let $x, h \in B$ be such that x is invertible and $|h| < 1/|x^{-1}|$. Observe that $x+h = (e+hx^{-1})x$. So, since x is invertible, invertibility of $x+h$ will be established if we can show that $e + hx^{-1}$ is invertible. For this use the result from the previous problem.

4. Show that the map $G(B) \to G(B) : x \mapsto x^{-1}$ is differentiable. Hint: Let $x \in G(B)$ and $h \in B$ be such that $|h| < 1/|x^{-1}|$. Look at

$$(x+h)^{-1} - x^{-1} = x^{-1}[(e + hx^{-1})^{-1} - e]$$

Set $y = -hx^{-1}$ and show that

$$(x+h)^{-1} - x^{-1} = x^{-1}[y + r]$$

where the remainder $r = y^2 + y^3 + \cdots$ has norm $\leq |y|^2 + |y|^3 + \cdots < |y|^2/(1 - |y|)$. Now show that

$$\lim_{h \to 0} \frac{|(x+h)^{-1} - x^{-1} - L_x h|}{|h|} = 0$$

where $L_x : B \to B$ is the linear map given by

$$L_x : B \to B : h \mapsto L_x h \stackrel{\text{def}}{=} -x^{-1}hx^{-1}$$

5. *The spectrum $\sigma(x)$ is not empty for every $x \in B$.*
Suppose $\sigma(x) = \emptyset$. The for every $\lambda \in \mathbf{C}$ the element $\lambda e - x$ is invertible. Let $f : B \to \mathbf{C}$ be any bounded linear functional. Then the function h on $\mathbf{C}$ given by

$$h(\lambda) = f\left((\lambda e - x)^{-1}\right)$$

is complex differentiable (i.e. holomorphic) everywhere. We have

$$(\lambda e - x)^{-1} = \frac{1}{\lambda}\left(e + (\lambda^{-1}x) + (\lambda^{-1}x)^2 + \cdots\right)$$

whenever $|\lambda^{-1}x| < 1$, i.e. for all complex λ for which $|\lambda| > |x|$. Moreover, for such λ, we have

$$|(\lambda e - x)^{-1}| \leq \frac{1}{|\lambda|}\frac{1}{(1 - |x|/|\lambda|)} = \frac{1}{|\lambda| - |x|}$$

So as $|\lambda| \to \infty$ the norm of $(\lambda e - x)^{-1}$ goes to 0. Since the linear functional f is continuous on B it follows that

$$\lim_{|\lambda|\to\infty} h(\lambda) = 0$$

Since h is also continuous (and hence bounded on any compact set) it follows that h is bounded. Then by Liouville's theorem it follows that h is constant. Since $\lim_{|\lambda|\to\infty} h(\lambda) = 0$, the constant value of h is actually 0. Looking back at the definition of h, this says that $f\left((\lambda e - x)^{-1}\right)$ is 0 for every $f \in B^*$ (and every $\lambda \in \mathbf{C}$). By the Hahn-Banach theorem it follows that $(\lambda e - x)^{-1}$ must be 0. But this is absurd since $(\lambda e - x)^{-1}(\lambda e - x) = e$.

6. The **Gelfand-Mazur theorem**. A complex Banach algebra in which every non-zero element is invertible is isometrically isomorphic to the Banach algebra $\mathbf{C}$.
Assume that B is a complex Banach algebra in which every non-zero element is invertible. Consider the map

$$F : \mathbf{C} \to B : \lambda \mapsto \lambda e$$

It is clear that this is a homomorphism of complex algebras and that it is an isometry. The substance of the result lies in the surjectivity of B. For this consider any element $x \in B$. We know that $\sigma(x)$. Take $\lambda \in \sigma(x)$. This means $\lambda e - x$ is not invertible. So $\lambda e - x$ must be 0. So $x = \lambda e$, i.e. $x = F(\lambda)$. Thus F is surjective.

1. Let R be a commutative ring with multiplicative identity e. A subset $S \subset R$ is an *ideal* of R if :(a) $0 \in S$, (b) $x + y \in S$ for every $x, y \in S$, and (c) $rx \in S$ for every $r \in R$ and $x \in S$. The ideal S is a *proper* ideal if $S \neq R$. It is a *maximal ideal* if it is a proper ideal and if the only ideals containing S are S itself and the whole ring R. The ideal S is a *prime* ideal if for every $x, y \in S$ if $xy \in S$ then at least one of x and y must be in S.

   (i) Let I be an ideal of R. For any $x \in R$ we write $x + I$ be the set of all elements of the form $x + i$ with i running over I. Let R/I be the set of all sets of the form $x + I$ with x running over R:

$$R/I \stackrel{\text{def}}{=} \{x + I : x \in R\}$$

Let

$$p : R \to R/I : x \mapsto x + I$$

For any elements $a, b \in R$ we have

$$p(a) = p(b) \text{ if and only if } a - b \in I$$

Show that if $x, x', y, y' \in R$ are such that $p(x) = p(x')$ and $p(y) = p(y')$ then $p(x + x') = p(y + y')$ and $p(xy) = p(yy')$.

Thus there are well-defined operations of addition and multiplication on R/I given by

$$p(x) + p(y) \stackrel{\text{def}}{=} p(x + y), \qquad p(x)p(y) \stackrel{\text{def}}{=} p(xy)$$

As is readily checked, these operations make R/I a ring and, of course, $p : R \to R/I$ is a ring homomorphism. Commutativity of R implies that R/I is commutative. If $e \in R$ is the identity of R then $p(e)$ is the multiplicative identity in R/I.

