NUCLEAR SPACE FACTS, STRANGE AND PLAIN

JEREMY J. BECNEL AND AMBAR N. SENGUPTA

Abstract. We present a scenic, but practical drive through nuclear spaces, stopping to look at unexpected results both for nuclear spaces and their strong and weak duals.

1. Introduction

The subject of nuclear spaces saw heavy activity in the 1960s and 1970s but, despite the publication of some deep and voluminous books on the subject, the field largely disappeared from mainstream functional analysis. However, nuclear spaces serve as a practical and useful setting in stochastic analysis. The books on the subject, with the exception of Gel'fand et al., tend to be heavy going on general theory. Our purpose in the present paper is to take a very concrete approach, firmly avoiding generality and focusing on concrete questions; in keeping with this spirit, the terms 'bornological' and 'barrelled' will not appear anywhere outside of this sentence! More regrettably, we do not explore tensor products. The guiding principle is to stay close to questions which arise in applications of the general theory to specific settings closer to hard analysis.

2. Topological Vector Spaces

By a *topological vector space* we shall mean a real or complex vector space, equipped with a Hausdorff topology for the which the operations of addition and multiplication are continuous. A general open set in such a space is the union of translates of neighborhoods of 0.

A topological vector space X is said to be *locally convex* if every neighborhood of 0 contains a neighborhood of 0 which is a convex set. By a

Date: December, 2009.

Key words and phrases. Nuclear Spaces.

The research of J. Becnel is supported by National Security Agency Young Investigators Grant MPO-BA331.

The research of A. Sengupta was supported by US National Science Foundation Grant DMS-0601141.

locally convex space we shall mean a locally convex topological vector space.

At first sight, a locally convex space appears ethereal, with little structure. Yet, remarkable, any such space can be understood in terms of normed linear spaces, as we explain below.

If U is a convex neighborhood of 0, inside it there is a convex neighborhood of 0 which is also *balanced*, i.e. mapped into itself under multiplications by scalars of magnitude ≤ 1 (see [3, Theorem 1.14(b)]).

For a convex balanced neighborhood of 0, the function

$$\rho_U : X \to [0, \infty) : v \mapsto \rho_U(v) = \inf\{t > 0 \,:\, v \in tU\} \tag{2.1}$$

is a *semi-norm* in the sense that

$$\rho_U(a+b) \leq \rho_U(a) + \rho_U(b), \qquad \rho_U(ta) = |t|\rho_U(a) \tag{2.2}$$

for all $a, b \in X$ and scalars t. The function ρ_U is uniformly continuous, because

$$|\rho_U(a) - \rho_U(b)| \leq \rho_U(a-b),$$

which is less than any given positive ϵ if $a - b \in \epsilon U$.

Each seminorm ρ_U generates a topology τ_U on X, consisting of all translates of scalar multiples of the unit ball $U = \{v \in X \,:\, \rho_U(v) < 1\}$. The topology on X is in fact the smallest topology which contains τ_U, for all convex balanced neighborhoods U of 0.

Warning: The topology τ_U is Hausdorff if and only if ρ_U is actually a norm, i.e. U does not contain any full lines!

Suppose X is a locally convex space, and U and V are convex balanced neighborhoods of 0 with $U \subset V$; using a smaller neighborhood of 0 results in larger magnitudes for vectors:

$$\rho_V \leq \rho_U \text{ if } U \subset V. \tag{2.3}$$

For $U \subset V$, the τ_V-open neighborhood rV of 0 contains the τ_U-open neighborhood rU of 0, for every $r > 0$, and so every τ_V-open set is the union of translates of suitable balls rU. Thus, using a smaller neighborhood of 0 results in a more refined topology:

$$\tau_V \subset \tau_U \text{ if } U \subset V. \tag{2.4}$$

If the convex balanced neighborhood U of 0 is 'finite' in every direction, i.e. 0 is the only vector for which all multiples lie inside U, then ρ_U is a *norm*, i.e. ρ_U is 0 only on the zero vector. Indeed, in this case the topology arises from ρ_U in the sense that every open set is the union of translates of scalar multiples of the unit ball $U = \{v \in X \,:\, \rho_U(v) < 1\}$.

On the other hand, if U is 'infinite in some direction', i.e. it contains all multiples of some non-zero vector, then ρ_U is 0 on any vector in that direction. As we shall see later, in many naturally occurring spaces, open neighborhoods of 0 contain infinite-dimensional subspaces, and so ρ_U can be 0 on large subspaces. Specifically, $\rho_U^{-1}(0)$ is the subspace consisting of all vectors all of whose multiples lie inside U. Nullifying this subspace yields the quotient space

$$X_U = X/\rho_U^{-1}(0), \tag{2.5}$$

on which ρ_U, of course, induces a genuine norm, not just a semi-norm. Note that ρ_U does indeed pass to a well-defined function on the quotient X_U, because if $v \in X$ and $z \in \rho_U^{-1}(0)$, then

$$\begin{aligned} \rho_U(v) &= \rho_U(v + z - z) \\ &\leq \rho_U(v + z) + \rho_U(z) = \rho_U(v + z) \\ &\leq \rho_U(v) + \rho_V(z) = \rho_U(v). \end{aligned} \tag{2.6}$$

If v is a non-zero vector in the locally convex space then (recalling that X is, by definition, Hausdorff) choosing a convex balanced neighborhood U of 0 for which $v \notin U$, we see that v survives to a non-zero vector in the quotient space X_U under the quotient map

$$p_U : X \to X_U : x \mapsto p_U(x). \tag{2.7}$$

If for a vector v, the projection $p_U(v)$ is 0 then $v \in \rho_U^{-1}(0)$ and so every multiple of v lies in U and, in particular, v itself lies in U; thus,

$$\ker p_U = \rho_U^{-1}(0) =\subset U. \tag{2.8}$$

Here we see a phenomenon that is very infinite-dimensional in flavor: every τ_U-neighborhood of 0 containing an entire non-zero subspace.

