Abstract Wiener Spaces in a few slides

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Introduction

Abstract Wiener spaces (AWS) form just the right setting for infinite-dimensional integration.

The framework was built by L. Gross [1], who also proved all the fundamental results concerning abstract Wiener spaces.

Standard Gaussian measure on $\mathbb{R}^N$

Standard Gaussian measure on $\mathbb{R}^N$ is the product of N copies of the one-dimensional standard Gaussian whose density is

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Thus standard Gaussian measure on $\mathbb{R}^N$ is the distribution of the $\mathbb{R}^N$-valued random variable

$$Z = Z_1 u_1 + \cdots + Z_N u_N,$$

where $u_1, \ldots, u_N$ form an orthonormal basis of $\mathbb{R}^N$ and $Z_1, \ldots, Z_N$ are iid standard Gaussian random variables.

Standard Gaussian measure for a Hilbert space

Let H be a separable infinite dimensional real Hilbert space. The standard Gaussian measure for H should be the distribution measure of the H-valued random variable

$$Z = Z_1 u_1 + Z_2 u_2 + \cdots , \tag{1}$$

where $u_1, u_2, \ldots$ form an orthonormal basis of H, and $Z_1, Z_2, \ldots$ are iid standard Gaussians, all defined on some $(\Omega, \mathcal{F}, \mathbb{P})$.

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The problem is: with probability 1 the series (1) is not convergent in H, because $\mathbb{P}[\sum_n Z_n^2 < \infty]$ is 0.
The *key idea then is to introduce a new norm* $|\cdot|$ *such that (1) is convergent with probability* 1 *as a series in the completion B of H relative to* $|\cdot|$*.*

Measurable norm

A norm $|\cdot|$ on H is a *measurable norm* if for any $\epsilon > 0$ there is a finite dimensional subspace F of H such that for any finite dimensional subspace F_1 of H orthogonal to F

$$\mathbb{P}[|Z| > \epsilon] < \epsilon$$

for any F_1-valued standard Gaussian variable Z.

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Then, assuming H is separable, by choosing a sequence of such subspaces, and orthonormal bases within each, we can construct an orthonormal basis $u_1, u_2, \ldots$ in H such that

$$\begin{aligned} \mathbb{P}[|Z_1 u_1 + \cdots Z_{k_1} u_{k_1}| > 1/2] &< 1/2 \\ \mathbb{P}[|Z_{k_1+1} u_{k_1+1} + \cdots Z_{k_2} u_{k_2}| > 1/4] &< 1/4 \\ \vdots &< \vdots \end{aligned} \tag{2}$$

for some $1 \leq k_1 < k_2 < \ldots$. (For details see Frame 16 below.)

Constructing a Gaussian variable

By Borel-Cantelli it follows that

$$Z = \left(Z_1 u_1 + \cdots + Z_{k_1} u_{k_1}\right) + \left(Z_{k_1+1} u_1 + \cdots Z_{k_2} u_{k_2}\right) + \cdots$$

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of H relative to $|\cdot|$.
If $f \in B^*$ then

$$\langle f, Z \rangle = \lim_{N \to \infty} \sum_{n=1}^{k_N} \langle f, u_n \rangle Z_n \qquad \mathbb{P}\text{-a.s.} \tag{3}$$

This is Gaussian, as we see below.

Variance bound

$$\mathbb{E}\left[\langle f, Z\rangle^2\right] \leq \liminf_{N\to\infty} \mathbb{E}\left[\left(\sum_{n=1}^{k_N} \langle f, u_n\rangle Z_n\right)^2\right] \tag{4}$$

using Fatou. Next since the Z_n's are independent Gaussians, each with $\mathbb{E}[Z_n^2] = 1$, we have

$$\mathbb{E}\left[\langle f, Z\rangle^2\right] \leq \sum_{n=1}^{\infty} \langle f, u_n\rangle^2. \tag{5}$$

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The inclusion $i : H \to B$ being continuous (Frame 14 below), for any $f \in B^*$ the restriction

$$f|H = f \circ i$$

is continuous linear on H and hence in H^*. Hence (5) is finite.