(ii) Suppose I is a maximal ideal of R. Show that then the commutative ring R/I is a *field*, i.e. every non-zero element has an inverse. Hint: Let $x \in R$ be such that $p(x)$ is a non-zero element of R/I, i.e. $x \in R$ is not in the ideal I. The set

$$Rx + I = \{rx + y : r \in R, y \in I\}$$

is clearly an ideal of R which contains I. Moreover, $Rx + I$ contains the element x which is not in I and so $Rx + I \neq I$. Since I is maximal, it follows then that $Rx + I$ equals the whole ring R. In particular, there is an element $y \in R$ and an element $a \in I$ such that $yx + a = e$. Apply p to this.

(iii) Let I be a ideal in R such that the quotient ring R/I is a field in which the multiplicative identity is not equal to 0. Show that I is maximal. Hint: Since $R/I \neq \{0\}$, the ideal I is proper. Let S be an ideal with $R \supset S \supset I$ and $S \neq I$. Choose $x \in S$ not in I. Then $p(x)$ is a non-zero element of R/I, where $p : R \to R/I : x \mapsto x + I$ is the projection map. So it has an inverse. Thus there is an element $y \in R$ such that $p(x)p(y) = p(e)$. This means $e - xy \in I$ and so $e - xy \in S$. But then $e = e - xy + xy \in S$.

In the following B is a complex Banach algebra which is assumed also to be *commutative*. An *ideal* in B is a subset $I \subset B$ which satisfies: (a) $x + y \in B$ for all $x, y \in I$, (b) $bx \in I$ for all $b \in B$ and $x \in I$. Note that taking $b = \lambda e$ for $\lambda \in \mathbf{C}$ in (b) shows, together with (a), that an ideal I is automatically a linear subspace of B. Recall the quotient

$$B/I = \{x + I : x \in B\}$$

and the projection map

$$p : B \to B/I : x \mapsto x + I$$

We have seen that B/I has a ring structure which makes p a ring homomorphism, and $p(e)$ is the identity element in B/I. Then the quotient B/I is also a complex vector space with multiplication by complex scalars λ defined by

$$\lambda p(x) \stackrel{\text{def}}{=} p(\lambda x)$$

This is well-defined because if $p(x) = p(y)$ then $x - y \in I$ and so $\lambda x - \lambda y = \lambda(x - y) \in I$ which means $p(\lambda x) = p(\lambda y)$. It is clear that B/I does become a vector space and indeed, together with the multiplication, B/I is a complex algebra and $p : B \to B/I$ a homomorphism of algebras (i.e. p is linear and $p(xy) = p(x)p(y)$ for all $x, y \in B$; $p(e)$ is the identity).

Any element B/I is of the form $p(x) = x + I$, for some $x \in B$. Thus it is a *translate* of the subspace I. Define

$$|p(x)| \stackrel{\text{def}}{=} \inf_{y \in p(x)} |y|,$$

the distance of $x + I$ from the origin. Since x itself belongs to $x + I$ it follows that

$$|p(x)| \leq |x|$$

2. We prove that if I is a closed proper ideal in B then $|\cdot|$ is a norm on B/I making it a complex Banach algebra.
   (i) For any $x, y \in B$,

   $$|p(x) + p(y)| \leq |p(x)| + |p(y)|$$

   Proceed as follows: Pick any $x' \in p(x) = x + I$ and $y' \in p(y) = y + I$. Then $p(x) = p(x')$ and $p(y) = p(y')$ and so $p(x + y) = p(x) + p(y) = p(x') + p(y') = p(x' + y')$. Therefore, $|p(x) + p(y)| = |p(x' + y')|$. So

   $$|p(x) + p(y)| \leq |x' + y'| \leq |x'| + |y'|$$

   Now take infimum over $x' \in p(x)$ and then over $y' \in p(y)$.

(ii) For any $x \in B$ and $\lambda \in \mathbf{C}$,

$$|\lambda p(x)| = |\lambda||p(x)|$$

Hint: Work as in (i), taking any $x' \in p(x)$ and showing that $|\lambda p(x)| = |p(\lambda x')| \leq |\lambda||x'|$ and taking inf over all $x' \in p(x) = x + I$. This shows $|\lambda p(x)| \leq |\lambda||p(x)|$. Now, for non-zero λ, write $p(x)$ on the right as $(1/\lambda)\lambda p(x)$.

(iii) Show that

$$|p(x)p(y)| \leq |p(x)||p(y)|$$

for every $x, y \in B$.

(iv) Show that if $I \neq B$ then $|p(e)| \neq 0$. Hint: Since I is a proper ideal it does not contain any invertible elements. The open ball of radius 1 around e consists entirely of invertible elements and so does not intersect I. So $e + I$ does not the open ball of radius 1 centered at 0. So $|p(e)| \geq ?$.

(v) Show that if $I \neq B$ then

$$|p(e)| = 1$$

Hint: Combine the observation obtained in proving (iv) with the inequality $|p(e)| \leq |e| = 1$. [Note also that if in (iii) we put $x = y = e$ then $|p(e)| \geq 1$ or $|p(e)| = 0$.]

(vi) Suppose that I is a *closed ideal* in B, i.e. suppose that I is an ideal and it is *closed* as a subset of B. If $|p(x)| = 0$ show that $p(x) = 0$. (Hint: If $|p(x)| = 0$ then every neighborhood of 0 contains a point of $x + I$, and so every neighborhood of x contains a point of I.)

The preceding parts show that if I is a closed ideal in B then the definition of $|p(x)|$ establishes a *norm* on the complex algebra B/I, and the map $p : B \to B/I$ is continuous.

(vii) Let $\epsilon > 0$ and $a, b \in B$. Suppose $|p(a) - p(b)| < \epsilon$. Then there is a $b' \in B$ such that $p(b') = p(b)$ and $|a - b'| < \epsilon$. Hint: Since $|p(a - b)| < \epsilon$, there is an element $x \in p(a - b) = a - b + I$ such that $|x| < \epsilon$. Since $x \in a - b + I$ there is an element $y \in I$ such that $x = a - b + y = a - (b - y)$.