One can think of $p_U(x)$ as a rough image of x, the roughness resulting from the fact that some information about x lying in $\ker p_U = \rho_U^{-1}(0)$ is discarded. If $U \subset V$ then $\rho_V \leq \rho_U$ which implies that $\rho_U^{-1}(0) \subset \rho_V^{-1}(0)$, and so $p_V(x)$ ought to be an even rougher image than $p_U(x)$, since more information about x is discarded. The 'full' picture for x is presumably contained in the collection of all the rough renditions $p_U(x)$. This is made precise in the notion of *projective limit* below in Theorems 2.1 and 2.2.

Thus, effectively, the space X is encoded in the family of normed linear spaces X_U, through the quotient maps p_U.

Observe that if a sequence $(x_n)_{n\geq 1}$ in a locally convex space X converges then it converges with respect to the topology τ_U, equivalently the semi-norm ρ_U, for every balanced convex neighborhood U of 0. This is simply because the topology τ_U is contained in the full topology τ of X. For the converse, assume that the topology τ of X has a local base $\mathcal{N}$ consisting of convex, balanced neighborhoods U of 0 for which ρ_U is a norm, i.e. that every neighborhood of 0 contains as subset some neighborhood in $\mathcal{N}$. Suppose $(x_n)_n$ is a sequence which is convergent with respect to τ_U, for every $U \in \mathcal{N}$, and let x_U denote the limit, which is unique since ρ_U is a norm. Working with two such neighborhoods U_1 and U_2 of 0, and noting that the topology $\tau_{U_1 \cap U_2}$ contains all the open sets in both τ_{U_1} and τ_{U_2}, it follows that x_U is, in fact, independent of the choice of U. Let x be the common value of all the x_U. Now if V is a neighborhood of 0 in τ then it contains as subset some $U \in \mathcal{N}$, and so $x_n - x \in U$ for large n, and hence $x_n - x \in V$ for large n. Thus, $x_n \to x$ with respect to the topology τ.

A sequence $(x_n)_{n\geq 1}$ in a topological vector space X is said to be *Cauchy* with respect to a topology τ if the differences $x_n - x_m$ are all eventually in any given τ-neighborhood of 0 for large n and m. Any convergent sequence is automatically Cauchy. A topological vector space is said to be *complete* if every Cauchy sequence converges.

In a locally convex space a sequence is Cauchy if and only if it is Cauchy with respect to the topology τ_U, for every convex balanced neighborhood U of 0, and this is equivalent to the sequence being Cauchy with respect to each semi-metric d_U, specified by

$$d_U(x, y) = \rho_U(x - y). \tag{2.9}$$

If U and V are convex balanced neighborhoods of 0 and $U \subset V$ then the identity map $X \to X$ induces a natural surjection p_{VU} on the quotients: it is the map

$$p_{VU} : X_U \to X_V : p_U(x) \mapsto p_V(x) \tag{2.10}$$

for all $x \in X$. This takes a picture $p_U(x)$ of x and produces a coarses picture $p_V(x)$. Note that p_{VU} is a linear contraction mapping.

If a locally convex space X is metrizable, then (Rudin [3, Theorem 1.24]) its topology is induced by a translation-invariant metric d for which open balls are convex, and so every neighborhood of 0 contains a locally convex neighborhood W (an open ball, for instance) for which ρ_W is a norm (of course, the topology τ_W induced by ρ_W is, in general, smaller than τ); if V is a convex balanced neighborhood of 0, with $V \subset W$, then ρ_V is also a norm.

Theorem 2.1. *Let X be a complete, metrizable, locally convex space. Let $\mathcal{B}$ be a set of all convex, balanced neighborhoods of* 0 *such that every* *neighborhood of* 0 *contains as subset some neighborhood in* $\mathcal{B}$*, i.e. suppose* $\mathcal{B}$ *is a local base at* 0*, consisting of convex balanced sets, for the topology of* X*. Suppose now that associated to each* $V \in \mathcal{B}$ *an element* $x_V \in X_V = X/\rho_V^{-1}(0)$*, such that*

$$p_{WV}(x_V) = x_W \tag{2.11}$$

for every $W \in \mathcal{B}$ *for which* $W \supset V$*. Then there exists a unique* $x \in X$ *such that* $p_W(x) = x_W$ *for every* $W \in \mathcal{B}$*. Moreover, for such* W*, the quotient* X_W *is a Banach space.*

Proof. Let d be a translation-invariant metric on X which induced the topology τ of X and for which all open balls are convex. Choose $W_n \in \mathcal{B}$ lying inside the open ball of d-radius $1/n$ centered at 0, and then let

$$V_n = W_1 \cap \ldots \cap W_n,$$

for all $n \in \{1, 2, ...\}$. As before, let $p_U : X \to X_U = X/\rho^{-1}(0)$ be the canonical projection. Given $x_V \in X_V$, for each $U \in \mathcal{B}$, satisfying the consistency condition (2.11), choose $x_n \in X$ for which $p_{V_n}(x_n) = x_{V_n}$, for every $n \in \{1, 2, 3, ...\}$. We claim that $(x_n)_{n\geq 1}$ is a Cauchy sequence: if W is any neighborhood of 0, choose $k \in \{1, 2, 3, ...\}$ such that $V_k \subset W$; then for any $n, m \geq k$ we have $V_k \supset V_n, V_m$, and so

$$\begin{aligned} p_{V_k}(x_n - x_m) &= p_{V_kV_n}\big(p_{V_n}(x_n)\big) - p_{V_kV_m}\big(p_{V_m}(x_m)\big) \\ &= p_{V_kV_n}(x_{V_n}) - p_{V_kV_m}(x_{V_m}) \\ &= 0. \end{aligned} \tag{2.12}$$

Hence $x_n - x_m$ is in V_k, and therefore also in W. Thus, $(x_n)_{n\geq 1}$ is Cauchy in X. Let x be the limit of this sequence. Then

$$\begin{aligned} p_{V_k}(x) &= \lim_{n\to\infty} p_{V_k}(x_n) \\ &= \lim_{n\to\infty} p_{V_kV_n}p_{V_n}(x_n) \\ &= \lim_{n\to\infty} p_{V_kV_n}(x_{V_n}) = \lim_{n\to\infty} x_{V_k} \\ &= x_{V_k}. \end{aligned} \tag{2.13}$$

Now, for any convex balanced neighborhood $W \in \mathcal{B}$, choose k such that $V_k \subset W$; then

$$p_W(x) = p_{WV_k}\big(p_{V_k}(x)\big) = p_{WV_k}(x_{V_k}) = x_W,$$

showing that x has the desired properties.