Variance

Then the convergence in (3) is in $L^2(\mathbb{P})$, and hence also in $L^1(\mathbb{P})$. Then

$$\mathbb{E}\left[\langle f, Z\rangle\right] = \lim_{N\to\infty} \sum_{n=1}^{k_N} \langle f, u_n\rangle \mathbb{E}[Z_n] = 0$$

and

$$\mathbb{E}\left[\langle f, Z\rangle^2\right] = \lim_{N\to\infty} \sum_{n=1}^{k_N} \langle f, u_n\rangle^2 = \|f\|_{H^*}^2$$

(Technically the element on the right is $f \circ i$, not just f.)

Gaussian nature

Then for any $t \in \mathbb{R}$ we have

$$e^{it\langle f,Z\rangle} = \lim_{N\to\infty} e^{it\sum_{n=1}^{k_N}\langle f,u_n\rangle Z_n}$$

in $L^1(\mathbb{P})$ [using $|e^{ia} - e^{ib}| \leq |a-b|$] and so applying $\mathbb{E}$ we obtain the characteristic function of Z:

$$\mathbb{E}\left[e^{it\langle f,Z\rangle}\right] = e^{it\cdot 0 - \frac{t^2}{2}\|f\|_{H^*}^2} \tag{6}$$

Thus $\langle f, Z\rangle$ is Gaussian with mean 0 and variance $\|f\|_{H^*}^2$.

AWS summary

Thus, starting with a measurable norm $|\cdot|$ on a real separable Hilbert space we have constructed a process Z with values in $B = \overline{H}$ such that

$$\langle f, Z \rangle \quad \text{is Gaussian with mean 0 and variance } \|f\|_{H^*}^2$$

for all $f \in B^* \subset H^*$.

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Thus, starting with a measurable norm $|\cdot|$ on a real separable Hilbert space we have constructed a process Z with values in $B = \overline{H}$ such that

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for all $f \in B^* \subset H^*$.
Definition: The distribution measure μ of Z is *Gaussian measure* on B.

Dense duals

Consider continuous linear

$$i : H \to B,$$

where H is Hilbert and B is Banach. Then

$$i^* : B^* \to H^* : f \mapsto i^*(f) = f \circ i$$

is continuous linear. If $\mathrm{Ran}(i^*)$ is not dense in H^* then all element of $i^*(B^*)$ vanish on some nonzero $v \in H$, and this means $f\big(i(v)\big) = 0$ for all $f \in B^*$ and hence $i(v) = 0$. So, if i is injective then i^* has dense range. Moreover, if $i(H)$ is dense in B then i^* is injective.

AWS summary II

Since $i^*(B^*)$ is dense in H^* and every $f \in B^*$ produces a Gaussian variable $\langle f, \cdot \rangle$ on B with mean 0 and L^2-norm $\|i^*(f)\|_{H^*}$ we can extend isometrically to a linear map

$$H^* \to L^2(B, \mu) : h \mapsto I(h)$$

where each $I(h)$ is Gaussian of mean 0 and variance $\|h\|^2_{H^*}$.

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where each $I(h)$ is Gaussian of mean 0 and variance $\|h\|^2_{H^*}$. By identifying $H \simeq H^*$ we can view I as being defined on H.
We can verify then that

$$\mathrm{Cov}\,(I(v), I(w)) = \langle v, w \rangle_H$$

for all $v, w \in H$.

Continuity of $i : H \to B$

Suppose i is not continuous. Then for any finite dimensional subspace F of H the restriction of i to $F^{\perp}$ is discontinuous (because $i|F$ is continuous). Consider any $\epsilon > 0$, and fix any $t > 0$. By discontinuity of $i|F^{\perp}$ there are vectors in $F^{\perp}$ of unit H-norm but of arbitrarily large $|\cdot|_B$-norm; hence there is a $v \in F^{\perp}$ with $\|v\|_H = 1$ and $|v|_B > \epsilon/t$. With Z the standard Gaussian on $\mathbb{R}$ we have

$$\mathbb{P}[|Zv|_B > \epsilon] \geq \mathbb{P}[|Z| > t] = 2\int_t^{\infty} (2\pi)^{-1/2} e^{-x^2/2} dx,$$

with the right side a positive value having nothing to do with ϵ. This contradicts the measurable norm property of $|\cdot|_B$

Measurable norm II

The condition for $|\cdot|$ to be a measurable norm on H is thus: for any $\epsilon > 0$ there is a finite-dimensional subspace $F \subset H$ such that

$$\mathrm{Gauss}_{F_1}[v \in F_1 \ : \ |v| > \epsilon] < \epsilon$$

for any finite dimensional subspace $F_1 \subset F^{\perp}$.