(viii) Let I be a closed proper ideal in B. Suppose $a_1, a_2, ...$ is a Cauchy sequence in B/I. Then there is a subsequence $a_{j_1}, a_{j_2}, ...$ such that $|a_{j_r} - a_{j_{r+1}}| < 2^{-r}$ for every $r \in \{1, 2, 3, ...\}$. Pick $x_1, x_2, ... \in B$ such that $p(x_i) = a_i$ for all i. Check that by (vii) we can choose $x'_{j_1}, x'_{j_2}, ...$ such that $p(x'_{j_r}) = p(x_{j_r})$ for all $r \in \{1, 2, 3, ...\}$ and such that

$$|x'_{j_{r+1}} - x'_{j_r}| < 2^{-r}$$

Since B is a Banach space, the sequence $(x'_{j_r})_r$ converges. Since $p : B \to B/I$ is continuous it follows then that the sequence $(p(x'_{j_r}))_r$ is convergent in B/I. Note that $p(x'_{j_r}) = a_{j_r}$ and so we have proven that the original Cauchy sequence (a_j) in B/I has a convergent subsequence. Since (a_j) is Cauchy and has a convergent subsequence it follows that (a_j) is itself convergent. Thus B/I *is a Banach space*, i.e. B/I *is a complex Banach algebra*.

We work with a complex commutative Banach algebra B.

It had been shown that the set $G(B)$ of all invertible elements in B is an open subset of B. A proper ideal I in B cannot contain any invertible elements (for if $x \in I$ is invertible then for any $y \in B$ we would have $y = (yx^{-1})x \in I$, which would mean $I = B$), i.e. is a subset of the closed set $G(B)^c$.

Zorn's lemma shows that every proper ideal of B is contained in a maximal ideal.

1. Let J be an ideal of B.
   (i) Check that the closure $\overline{J}$ is also an ideal.

   (ii) Show that if J is a proper ideal then so is its closure $\overline{J}$.

   (iii) Show that if J is a maximal ideal then J is closed. Hint: Consider the ideal $\overline{J}$. It is an ideal which contains J. Since J, being maximal, is proper, (ii) implies that $\overline{J}$ is a proper ideal.

A mapping $\phi : B \to \mathbf{C}$ is a *complex homomorphism* if f is linear and satisfies $f(xy) = f(x)f(y)$ for all $x, y \in B$. Note that then $f(x) = f(xe) = f(x)f(e)$ for every $x \in B$, and so either $f(e) = 1$ or $f(x) = 0$ for every $x \in B$. The set of all *non-zero* complex homomorphisms $B \to \mathbf{C}$ will be denoted Δ and is the *Gelfand spectrum* of the algebra B.

2. Let J be a maximal ideal of B. Show that there is a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $J = \ker h$. Hint: Consider B/J. This is a field because J is a maximal ideal, and, moreover, since J is a closed proper ideal in B, B/J is also a Banach algebra. Therefore, by Gelfand-Mazur, there is an isometric isomorphism $j : B/J \to \mathbf{C}$. Let $p : B \to B/J : x \mapsto p(x) = x + J$ be the usual projection homomorphism. Work with $h = j \circ p$.

3. Let $h_1, h_2 : B \to \mathbf{C}$ be complex homomorphisms such that $\ker h_1 = \ker h_2$. Show that $h_1 = h_2$. Hint: Write any $x \in B$ as $x = [x - h_1(x)e] + h_1(x)e$, and observe that $x - h_1(x)e \in \ker h_1$. Now calculate $h_2(x)$.

4. An element $y \in B$ is not invertible if and only if there is a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $h(y) = 0$.

   (Easy half) Suppose y is invertible. Then for any $h \in \Delta$ we have $h(y)h(y^{-1}) = h(yy^{-1}) = h(e) = 1$ and so $h(y)$ can't be 0.

   For the converse (harder half), suppose $y \in B$ is not invertible. Then the set $By = \{xy : x \in B\}$ is a *proper* ideal of B. Let J be a maximal ideal with $J \supset Bh$ (existence of J follows by an application of Zorn's lemma). By (2) there exists a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $J = \ker h$. Since $y \in By \subset J$ it follows that $h(y) = 0$, which is what we wished to prove.

5. Let $x \in B$. Prove that a complex number λ belongs to the spectrum $\sigma(x)$ if and only if there is a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $h(x) = \lambda$.

6. Let $h : B \to \mathbf{C}$ be a complex homomorphism. Show that h is continuous and, viewed as a linear functional on B, has norm $|h| \leq 1$, the norm being equal to 1 if $h \neq 0$. Hint: Combine the easy half of (5) with the fact that $\sigma(x) \subset \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$.

7. Let $h : B \to \mathbf{C}$ be a non-zero complex homomorphism. Then $\ker h$ is a maximal ideal in B.

   Since h is a non-zero homomorphism, $h(e) = 1$ and so $h(\lambda e) = \lambda$, which shows that h is surjective. So $B/\ker h \simeq \mathbf{C}$, and the latter is a field. So the ideal $\ker h$ must be maximal. This is a pure algebra result and uses nothing about the norm on B.

The preceding discussions establishes

a one-to-one correspondence $h \mapsto \ker h$ *between the set* Δ *of all non-zero complex homomorphisms* $B \to \mathbf{C}$ *and the set of all maximal ideals of* B.

An *involution* $*$ on a complex algebra B is a map $* : B \to B$ for which

(i) $*(a + b) = *a + *b$ for all $a, b \in B$

(ii) $*(\lambda a) = \overline{\lambda} * a$ for all $\lambda \in \mathbf{C}$ and $a \in B$

(iii) $(xy) = y^*x^*$ for all $x, y \in B$

(iv) $(x^*)^* = x$ for all $x \in B$. An element $a \in B$ is *hermitian* if $a = a^*$.