Let us check that x is unique. To this end suppose $x \in X$ satisfies $p_W(x) = 0$ for every $W \in \mathcal{B}$, i.e. x lies in every such neighborhood. But then x lies in every open ball centered at 0 relative to the metric d, and so x must be 0.

The completeness of X_W follows from (a somewhat overlooked consequence of) the open mapping theorem, see Rudin [3, Theorem 2.11(iii)], which guarantees that the continuous linear image of a complete metrizable space inside a topological vector space is complete. □

Finally, we can prove the fundamental projective limit nature of a complete, metrizable, locally convex space.

Theorem 2.2. *Suppose X is a complete, metrizable, locally convex space, let $p_V : X \to X_V = X/\rho_V^{-1}(0)$ be the usual projection. Let $\mathcal{B}$ be a set of all convex, balanced neighborhoods of 0 such that every neighborhood of 0 contains as subset some neighborhood in $\mathcal{B}$, i.e. suppose $\mathcal{B}$ is a local base at 0, consisting of convex balanced sets, for the topology of X. Assume Y is a topological vector space and, for each $V \in \mathcal{B}$ there is a continuous linear map $f_V : Y \to X_V$ such that $p_{WV} f_V = f_W$ for every $W \supset V$ in $\mathcal{B}$. Then there is a unique continuous linear map $f : Y \to X$ such that*

$$p_V f = f_V \tag{2.14}$$

for every $V \in \mathcal{B}$.

Proof. From the previous theorem, we have, for any $y \in Y$, the existence of a unique $f(y) \in X$ for which $p_V\big(f(y)\big) = f_V(y)$ for every $V \in \mathcal{B}$. The uniqueness of $f(y)$ for each $y \in Y$ also implies that f is linear. It remains to show that f is continuous. Since $f_V = p_V f$ is continuous it follows that f^{-1} of any open ρ_V-ball is open in Y, for every $V \in \mathcal{B}$. If U is a neighborhood of 0 in X then $U \supset V$, for some $V \in \mathcal{B}$, and since $V = \{x \in X : \rho_V(x) < 1\}$, it follows that $f^{-1}(V)$ is open in Y, i.e. $f^{-1}(U)$ contains an open neighborhood of 0. Thus, f is continuous. □

Thus, a complete, metrizable, locally convex space is obtainable as a projective limit of Banach spaces. In the next section we turn to nuclear spaces, where this specialization is carried just a bit further to Hilbert spaces.

For a topological vector space X over the field F (which, for us, is always $\mathbb{R}$ or $\mathbb{C}$), the dual space X' is vector space of all continuous linear functionals on X. If X is locally convex then the Hahn-Banach theorem guarantees that $X' \neq \{0\}$. There are several topologies of interest on X'. For now let us note the two extreme ones:

- The *weak topology* on X' is the smallest topology on X' for which the evaluation map
$$X' \to F : x' \mapsto \langle x', x\rangle$$
is continuous for all $x \in X$. This topology consists of all unions of translates of sets of the form
$$B(D;\epsilon) = \{x' \in X' : \sup_{x\in D} |\langle x', x\rangle| < \epsilon\}$$
with D running over finite subsets of X, and ϵ over $(0, \infty)$.
- The *strong topology* on X' consists of all unions of translates of sets of the form
$$B(D;\epsilon) = \{x' \in X' : \sup_{x\in D} |\langle x', x\rangle| < \epsilon\}$$
with D running over all bounded subsets of X and ϵ over $(0, \infty)$.

3. The nuclear space structure

We work with a real complex topological vector space $\mathcal{H}$ equipped with the following structure. There is a sequence of inner-products $\langle \cdot, \cdot \rangle_p$, for $p \in \{0, 1, 2, 3, ...\}$, on $\mathcal{H}$ such that
$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \cdots \tag{3.1}$$
The completion of $\mathcal{H}$ in the norm $\|\cdot\|_0$ is denoted H_0, and inside this Hilbert space the completion of $\mathcal{H}$ with respect to $\|\cdot\|_p$ is a dense subspace denoted H_p. We assume that H_0 is separable, and that $\mathcal{H}$ is the intersection of all the spaces H_p. Thus,
$$\mathcal{H} = \bigcap_{p=0}^{\infty} H_p \subset \cdots \subset H_2 \subset H_1 \subset H_0. \tag{3.2}$$
Furthermore, we assume that each inclusion $H_{p+1} \to H_p$ is a Hilbert-Schmidt operator, i.e. there is an orthonormal basis $v_1, v_2, ...$ in H_{p+1} for which
$$\sum_{n=1}^{\infty} \|v_n\|_p^2 < \infty. \tag{3.3}$$
The topology on $\mathcal{H}$ is the *projective limit topology* from the inclusions $\mathcal{H} \to H_p$, i.e. it is induced by the norms $\|\cdot\|_p$. Thus an open set in this topology is the union of $\|\cdot\|_p$-balls with p running over $\{0, 1, 2, 3, ...\}$.

All these assumptions make $\mathcal{H}$ a *nuclear space*. An infinite-dimensional topological vector space is never locally compact, but a nuclear space is an excellent substitute in the infinite-dimensional case: the definition ensures, in particular, that an open ball in H_{p+1} has compact closure in H_p.

Note that if τ_p is the topology on $\mathcal{H}$ given by $\|\cdot\|_p$, then by (3.1), the identity map

$$(\mathcal{H}, \tau_{p+1}) \to (\mathcal{H}, \tau_p)$$

is continuous, for $p \in \{0, 1, 2, ...\}$, and so

$$\tau_0 \subset \tau_1 \subset \dots . \tag{3.4}$$

The inclusions here are strict if $\mathcal{H}$ is infinite-dimensional, because of the Hilbert-Schmidt assumption made above.