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In analysis we often work with multiple norms that are continuous, in one way or another, with respect to each other. The notion of *measurable norm* is distinct and unique in that it is a relationship between norms that involves the measure of balls.

Orthonormal basis detail

Let $i : H \to B$ be an AWS, where H is separable. Let $x_1, x_2, \ldots$ be dense in H. Choose a finite dimensional subspace $S_1 \subset H$ such that

$$\mathrm{Gauss}_T[|v|_B > 1/2] < 1/2$$

where T is any finite dimensional subspace of $S_1^\perp$. Let

$$F_1 = \text{span of } S_1 \text{ and } x_1.$$

Next choose finite dimensional subspace $S_2 \subset F_1^\perp$ with

$$\mathrm{Gauss}_T[|v|_B > 1/2] < 1/2$$

where T is any finite dimensional subspace of $S_2^\perp$. Let

$$F_2 = \text{span of } S_2 \text{ and } x_2.$$

Define F_n for every positive integer n thus. Then $\cup_n F_n$ is dense, containing each x_i. Choose an orthonormal basis of F_1, extend to an orthonormal basis of F_2, and so on.

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assumed nondenerate. Assume also that each f has mean 0.

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assumed nondenerate. Assume also that each f has mean 0. Thus

$$B^* \subset L^2(B, \mu).$$

The *Cameron-Martin space* H for (B, μ) is the Hilbert space:

$$H = H^* = \overline{B^*} \subset L^2(\mu)$$

Gaussian measure on Banach spaces

For any $\phi \in B^*$ and $v \in H$ we have the $L^2(\mu)$ pairing

$$\langle \phi, v \rangle_\mu = \int_B \phi(x) v(x)\, d\mu(x), \tag{7}$$

This is the evaluation of $\phi \in B^*$ on the B-valued integral

$$\int_B x v(x)\, d\mu(x) \in B \tag{8}$$

(Badly disappointing, to have to use the Bochner integral, but see Driver's notes [4] for a detailed development of the Bochner integral.) From $B^* \subset H^*$ we cannot just conclude that H lies inside B. But we can imbed H into B using:

$$i : H \to B : v \mapsto \int_B x v(x)\, d\mu(x), \tag{9}$$

Other results

Among the vast array of other results we note:

- If μ is centered nondegenerate Gaussian on a separable Banach space B then $i : H \to B$ is an AWS, where H is the Cameron-Martin space.

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- If μ is centered nondegenerate Gaussian on a separable Banach space B then $i : H \to B$ is an AWS, where H is the Cameron-Martin space.
- Wiener measure on the sup-normed Banach space $C_0([0,1];\mathbb{R}^d)$, with initial value 0, is Gaussian measure whose Cameron-Martin space works out to be the Hilbert space of all absolutely continuous functions $f \in C_0([0,1];\mathbb{R}^d)$ for which the a.e.-defined derivative f' is in $L^2([0,1];\mathbb{R}^d)$.

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- If $i : H \to B$ is an AWS in which the norm on B is actually a Hilbert norm then i is Hilbert-Schmidt. Conversely any HS injection gives an AWS.

Bibliography I

There is a huge literature, but the following is a list I consulted ($\neq$ read) for these notes in decreasing order of depth of consultation:

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Nate Eldredge, online notes at Cornell:
`http://www.math.cornell.edu/~neldredge/7770/`

Bruce Driver's online notes on probability theory:
follow link in [3]

Bibliography II

Daniel W. Stroock, *Abstract Wiener Spaces, Revisited*, Communications on Stochastic Analysis Vol. 2, No. 1 (2008) 145-151.