On a complex *Banach* algebra we also require an involution $*$ to satisfy

(v) $|xy| \leq |x||y|$ for all $x, y \in B$.

Observe that for the identity e, we have $e^* = ee^*$ and so taking $*$ of this we get $(e^*)^* = (e^*)^*e^*$, which says $e = ee^*$. Thus

$$e = e^*$$

A *B*-algebra* is a complex Banach algebra B on which there is an involution $*$ for which

$$|xx^*| = |x|^2 \qquad \text{for all } x \in B$$

1. Let B be a complex Banach algebra with involution.

   (i) Show that

   $$\text{if } B \text{ is a B*-algebra then } |x| = |x^*| \text{ for all } x \in B$$

   (ii) Suppose $|y^*y| = |y|^2$ for all $y \in B$. Show that $|y| = |y^*|$ for all $y \in B$.

(iii) Suppose $|y^*y| = |y|^2$ for all $y \in B$. Show that $|xx^*| = |x|^2$ for all $x \in B$.

2. Let B be a B*-algebra.

(i) Show that if $y \in B$ is hermitian and s is any real number then

$$|se + iy|^2 = |s^2e + y^2|$$

(ii) Show that $e + iy$ is invertible for every hermitian $y \in B$. Proceed as follows: Suppose $e + iy$ is not invertible. Then for every $\lambda \in \mathbf{R}$, $(\lambda + 1)e - (\lambda e - iy))$ is not invertible, i.e. $(\lambda+1) \in \sigma(\lambda e - iy)$. So $|\lambda+1| \leq |\lambda e - iy|$. By (i), this implies $(\lambda + 1)^2 \leq |\lambda^2 e + y^2|$ and the latter is $\leq \lambda^2 + |y^2|$. This would be true for every real number λ. Show that this is impossible.

3. Let B be a complex algebra with involution $*$.

(i) If $e + x^*x$ is invertible for every $x \in B$ then show that $e + iy$ is invertible for every hermitian $y \in B$. Hint: Note that $(e + iy)(e - iy) = e + y^2 = e + y^*y$.

(ii) If $e+iy$ is invertible for every hermitian $y \in B$ then $\sigma(a) \subset \mathbf{R}$ for every hermitian $a \in B$. Hint: Consider any complex number $\lambda = \alpha + i\beta$ with $\beta \neq 0$. Check that $\lambda e - a = i\beta(e+iy)$ for some *hermitian* element y. By (i) then $\lambda e - a$ is invertible and so $\lambda \notin \sigma(a)$.

(iii) If $e + x^*x$ is invertible for every $x \in B$ then $\sigma(y^*y) \subset [0, \infty)$ for every $y \in B$. Proceed as follows: Let $k > 0$ and show that $(-k)e - y^*y$ is invertible by writing it as $(-k)[e + x^*x]$ where $x = k^{-1/2}y$.

3. Let B be a complex commutative Banach algebra with an involution $*$. Show that the following are equivalent:
   (a) $e + x^*x$ is invertible for every $x \in B$
   (b) every hermitian element has real spectrum
   (c) $\hat{x^*} = \overline{\hat{x}}$ for every $x \in B$.
   (d) $J^* = J$ for every maximal ideal J of B.

   (a) implies (b) is from the previous problem.
   Now suppose (b) holds. Let $x \in B$. Then $a = x + x^*$ and $b = i(x - x^*)$ are hermitian. So their spectra are real. So $\hat{a}$ and $\hat{b}$ are real-valued. Thus $f = \hat{x} + \hat{x^*}$ and $g = i(\hat{x} - \hat{x^*})$ are *real-valued*. Now $\hat{x} = (f - ig)/2$ and $\hat{x^*} = (f + ig)/2$. It follows that $\hat{x^*} = \overline{\hat{x}}$.
   Assume (c). Let J be a maximal ideal. Then $J = \ker h$ for some $h \in \Delta$ (i.e. h is a non-zero complex homomorphism $B \to \mathbf{C}$). Let $x \in B$. Then

$$h(x^*) = \hat{x^*}(h) = \overline{\hat{x}(h)} = \overline{h(x)} = 0$$

   and so $x^* \in kerh = J$.
   Now suppose (d) holds. We prove (c). Let $x \in B$. Consider any $h \in \Delta$. Then $x - h(x)e \in \ker h$. Since $\ker h$ is a maximal ideal, (d) implies that $x^* - \overline{h(x)}e$ is also in $\ker h$. So $h(x^* - \overline{h(x)}e) = 0$ and this implies $h(x^*) = \overline{h(x)}$. This holds for all $h \in \Delta$. So (c) holds.
   Finally we show that (c) implies (a). Assume (c). Let $x \in B$. Then the Gelfand transform of $e + x^*x$ is $1 + |\hat{x}|^2$ which never has the value zero. So 0 is not in the spectrum of $e + x^*x$ and so $e + x^*x$ is invertible.

Let B be a complex, commutative B^* algebra, with Δ its Gelfand spectrum. Then, as we have seen in class,

(i) the Gelfand transform $B \to C(\Delta) : x \mapsto \hat{x}$ satisfies

$$\hat{x^*} = \overline{\hat{x}}$$

for every $x \in B$;

(ii) the spectral radius $\rho(x)$ equals the norm $|x|$ for every $x \in B$.