A linear functional on $\mathcal{H}$ is continuous with respect to $\|\cdot\|_p$ if and only if it extends to a (unique) continuous linear functional on H_p. Thus we may and will make the identification

$$\mathcal{H}'_p \simeq H'_p, \tag{3.5}$$

where the left side is the vector space of all linear functionals on $\mathcal{H}$ which are continuous with respect to the norm $\|\cdot\|_p$. Virtually always, the space H'_p will be equipped with the *strong topology*, i.e. the one which arises from identification of H'_p with H_p by associating to each $x \in H_p$ the linear functional $\langle\cdot, x\rangle_p$. Then H'_p is a Hilbert space, isometrically isomorphic to H_p. We use the notation

$$H_{-p} = H'_p, \tag{3.6}$$

for $p \in \{0, 1, 2, ...\}$, and, of course, this continues to be meaningful as an identification even for negative integers p. We have the norm

$$\|x\|_{-p} = \sup_{x \in H_p; \|x\| \leq 1} |\langle x', x\rangle| \tag{3.7}$$

The topological dual of $\mathcal{H}$ is

$$\mathcal{H}' = \bigcup_{p=0}^{\infty} H'_p \supset \cdots \supset H'_2 \supset H'_1 \supset H'_0 \simeq H_0. \tag{3.8}$$

It is readily checked that the inclusion $H'_{p+1} \to H'_p$ is the adjoint of the inclusion map $H_p \to H_{p+1}$.

In addition to the weak and strong topologies, there is also the *inductive limit* topology on $\mathcal{H}'$, which is the largest locally convex topology for which the inclusions $H'_p \to \mathcal{H}'$ are all continuous.

Fact 3.1. *The strong topology and the inductive limit topology on $\mathcal{H}'$ are the same.*

For a proof see [1].

3.1. Hilbert-Schmidt facts. We will use Hilbert-Schmidt operators, and recall here some standard facts about them. For a linear operator $T : H \to K$ between Hilbert spaces, define the Hilbert-Schmidt norm to be

$$\|T\|_{\mathrm{HS}} \stackrel{\mathrm{def}}{=} \Big\{\sum_{v\in\mathcal{B}} \|Tv\|^2\Big\}^{1/2} < \infty. \tag{3.9}$$

where $\mathcal{B}$ is some orthonormal basis of H; if $H = 0$ the Hilbert-Schmidt norm of T is, by definition, 0. If $\|T\|_{\mathrm{HS}} < \infty$ then T is said to be a *Hilbert-Schmidt operator*.

Fact 3.2. *For a linear operator $T : H \to K$ between Hilbert spaces, the Hilbert-Schmidt norm given in (3.9) is independent of the choice of the orthonormal basis $\mathcal{B}$. If T is Hilbert-Schmidt, then:*

(i) *T is bounded and $\|T\| \le \|T\|_{\mathrm{HS}}$;*
(ii) *the adjoint $T^* : K \to H$ is also Hilbert-Schmidt and $\|T^*\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$;*
(iii) *the image of any closed ball in H under T is a compact subset of K;*
(iv) *if T is injective then H is separable;*
(v) *if $H \neq 0$ then there is an orthonormal basis of H consisting of eigenvectors of T^*T.*

Proof. . These results are standard, but we prove (iii) for entertainment and because we will need the ideaa in the proof later. Let $D(R)$ be the closed ball of radius $R \in (0, \infty)$ in H. We will show that T maps $D(R)$ to a compact subset of K. Since K is a complete metric space, compactness of a set C is equivalent to it being closed and totally bounded (which means that for any $\epsilon > 0$, the set C can be covered by a finite collection of balls of radius ϵ).

Let $\epsilon > 0$. Choose an orthonormal basis $\mathcal{B}$ in H (if $H = 0$ the result is trivial, so we assume $H \neq 0$). Since $\sum_{u\in\mathcal{B}} \|Tu\|_K^2 < \infty$, there is a non-empty finite set $\mathcal{B}_f \subset \mathcal{B}$ such that

$$R^2 \sum_{u\in\mathcal{B}-\mathcal{B}_f} \|Tu\|_K^2 < \epsilon^2/4.$$

Let L_f be the linear span of $\mathcal{B}_f$, and $D(R)_f$ the orthogonal projection of $D(R)$ onto L_f. Then $D(R)_f$ is compact, being the closed ball of finite radius R with respect to the norm on the finite-dimensional space L_f induced by $\|\cdot\|_H$, and so $D(R)_f$ is also totally bounded. Therefore, there is a non-empty finite subset F of $D(R)_f$ such that for any $x \in D(R)$, there is some $y_F \in F$ whose distance from $x_f = \sum_{u\in\mathcal{B}_f} \langle x, u\rangle u$ is $<$ $(1 + \|T\|_{\mathrm{HS}})^{-1}\epsilon/2$. Hence,

$$\begin{aligned}
\|Tx - Ty_F\|_K &\leq \left\|Tx - Tx_f\right\|_K + \left\|Tx_f - Ty_F\right\|_K \\
&\leq \left\{\sum_{u\in\mathcal{B}-\mathcal{B}_f} |\langle x, u\rangle|^2\right\}^{1/2} \left\{\sum_{u\in\mathcal{B}-\mathcal{B}_f} \|Tu\|_K^2\right\}^{1/2} \\
&\qquad + \|T\|_{\mathrm{HS}}\|x_f - y_F\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned} \tag{3.10}$$

Hence the image of $D(R)$ under T is totally bounded.

Next we check that T maps $D(R)$ onto a closed set in K. Consider a sequence $x_1, x_2, ... \in D(R)$ such that $(Tx_n)_{n\geq 1}$ converges to some $y \in K$. The image of $D(R)$ under T is bounded and hence contained inside some closed ball of radius $R' \in (0, \infty)$ in K, and since a closed ball in a normed linear space is weakly compact (by the Banach-Alaoglu theorem) there is a subsequence (x_{n_k}) such that (Tx_{n_k}) converges weakly to some $y \in K$. Then, for any $w \in K$, we have

$$\langle y, w\rangle_K = \lim_{k\to\infty} \langle Tx_{n_k}, w\rangle_K = \lim_{k\to\infty} \langle x_{n_k}, T^*w\rangle_H = \langle x, T^*w\rangle_H = \langle Tx, w\rangle_K \tag{3.11}$$