Fact (ii) was proven first for hermitian elements in any B^* algebra and then, using the Gelfand transform, for all elements in a commutative B^* algebra. If $a \in B$ is hermitian then

$$\rho(a) = \lim_{n\to\infty} |a^n|^{1/n}$$

while $|a^2| = |aa^*| = |a|^2$ which implies $|a^{2^k}| = |a|^{2^k}$, and so, letting $n \to \infty$ through powers of 2 we get

$$\rho(a) = |a|$$

for every hermitian a in any B^* algebra. For a commutative B^* algebra B we have for a general $x \in B$,

$$\rho(xx^*) = |\hat{xx^*})|_{\sup} \leq |\hat{x}|_{\sup}|\hat{x}^*|_{\sup} = \rho(x)\rho(x^*) \leq \rho(x)|x^*|$$

Since xx^* is hermitian, $\rho(xx^*) = |xx^*|$, which is equal to $|x||x^*|$. So we have

$$|x| \leq \rho(x)$$

But we already know the opposite inequality. So $\rho(x) = |x|$.

By (i) and (ii) and other properties we have studied before, the Gelfand transform is a $*$–algebra homomorphism and is also an isometry. Its image $\hat{B}$ in $C(\Delta)$ is therefore a subalgebra of $C(\Delta)$ which is preserved under conjugation. Moreover, since the Gelfand transform is an isometry it follows that $\hat{B}$ is a *closed* subset of $C(\Delta)$: for if $x_n \in B$ are such that $\hat{x}_n \to f$ for some $f \in C(\Delta)$ then $(\hat{x}_n)_n$ is Cauchy in $C(\Delta)$ and so, by isometricity, $(x_n)_n$ is Cauchy in B and so is convergent, say to x and then by continuity of $\hat{}$ it follows that $f = \hat{x}$, and so f is in the image of the Gelfand transform. Finally, $\hat{B}$ separates points of Δ because if h_1 and h_2 are distinct elements of Δ, then, by definition of Δ, there must be some $x \in B$ for which $h_1(x) \neq h_2(x)$, i.e. $\hat{x}(h_1) \neq \hat{h}(x_2)$.

The Stone-Weierstrass theorem now implies that

$$\hat{B} = C(\Delta)$$

This proves the **Gelfand-Naimark** theorem:

Theorem. For a complex commutative B^*–algebra B, the Gelfand transform is an isometric isomorphism of B onto $C(\Delta)$, where Δ is the Gelfand spectrum of B.

1. Let H be a complex vector space and $F : H \times H \to \mathbf{C}$ a mapping such that $F(x, y)$ is linear in x and conjugate-linear in y.

(i) Prove the polarization formula

$$F(x,y) = \frac{1}{4}F(x+y,x+y) - \frac{1}{4}F(x-y,x-y) + \frac{i}{4}F(x+iy,x+iy) - \frac{i}{4}F(x-iy,x-iy) \tag{1}$$

(ii) Use this to prove that

$$\sup_{x,y\in H,|x|,|y|\leq 1} |F(x,y)| \leq 4 \sup_{v\in H,|v|\leq 1} |F(v,v)| \tag{2}$$

[Hint: In (1), the first term equals $F(a, a)$ with $a = (x + y)/2$ and $|a| \leq 1$ if $|x|, |y| \leq 1$. Similarly for the other terms.]

(iii) If $y \in H$ then show that

$$\sup_{v \in H, |v| \leq 1} |(y, v)| = |y|$$

(iv) If $T : H \to H$ is a linear map for which $\sup_{v \in H, |v| \leq 1} |(Tv, v)| < \infty$, show that T is a bounded linear map and

$$|T| \leq 4 \sup_{v \in H, |v| \leq 1} |(Tv, v)|$$

(Recall that the norm of T is $|T| = \sup_{x \in H, |x| \leq 1} |Tx|$.)

2. Let H be a complex Hilbert space and $F : H \times H \to \mathbf{C}$ a map such that $F(x, y)$ is linear in x, conjugate linear in y, and $\sup_{x,y \in H, |x|,|y| \leq 1} |F(x, x)| < \infty$.

(i) Fix $x \in H$, and consider

$$\phi_x : H \to \mathbf{C} : y \mapsto \overline{F(x, y)}.$$

Show that this is a bounded linear functional. Consequently, there exists a *unique* element $Tx \in H$ such that $\phi_x(y) = (Tx, y)$ for every $y \in H$. Thus for each $x \in H$ there exists a unique element $Tx \in H$ such that

$$F(x, y) = (Tx, y) \qquad \text{for all } y \in H$$

(ii) Let $x, x' \in H$ and $a, b \in \mathbf{C}$. Show that

$$(aTx + bTx', y) = F(ax + bx', y) \qquad \text{for all } y \in H$$

Then by the uniqueness property noted in (i) it follows that

$$T(ax + bx') = aTx + bTx'$$

Thus $T : H \to H$ is *linear*.

(iii) Show that the map $T : H \to H$ is a *bounded* linear map. [Hint: Use 1(iii) and (ii).]

3. Let X be a non-empty set and $\mathcal{B}$ a σ–algebra of subsets of X.
   (i) Suppose $\lambda_1, ..., \lambda_n$ and $\lambda'_1, ..., \lambda'_m$ are finite measures on $\mathcal{B}$ and $a_1, ..., a_n, a'_1, ..., a'_m$ are complex numbers such that

$$\sum_{j=1}^{n} a_j \lambda_j = \sum_{j=1}^{m} a'_j \lambda'_j$$

Then show that for any bounded $\mathcal{B}$–measurable function $f : X \to \mathbf{C}$,

$$\sum_{j=1}^{n} a_j \int_X f \, d\lambda_j = \sum_{j=1}^{m} a'_j \int_X f \, d\lambda'_j$$

[Hint: There is a sequence of measurable simple functions s_N such that $s_N(x) \to f(x)$ uniformly for $x \in X$ as $N \to \infty$.] If μ is the complex measure given by

$$\mu = \sum_{j=1}^{n} a_j \lambda_j$$

then we define

$$\int f \, d\mu \stackrel{\text{def}}{=} \sum_{j=1}^{n} a_j \int_X f \, d\lambda_j$$

for all bounded measurable functions f on X. The fact proven above says that this definition is independent of the particular choice of a_j and λ_j used to express μ.

(ii) If $b_1, ..., b_k$ are complex numbers and $\mu_1, ..., \mu_k$ are complex measures, each of the type described in (i), and μ is the complex measure given by

$$\mu = \sum_{j=1}^{n} b_j \, \mu_j$$

then show that

$$\int f \, d\mu = \sum_{j=1}^{n} b_j \int f \, d\mu_j$$

for all bounded measurable functions f on X.

(iii) Suppose now that X is a compact Hausdorff space and $\mathcal{B}$ is the Borel σ–algebra. Let μ_1, μ_2 be complex measures on $\mathcal{B}$, each μ_i being a complex linear combination of finite regular Borel measures λ_{ij} on $\mathcal{B}$. Show that if

$$\int f \, d\mu_1 = \int f \, d\mu_2 \qquad \text{for all } f \in C(X)$$

then

$$\mu_1 = \mu_2$$

Hint: Write $\mu_1 = \sum_j a_j \lambda_j$ and $\mu_2 = \sum_i a'_i \lambda'_i$, where the a_i, a'_j are complex numbers and λ_i, λ'_j are finite regular Borel measures. The $\lambda = \sum_i \lambda_i + \sum_j \lambda'_j$ is a finite regular Borel measure. Let g be any bounded Borel function. Then there is a sequence of continuous functions $g_n \in C(X)$ such that $g_n(x) \to g(x)$ for λ–a.e. x and $|g_n|_{\text{sup}} \leq |g|_{\text{sup}}$. Then the same holds a.e. for each λ_i and each λ'_j. Now use the dominated convergence theorem. Finally, set $g = 1_A$ for any Borel set $A \subset X$.

4. Let H be a complex Hilbert space, X a compact Hausdorff space, $\mathcal{B}$ its Borel $\sigma-$algebra. Suppose that for each x we have a finite regular Borel measure $\mu_{x,x}$ on $\mathcal{B}$. Define, for every $x, y \in H$,

$$\mu_{x,y} = \frac{1}{4}\mu_{x+y,x+y} - \frac{1}{4}\mu_{x-y,x-y} + \frac{i}{4}\mu_{x+iy,x+iy} - \frac{i}{4}\mu_{x-iy,x-iy} \tag{3}$$

This is a complex measure which is a linear combination of *finite* regular Borel measures. *Assume that* $\int f\, d\mu_{x,y}$ *is linear in* x *and conjugate linear in* y *for every* $f \in C(X)$.

(i) Show that $\mu_{x,y}$ is linear in x and conjugate linear in y.
Hint: Let $x, x', y \in H$ and $a \in \mathbf{C}$. Then, by hypothesis, $\int f\, d\mu_{ax+x',y}$ equals $a \int f\, d\mu_{x,y} + \int f\, d\mu_{x',y}$, for every $f \in C(X)$, i.e. $\int f\, d\mu_{ax+x',y} = \int f\, d\big(a\mu_{x,y} + \mu_{x',y}\big)$ for every $f \in C(X)$. Now use 3(iii).

(ii) Show that

$$\sup_{x,y \in H, |x|,|y| \le 1} \left| \int g\, d\mu_{x,y} \right| \le 4|g|_{\text{sup}} \sup_{v \in H, |v| \le 1} \mu_{v,v}(X)$$

for every bounded Borel function g on X.

(iii) Assume that $\sup_{v\in H,|v|\le 1}\mu_{v,v}(X) < \infty$. Show that for every bounded Borel function g on X there is a *unique* bounded linear operator $\Phi(g) : H \to H$ such that

$$\big(\Phi(g)x, y\big) = \int_X g\, d\mu_{x,y}$$

(iv) Assume that $\sup_{v\in H,|v|\le 1}\mu_{v,v}(X) < \infty$. Show that the mapping $g \mapsto \Phi(g)$ is linear. Hint: Let g, h be bounded Borel functions and a any complex number. Show that $\big(\Phi(ag+h)x, y\big)$ equals $a\big(\Phi(g)x, y\big) + \big(\Phi(h)x, y\big)$, i.e. is equal to $\big([a\Phi(g) + \Phi(h)]x, y\big)$. Now use the uniqueness of $\Phi(ag+h)$.

(v) Assume that $\sup_{v\in H,|v|\le 1}\mu_{v,v}(X) < \infty$. Assume also that $\Phi(\overline{f}) = \Phi(f)^*$ and $\Phi(fg) = \Phi(f)\Phi(g)$ hold for all $f, g \in C(X)$. Show that for any $x \in H$, the linear mapping

$$C(X) \to H : f \mapsto \Phi(f)x$$

satisfies

$$|\Phi(f)x| = |f|_{L^2(\mu_{x,x})}$$

for all $f \in C(X)$. Hint: $|\Phi(f)x|^2 = \big(\Phi(f)x, \Phi(f)x\big) = \big(\Phi(f)^*\Phi(f)x, x\big)$.