Thus, the weak limit y of (Tx_{n_k}) is Tx. Since x necessarily lies in $D(R)$, we conclude that y is in the image of $D(R)$ under T, and so $T\big(D(R)\big)$ is a closed subset of K. Being closed and totally bounded, it is compact. □

Some unexpected facts about nuclear spaces are connected with the following observation:

Fact 3.3. *Let $j : H \to K$ be Hilbert-Schmidt map between Hilbert spaces. Then there is a sequence of points in H which does not converge in H but whose image by j in K is convergent.*

Proof. An orthonormal sequence $e_1, e_2, ...$ in H is not Cauchy and hence not convergent, but the sequence $\big(j(e_n)\big)_{n\geq 1}$ converges to 0 in K because $\sum_{n=1}^{\infty} \|j(e_n)\|_K^2$ is convergent. □

4. Balls and Cubes

In this section, as usual, p denotes an element of $\{0, 1, 2, 3, ...\}$. We work, as before, with the space $\mathcal{H}$, equipped with inner-products $\langle\cdot,\cdot\rangle_p$, with the Hilbert-Schmidt condition explained in the context of (3.2).

4.1. Open and Closed Balls. We examine properties of the 'open' balls

$$B_p(R) = \{y \in H_p : \|y\|_p < R\} \tag{4.1}$$

and the 'closed' balls

$$D_p(R) = \{y \in H_p : \|y\|_p \leq R\} \tag{4.2}$$

for every $p \in \mathbb{Z}$ and $R \in (0, \infty)$.

Fact 4.1. *$B_p(R) \cap \mathcal{H}$ is open in $\mathcal{H}$, for all $p \in \{0, 1, 2, ...\}$.*

Proof. . This is because the inclusion $\mathcal{H} \to H_p$ is continuous from the definition of the topology on $\mathcal{H}$. □

Fact 4.2. *$B_p(R) \cap \mathcal{H}$ is not bounded in $\|\cdot\|_{p+1}$-norm. It is not bounded in $\mathcal{H}$.*

Proof. Let $v_1, v_2, ...$ be an orthonormal basis of H_{p+1} lying inside $\mathcal{H}$ (choose a maximal $\langle \cdot, \cdot \rangle_p$-orthonormal set in $\mathcal{H}$; this is necessarily also maximal in H_{p+1}, and hence an orthonormal basis of H_p). Then, since the inclusion $H_{p+1} \to H_p$ is Hilbert-Schmidt, the sum $\sum_{n=1}^{\infty} \|v_n\|_p^2$ is finite, and so the lengths $\|v_n\|_p$ tend to 0. Thus the vectors

$$\frac{R}{2} \|v_n\|_p^{-1} v_n, \tag{4.3}$$

which are all in $B_p(R)$, have $\langle \cdot, \cdot \rangle_{p+1}$-norm going to ∞. In particular, $B_p(R) \cap \mathcal{H}$ is not bounded in H_{p+1}, and hence also not bounded in $\mathcal{H}$. □

The next observation provides some compact sets in $\mathcal{H}$. A subset D of a topological vector space X is said to be *bounded* if for any neighborhood W of 0 in X, the set D is contained in tW for all scalars t with large enough magnitude. Thus, for instance any open or closed ball in a normed linear space is bounded.

Fact 4.3. *The set $\cap_{p \geq 0} D_p(r_p)$ is compact in $\mathcal{H}$, for any $r_1, r_2, ... \in (0, \infty)$.*

Proof. . A closed ball $D_p(r) \cap \mathcal{H}$ is closed in the topology $\tau_p \subset \tau$, and so it is a closed subset of $\mathcal{H}$. If W is a neighborhood of 0 in $\mathcal{H}$, then there is a $q \in \{0, 1, 2, ..\}$ and an $r \in (0, \infty)$ such that $B_q(r) \subset W$; hence, $\cap_{p \geq 0} D_p(r_p) \subset tB_q(r) \subset tW$ if $|t| > r_q/r$. Thus, $\cap_{p \geq 0} D_p(r_p)$ is closed and bounded and hence, by the Heine-Borel property proved below in Fact 5.3, it is compact in $\mathcal{H}$ □

We turn to balls in the dual spaces H_p' and $\mathcal{H}'$. Unless stated otherwise, we equip each Hilbert space dual $H_p' \simeq \mathcal{H}_p'$ with the strong topology, which is the same as the Hilbert-space topology.

Fact 4.4. *$\mathcal{H}'$ contains no non-empty open subset lying entirely inside H_p', for $p \in \{0, 1, 2, ...\}$. More generally, in a topological vector space a proper subspace has empty interior.*

Proof. . If a proper subspace Y of a topological vector space X contains an open set U with a point y lying in U, then $-y + U$ is a neighborhood of 0 in X lying entirely inside Y, but then the union of all multiples of $U - y$ would be all of X, and hence Y would be all of X. □

Fact 4.5. *$B_{-1}(R) \cap H_0$ is open in H_0. More generally, $B_{-q}(R) \cap H_p'$ is open in H_p' for $p < q$, with $p, q \in \{0, 1, 2, ...\}$.*

Proof. . This is because the inclusion $\mathcal{H} \to H_p$ is continuous. The inclusion $j : H_q \to H_p$ induces $j^* : H_p' \to H_q'$ which is also continuous and so $(j^*)^{-1}(B_{-q}(R))$ is open in H_p', i.e. $B_{-q}(R) \cap H_p'$ is open in H_p'. □

Fact 4.6. *$B_0(R)$ is not open in H_1' in the strong (Hilbert) topology. More generally, $B_{-p}(R)$ is not strongly open in H_q' for $p < q$.*

Proof. . By Fact 4.4. □

Fact 4.7. *$D_{-p}(R)$ is weakly, and hence strongly, closed in $\mathcal{H}'$. If $p, q \in \{0, 1, 2, ...\}$, with $q > p$, and $R \in (0, \infty)$, then $D_{-p}(R)$ is compact in the Hilbert-space H_q'. Moreover, $D_{-p}(R)$ is strongly, and hence weakly, compact as well as sequentially compact in $\mathcal{H}'$.*

Proof. . The inclusion map $H_{p+1} \to H_p$ being Hilbert-Schimdt, so is the adjoint inclusion $H_p' \to H_{p+1}'$, and so $D_{-p}(R)$ is compact inside H_{p+1}' with the strong (Hilbert-space) topology. By continuity of the inclusion map $H_q' \to \mathcal{H}'$ it follows that $D_{-p}(R)$ is compact in H_q' with respect to the strong topology.