(vi) Assume the hypotheses of (v). Since $C(X)$ is a dense subspace of $L^2(\mu_{x,x})$, it follows from (v) that Φ extends to a linear isometry

$$L^2(X, \mu_{x,x}) \to H : g \mapsto \Phi(g)x$$

(vii) Assume the hypotheses of (v) and assume also that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in C(X)$. Now let h, k be bounded Borel functions on X. Let $x \in H$. Then $h, k \in L^2(\mu_{x,x})$ and so there exist sequences of functions $h_n, k_n \in C(X)$ converging pointwise $\mu_{x,x}$–a.e. to h, k, respectively, and within $|h_n|_{\sup} \leq |h|_{\sup}$ and $|k_n|_{\sup} \leq |k|_{\sup}$. Then, by dominated convergence, h_n, k_n converge in $L^2(\mu_{x,x})$ to h, k, respectively. Moreover, $h_n k_n$ also converges $\mu_{x,x}$–a.e. to hk and $|h_n k_k|_{\sup} \leq |h|_{\sup}|k|_{\sup}$. Then, by dominated convergence, $h_n, k_n, h_n k_n$ converge in $L^2(\mu_{x,x})$ to h, k, hk, respectively. Similarly, $\overline{h}_n$ converges to h. Consider

$$(\Phi(h_n)x, x) = (x, \Phi(h_n)^* x) = \big(x, \Phi(\overline{h}_n)x\big)$$

and

$$(\Phi(h_n k_n)x, x) = (\Phi(h_n)\Phi(k_n)x, x) = \big(\Phi(k_n)x, \Phi(\overline{h_n})x\big)$$

Let $n \to \infty$ to show that

$$\Phi(\overline{h}) = \Phi(h)^*$$

and

$$\Phi(hk) = \Phi(h)\Phi(k)$$

for all bounded Borel functions h, k on X.

(viii) All hypotheses as before. For any Borel set $A \subset X$ show that the operator

$$E(A) \stackrel{\text{def}}{=} \Phi(1_A)$$

is an orthogonal projection. From the isometry property in (vi) it follows that E is a *projection-valued measure* on the Borel σ–algebra of X.

(ix) All hypotheses as before. Now (iii) shows that

$$\mu_{x,y}(A) = (E(A)x, y)$$

By definition, if g is a bounded Borel function on X then $\int g\, dE$ is the unique operator on H for which $\big(\big(\int g\, dE\big)x, x\big)$ equals $\int g\, dE_{x,x}$. Therefore, by (iii),

$$\int g\, dE = \Phi(g)$$

5. Let H be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H, X a compact Hausdorff space, $\mathcal{B}$ its Borel σ–algebra, and suppose that

$$\Phi : C(X) \to B(H) : f \mapsto \Phi(f)$$

is an algebra homomorphism with $\Phi(\overline{f}) = \Phi(f)^*$ and $|\Phi(f)| = |f|_{\text{sup}}$ for all $f \in C(X)$. For each $x \in H$, let $L_{x,x} : C(X) \to \mathbf{C}$ the mapping given by

$$L_{x,x} : C(X) \to \mathbf{C} : f \mapsto_{x,x} f \stackrel{\text{def}}{=} \big(\Phi(f)x, x\big)$$

Clearly, $L_{x,x}$ is a linear functional.

(i) Check that

$$L_{x,x}(\overline{f}) = \overline{L_{x,x} f}$$

for all $x \in H$ and $f \in C(X)$. Thus if f is real-valued then $L_{x,x}f$ is a real number, and so $L_{x,x}$ restricts to a real-linear map $C^{\text{real}}(X) \to \mathbf{R}$.

(ii) Show that if $f \in C(X)$ is non-negative then $L_{x,x}f \geq 0$ for all $x \in H$. Hint: Show that $L_{x,x}f = |\Phi(f^{1/2})|^2$.

(iii) From the observations noted above it follows by the Riesz-Markov theorem that for each $x \in H$ there is a *unique* regular Borel measure $\mu_{x,x}$ on X such that

$$\int f \, d\mu_{x,x} = \big(\Phi(f)x, x\big) \tag{4}$$

for every $f \in C^{\mathrm{real}}(X)$. Because both sides of (4) are complex-linear in f it follows that (4) holds for all $f \in C(X)$.
Now for any $x, y \in H$ let $\mu_{x,y}$ be the complex measure on $\mathcal{B}$ given by

$$\mu_{x,y} = \frac{1}{4}\mu_{x+y,x+y} - \frac{1}{4}\mu_{x-y,x-y} + \frac{i}{4}\mu_{x+iy,x+iy} - \frac{i}{4}\mu_{x-iy,x-iy} \tag{5}$$

Show that then

$$\int f \, d\mu_{x,y} = \big(\Phi(f)x, y\big) \tag{6}$$

for every $f \in C(X)$.

6. Let H be a complex Hilbert space and B a commutative subalgebra of $B(H)$ such that $T^* \in B$ for every $T \in B$ and B is a closed subset of $B(H)$ (in the norm topology). Then B is itself a commutative B^*–algebra. Let Δ be its Gelfand spectrum. By Gelfand-Naimark, the Gelfand transform

$$B \to C(\Delta) : T \mapsto \hat{T}$$

is an isometric $*$–isomorphism. Let

$$\Phi : C(\Delta) \to B : f \mapsto \Phi(f)$$

be its inverse. Applying the preceding results to this situation we see that there is a projection valued measure E on the Borel σ–algebra of Δ such that

$$\Phi(f) = \int f \, dE$$

for every continuous function f on Δ. Thus

$$T = \int_{\Delta} \hat{T} \, dE$$

for every $T \in B$. This is the *spectral resolution* of the operator T. Note that since T and T^* both belong to the commutative algebra B, the operator T must be normal. Conversely, for any bounded normal operator T on H we can take B to be the closure of the set of all operators which can be expressed as polynomials $p(T, T^*)$ in T and T^*.

7. Let the setting be as in Problem 6. Suppose E' is also a projection valued measure on the Borel sigma-algebra of Δ such that

$$\Phi(f) = \int f \, dE'$$

for every continuous function f on Δ. Assume that E' is regular in the sense that $E'_{x,x}$ is a regular Borel measure for each $x \in H$. Show that $E' = E$. [Hint: Show that $E'_{x,x} = E_{x,x}$ for every $x \in H$, and then see what this says about $\big(E'(A)x, x\big)$.]