Continuity of the inclusion map $H_{p+1}' \to \mathcal{H}'$ also implies that $D_{-p}(R)$ is compact in $\mathcal{H}'$ with respect to the strong topology.

Since $D_{-p}(R)$ is compact in the metric space H_q' it is sequentially compact in H_q', i.e. any sequence on $D_{-p}(R)$ has a subsequence which is convergent in H_q'. From continuity of the inclusion $H_q' \to \mathcal{H}'$ it follows that such a subsequence also converges in $\mathcal{H}'$. Thus, $D_{-p}(R)$ is strongly, and hence also weakly, sequentially compact in $\mathcal{H}'$. □

4.2. Cubes.

Fact 4.8. *If H is a separable Hilbert space then the closed unit cube, relative to any orthonormal basis, in the dual H^* is not compact. This is in contrast to the closed unit ball in H^*.*

Proof. . The closed unit cube in H^* relative to an orthonormal basis $e_1, e_2, ...$ is

$$C = \{v^* \in H^* : |\langle v^*, e_j\rangle| \leq 1 \text{ for every } j \in \{1, 2, 3, ...\}\}$$

Let

$$c_N = e'_1 + \cdots + e'_N,$$

where $e'_j \in H^* : v \mapsto \langle v, e_j\rangle$. Then each c_N lies in C. If C were weakly compact then there would be a subsequence of $(c_N)_{N\geq 1}$ which would converge weakly to a point $c \in C$. Now for any $j \in \{1, 2, 3, ...\}$, we have

$$c_N(e_j) = 1 \text{ for all } N \geq j.$$

Consequently,

$$c(e_j) = 1 \text{ for all } j \in \{1, 2, 3, ...\}.$$

But then

$$\sum_{j=1}^{\infty} |c(e_j)|^2 = \infty,$$

and so there is no such c in H^*. □

Fact 4.9. *Suppose $e_1, e_2, ...$ is an orthonormal basis of H_0 which lies in $\mathcal{H}$, and assume that the vectors $\lambda_n^{-p} e_n$ form an orthonormal basis of H_p, where $\lambda_n = \|e_n\|_1$. Assume also that*

$$1 \leq \lambda_1 < \lambda_2 < \cdots \qquad \text{and } \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$$

Let C be the closed cube in $\mathcal{H}'$ given by

$$C = \{x' \in \mathcal{H}' : |\langle x', e_n\rangle| \leq \lambda_n^n \text{ for all } n \in \{1, 2, 3, ...\}\}$$

Then C contains strong neighborhood of 0 and hence is not compact. However, $C \cap F$ is compact for every finite-dimensional subspace of H_0.

Proof. . Let $x' \in B_{-p}(\epsilon_p)$, for some $p \in \{0, 1, 2, 3, ...\}$, and $\epsilon_p > 0$. Then

$$\sum_{n=1}^{\infty} |\langle x', \|e_n\|_p^{-1} e_n\rangle|^2 = \|x'\|_{-p}^2 < \epsilon_p^2$$

and so, for every n,

$$|\langle x', e_n\rangle| \leq \epsilon_p \|e_n\|_p$$

To make this $\leq \lambda_n^n$, we should take

$$\epsilon_p \leq \|e_n\|_p^{-1}\lambda_n^n = \lambda_n^{n-p},$$

and this should hold for all $n \in \{1, 2, 3, ...\}$. Thus, we could take $\epsilon_0 = 1$, and, for $p \in \{1, 2, 3, ...\}$:

$$\epsilon_p = \min\{\lambda_1^{1-p}, \lambda_2^{2-p}, ..., \underbrace{\lambda_p^{p-p}}_{1}\},$$

wherein we have dropped all the later terms λ_n^{n-p} with $n > p$, as these are all ≥ 1. Thus, with this choice of ϵ_p, the cube

$$C = \{x' \in \mathcal{H}' \,:\, |\langle x', e_n\rangle| \leq \lambda_n^n \text{ for all } n \in \{1, 2, 3, ...\}\}$$

contains each open ball $B_{-p}(\epsilon_p)$ and hence also the convex hull of their union, which is a neighborhood of 0 in the inductive limit topology.

Now consider a finite-dimensional subspace F of $H_0 \subset \mathcal{H}'$, and v' a point in $F \cap C$. Then

$$|\hat{e}_n(v')| \leq \lambda_n^n$$

for all $n \in \{1, 2, 3, ...\}$. Let us *assume for the moment that there is an orthonormal basis* $f_1, ..., f_d$ *of* $F \subset H_0$ *and a fixed* $r \in \{1, 2, 3, ...\}$ *such that each* $\langle f_j, \cdot\rangle_0$ *on* F *is a linear combination of the functionals* $\hat{e}_1, ..., \hat{e}_r$, say

$$\langle f_j, \cdot\rangle_0 = \sum_{n=1}^{r} a_{jn}\hat{e}_n.$$

Then

$$|\langle f_j, v'\rangle| \leq R_j \stackrel{\text{def}}{=} \sum_{n=1}^{r} |a_{jn}|\lambda_n^n < \infty.$$

Hence,

$$\|v'\|_0^2 \leq R_1^2 + \cdots + R_d^2 < \infty.$$

This shows that $C \cap F$ is a bounded subset of F. Since C is weakly closed and so is any finite-dimensional subspace, $C \cap F$ is weakly closed in $\mathcal{H}'$ and hence weakly closed in the induced topology on F; but on the finite-dimensional subspace F there is only one topological vector space structure, and so $C \cap F$ is closed. Being closed and bounded in F, it is compact.