In the following, H is a complex Hilbert space with a Hermitian inner-product $(\cdot, \cdot)$.

1. Let $D \subset H$ be a subspace of H which is *dense* in H.

(i) Suppose $B : D \to H$ is a bounded linear mapping, i.e. B is linear and the norm $|B| = \sup_{x \in D, |x| \leq 1|} |Bx|$ is finite. Show that there is a unique bounded linear operatot $B' : H \to H$ which restricts to B on D. [Hint: Let $x \in H$. Since D is dense, there is a sequence of points $x_n \in D$ with $x_n \to x$ as $n \to \infty$. Show that the sequence of points Bx_n is Cauchy. Show also that the limit of the sequence (Bx_n) is the same if x_n is replaced by any other sequence of points converging to x.]

(ii) Let D be a dense subspace of H and $T : D \to H$ a linear mapping. For $x, y \in H$ consider the inner-product (Tx, y). For fixed y, this is a linear functional of $x \in D$ but in general we can't expect it to be a a bounded linear functional. But for certain values y, the linear map $T_y : D \to \mathbf{C} : x \mapsto (Tx, y)$ will be bounded linear; for example, for $y = 0$. Let D' be the set of all $y \in H$ such that $T_y : D \to \mathbf{C}$ is a bounded linear functional. Check that D' is a subspace of H.

Definition of the adjoint T^*. Let $T : D \to H$ be a linear operator with dense domain D. As in (ii), let D' be the set of all $y \in H$ for which $T_y : D \to \mathbf{C} : x \mapsto (Tx, y)$ is a bounded linear operator on (the dense subspace) D. So by (i), there is a unique extension of T_y to a bounded linear map $T'_y : H \to \mathbf{C}$. Now we also know that any bounded linear functional on H is given by inner-product with a unique vector of H. Thus there is a unique vector $w \in H$ such that $T'_y(x) = (x, w)$ for all $x \in H$. In particular, $(Tx, y) = (x, w)$ for every $x \in D$. This vector w is denoted T^*y. Thus the defining property of T^*y is:

$$(Tx, y) = (x, T^*y) \tag{1}$$

holding for all $x \in D$. Note that T^*y is meaningful only for $y \in D'$. Thus on D', we have the mapping $T^* : D' \to H$. Since (1) specifies T^*y uniquely, it follows readily that T^* is in fact a linear map. We write $D(T^*)$ for D', to indicate that it is the domain of the linear operator T^*.

(iii) Let $T : D \to H$ be a densely defined operator. Show that the operator T^* is *closed* in the following sense: if $y_n \in D(T^*)$ is any sequence of points in D converging to some point $y \in H$ *and* if, further, the sequence of elements T^*y_n also converges then y actually lies in $D(T^*)$ and $T^*y_n \to T^*y$ as $n \to \infty$. [Hint: Since $y_n \to y$, we have $(Tx, y_n) \to (Tx, y)$. Rewrite (Tx, y_n) using T^*y_n. Let z be the limit of the sequence of elements T^*y_n. Show that $(Tx, y) = (x, z)$. Examine now what this says about y and what it says about z.]

2. Let E be a spectral measure for a measurable space $(\Omega, \mathcal{B})$ with values being orthogonal projection operators in a complex Hilbert space H. Let $f : \Omega \to \mathbf{C}$ be a measurable function not necessarily bounded). Let

$$D_f = \{x \in H : \int |f|^2 \, dE_{x,x} < \infty\}$$

(i) For any $x, y \in H$ and any measurable set A, show that

$$E_{x+y,x+y}(A) \leq 2E_{x,x}(A) + 2E_{y,y}(A)$$

[Hint: First recall that $E_{v,v}(B) = |E(B)v|^2$. Next, for any vectors $a, b \in H$ we have the Cauchy-Schwarz inequality $|(a,b)| \leq |a||b|$ which leads to the inequality $|a+b|^2 \leq |a|^2 + |b|^2 + 2|a||b|$. This, together with $(|a| - |b|)^2 \geq 0$ implies that $|a+b|^2 \leq 2|a|^2 + 2|b|^2$.]

(ii) Show that D_f is a *linear* subspace of H, i.e. if $x, y \in D_f$ then $x + y \in D_f$ and $ax \in D_f$ for every $a \in \mathbf{C}$.

(iii) Let $A_n = \{p \in \Omega : |f(p)| \leq n\}$. Consider any vector x in the range of the projection $E(A_n)$. Show that

$$E_{x,x}(A) = E_{x,x}(A \cap A_n)$$

for every $A \in \mathcal{B}$. [Hint: What is $E(A_n)x$?]

(iv) With notation as above, show that

$$\int s\,dE_{x,x} = \int_{A_n} s\,dE_{x,x}$$

for every measurable simple function s on Ω.

(v) With notation as above, show that

$$\int |f|^2\,dE_{x,x} = \int_{A_n} |f|^2\,dE_{x,x}$$

Note that the right side is $\leq n^2 E_{x,x}(\Omega) = n^2|x|^2 < \infty$, and so $x \in D_f$.

(vi) Let y now be any vector in H. Let $y_n = E(A_n)y$, which is thus an element in the range of $E(A_n)$ and therefore in D_f. Show that $y_n \to y$, as $n \to \infty$. [Hint: Show that $|y_n - y|^2 = E_{y,y}(A_n^c)$.] This show that the subspace D_f is *dense* in H.

(vii) Let $x \in H$. For any bounded measurable function g which is in $L^2(\Omega, \mathcal{B}, E_{x,x})$ let $T_x g = (\int g\, dE)x$, an element of H. Show that $|T_x g| = |g|_{L^2(E_{x,x})}$.