It remains to prove the algebraic statement assumed earlier. Suppose that the algebraic linear span of the functionals $\hat{e}_n$ on F is not the entire dual F^* of F; then they span a proper subspace of F^* and hence there is a non-zero vector $f \in F$ on which they all vanish, but this would then be a vector in H_0 whose inner-product with every e_n is 0 and hence f

would have to be 0. Thus the algebraic linear span of the functionals $\hat{e}_n$ on F is the entire dual F^*, and so, in particular, any element of F^* is a finite linear combination of the functionals the functionals $\hat{e}_n$. This proves that the assumption made earlier is correct. □

5. Facts about the Nuclear Space Topology

We work with an infinite-dimensional nuclear space $\mathcal{H}$ with structure as detailed in the context of (3.1). While there are some treacherous features, such as the one in Fact 5.1, a nuclear space has some very convenient properties which make them almost as good as finite-dimensional spaces.

Fact 5.1. *There a no non-empty bounded open set in $\mathcal{H}$.*

Proof. . Suppose U is a bounded open set in $\mathcal{H}$ containing some point y. Then $U - y$ is a bounded open neighborhood of 0 in $\mathcal{H}$ and so contains some ball $B_p(R) \cap \mathcal{H}$, with $p \in \{0, 1, 2, ...\}$. Then by Fact 4.2 this is impossible. □

One consequence of the preceding observation is that *a Banach space if not a nuclear space.*

Fact 5.2. *The topology on $\mathcal{H}$ is metrizable but not normable.*

Proof. . If there were a norm then the unit ball in the norm would be a bounded neighborhood of 0, contradicting Fact 5.1. The translation-invariant metric on $\mathcal{H}$ is given by

$$d(x, y) = \sum_{p=0}^{\infty} 2^{-p} \min\{1, \|x - y\|_p\} \tag{5.1}$$

induces the topology on $\mathcal{H}$. □

For the following, recall that in a locally convex space, a subset B if said to be bounded if for any U, the set B lies inside some multiple tU of U. If $f : X \to Y$ is a continuous linear map between topological vector spaces and B is a bounded subset of X then $f(B)$ is a bounded sunbset of Y, for if V is a neighborhood of 0 in Y then $B \subset tf^{-1}(V) = f^{-1}(tV)$ for some scalar t and hence $f(B) \subset tV$.

Fact 5.3. *$\mathcal{H}$ has the Heine-Borel property, i.e. every closed and bounded set is compact.*

Proof. . Let B be a closed and bounded subset of $\mathcal{H}$. Take any $p \in \{0, 1, 2, ..\}$. Since the inclusion $\mathcal{H} \to H_{p+1}$ is continuous, B is bounded in H_{p+1}. Since the inclusion $H_{p+1} \to H_p$ is Hilbert-Schmidt it follows, as in the proof of Fact 4.7, that B, as a subset of H_p, is contained in a compact set. Let $(x_n)_{n\geq 1}$ be a sequence of points in B. Then it follows that there is a subsequence $(x^{(p)})_{n\geq 1}$ which converges in H_p. Applying the Cantor process of extracting repeated subsequences, there is a subsequence $(x')_{n\geq 1}$ which is convergent in H_p for every $p \in \{0, 1, 2, ...\}$. By contunity of the inclusions $H_{p+1} \to H_p$ it follows that the limit is the same for every p, i.e. there is a point $y \in \cap_{p\geq 0} H_p = \mathcal{H}$, such that the subsequence (x'_n) converges to y in all the H_p. If V is a neighborhood of 0 in $\mathcal{H}$ then there is a $p \in \{0, 1, 2, ...\}$ and an $\epsilon > 0$ such that the ball $B_p^{\mathcal{H}}(\epsilon)$ lies inside $\{x \in \mathcal{H} : \|x\|_p < \epsilon\} \subset V$. Since $x'_n \to y$ in H_p we have $x'_n - y \in B_p^{\mathcal{H}}(\epsilon) \subset V$ for large n, and so $x'_n \to y$ in $\mathcal{H}$. Thus, a closed and bounded subset of $\mathcal{H}$ is sequentially compact; since $\mathcal{H}$ is metrizable, sequential compactness is equivalent to compactness for any subset of $\mathcal{H}$. □

6. Facts about the Dual Topologies

We turn now to the dual of an infinite-dimensional nuclear space $\mathcal{H}$, and continue with the notation explained earlier in the context of (3.1).

Let us keep in mind that thanks to the Hahn-Banach theorem, the dual of a non-trivial locally convex space is again non-trivial. Indeed, the dual of an infinite-dimensional locally convex space is infinite-dimensional.

If a topology is metrizable then it has a countable local base: any neighborhood U of any point $p \in X$ contains a neighborhood of the form $p + B(0; 1/n)$ for large n.

The following facts are proved in Rudin [3, Theorem 1.24, Theorem 1.32]:

(1) If X has a countable local base then there is a metric d on X which induces the topology of X, is invariant under translations, and every open ball centered at 0 is mapped into itself under multiplication by a scalar of magnitude ≤ 1; moreover, if X is locally convex then d can be chosen so that every open ball is convex.

(2) A sequence in X is Cauchy if and only if it is Cauchy relative to some (hence any) invariant metric which induces the topology on X.

(3) If X is a complete topological vector space with a countable local base, then there is an invariant complete metric on X. A linear functional on a metrizable topological vector space is continuous if and only if it is bounded (which means it maps bounded sets to bounded sets; a subset B of a topological vector space is said to be bounded if for any neighborhood U of 0 the set B lies inside tU for all scalars t with large enough $|t|$).

(4) *Banach-Steinhaus theorem* (uniform boundedness principle): If S is a non-empty set of continuous linear functionals on a complete, metrizable, topological vector space X, and if, for each $x \in X$ the set $\{f(x) : f \in S\}$ is bounded, then for every neighborhood W of 0 in the scalars, there is a neighborhood U of 0 in X such that $f(U) \subset W$ for all $f \in S$. As consequence, with X as stated, *if $f_1, f_2, ... \in X'$ are such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$ then f is in X'.*

Fact 6.1. *If X is a complete, metrizable topological vector space, then the dual space X' is complete with respect to both strong and weak topologies.*

Proof. . Let $\mathcal{A}$ be a non-empty collection of subsets of X, closed under finite unions. Let $\tau_{\mathcal{A}}$ be the topology on X whose open sets are unions of translates of sets of the form

$$B(A; \epsilon) = \{x' \in X' : \sup_{x \in A} |\langle x', x\rangle| < \epsilon\}$$

with A running over $\mathcal{A}$ and ϵ over $(0, \infty)$. If $\mathcal{A}$ is the set of all finite subsets of X then $\tau_{\mathcal{A}}$ is the weak topology; if $\mathcal{A}$ is the set of all bounded subsets of X then $\tau_{\mathcal{A}}$ is the strong topology.

Suppose $(f_n)_{n\geq 1}$ is Cauchy sequence in X' for the topology $\tau_{\mathcal{A}}$. This means that, for any $A \in \mathcal{A}$, the sequence of functions $f_n|A$ is uniformly Cauchy, and hence uniformly convergent, for every $A \in \mathcal{A}$. Let $f(x) = \lim_{n\to\infty} f_n(x)$. By Banach-Steinhaus, f is continuous. A $\tau_{\mathcal{A}}$-neighborhood U of f contains a set of the form $f + B(A; \epsilon)$, and the uniform convergence $f_n \to f$ on A implies that $f_n - f \in B(A; \epsilon)$ for large n, and so $f_n - f \in U$ for large n. Thus, $f_n \to f$ in $\tau_{\mathcal{A}}$. □

Fact 6.2. *Any weakly open neighborhood of* 0 *in the dual of an infinite dimensional locally convex topological vector space contains an infinite-dimensional subspace.*

Proof. . If Y is a vector space over a field F, and $\hat{x}_1, ..., \hat{x}_N : Y \to F$ are linear functionals, then the mapping

$$Y \to F^N : y \mapsto \big(\hat{x}_1(y), ..., \hat{x}_N(y)\big)$$

has kernel infinite-dimensional if Y is infinite-dimensional. We apply this to $Y = X'$. A weakly open neighborhood of 0 in X' contains contains a set of the form

$$B = \{f \in X' \,:\, \hat{x}_1(f) \in W_1, ..., \hat{x}_N(f) \in W_N\},$$

for some $x_1, ..., x_N \in X$, and open neighborhoods $W_1, ..., W_N$ of 0 in ths scalars, and so B contains the $\ker(\hat{x}_1, ..., \hat{x}_N)$. □

Fact 6.3. *The weak dual of an infinite-dimensional Banach space is not metrizable.*

Proof. . Let X be a Banach space. Since it is normed, there is a bounded neigborhood D_1 of 0 (say the unit ball). For $x \in X$, let $\hat{x}$ be the linear functional on X' which is evaluation at x. Let D_1' be the unit ball in X', i.e. the set of all $f \in X'$ for which $\sup_{x \in D_1} |f(x)| \leq 1$. Thus, D_1' is the intersection of the sets $\hat{x}^{-1}(W_1)$, where W_1 is the closed unit ball in the scalars, with x running over B_1. Hence, B_1' is weakly closed. Observe that $\cup_{n \geq 1} nB_1'$ is all of X', for if $f \in X'$ then $f(D_1)$ is a bounded set of scalars and so $\sup_{x \in D_1} |f(x)| \leq n$, i.e. $f \in nDB_1'$, for some $n \in \{1, 2, ...\}$. Now X', with the weak topology, is complete; if it were metrizable, then the Baire category theorem would imply that X' could not be the union of a countable collection of nowhere dense subsets. Thus it will suffice to show that D_1' has empty interior in the weak topology. If, to the contrary, B_1' contains a nonempty weakly open subset, then $D_1' - D_1'$ contains a weakly open neighborhood of 0, and hence an infinite-dimensional subspace of X'. Now $D_1' - D_1' \subset 2D_1'$, and so it would follow that $2D_1'$, and hence D_1' itself, contains an infinite-dimensional subspace. In particular, there is a non-zero $g \in X'$ such that all scalar multiples of g belong to D_1'; but this is impossible from the definition of D_1'. □

In a similar vein there is the following negative result:

Fact 6.4. *The weak topology and the strong topology of the dual of an infinite-dimensional nuclear space are not metrizable.*

Proof. Let X be a nuclear space, with topology generated by norms $\| \cdot \|_p$ for $p \in \{0, 1, 2, ...\}$. Let $D_p(R)$ be the subset of X given by the closed ball of radius R, center 0, for the $\| \cdot \|_p$-norm, and let $D_p'(R)$ be the set of all linear functionals f on X which map $D_p(1)$ into scalars of magnitude $\leq R$. Each set $D'_p(R)$, being the intersection of weakly closed sets, is a weakly closed set, and hence also strongly closed, in X'. The union of the sets $D'_p(n)$ with $p \in \{0, 1, 2, ...\}$ and $n \in \{1, 2, ...\}$ is all of X'. Now $D'_p(n)$ lies in the proper subspace X'_p of X', and so has empty interior in any topology on X' which makes X' a topological vector space. Thus, X', which is a complete topological vector space with respect to both weak and strong topologies, is the countable union of nowhere dense sets, and hence the weak and strong topologies on X' are not metrizable. □

In a sense, sequences do not detect the difference between the weak and strong topologies on the dual space:

Fact 6.5. *A sequence in the dual of a nuclear space is weakly convergent if and only if it is strongly convergent.*

Proof. Since the weak topology is contained in the strong topology, strong convergence implies weak convergence. For the converse, let $(x'_n)_{n\geq 1}$ be a sequence in the dual X' of a nuclear space X, converging weakly to $x' \in X'$. By Banach-Steinhaus, $\{x'_1, x'_2, x'_3, ...\}$ is a uniformly continuous set of functions on X and hence the convergence $x'_n \to x'$ is uniform on compact subsets of X. The closure $\overline{D}$ of the bounded set D is bounded (see Rudin [3, Theorem 1.13(f)]) and hence also compact by the Heine-Borel property for nuclear spaces Fact 5.3. So the sequence $(x'_n)_{\geq 1}$ is uniformly convergent on $\overline{D}$, and hence also on D, and therefore it is strongly convergent as a sequence in X'. □

7. Continuous Functions

Linear functionals automatically stand a better chance of having continuity properties in locally convex spaces because they map convex sets to convex sets. In this section we look at some examples of nonlinear functions.

Fact 7.1. *If X is an infinite-dimensional topological vector space then the only continuous function on X having compact support is* 0.

Proof. This is because any compact set in X has empty interior, since X, being infinite-dimensional, is not locally compact. □

Fact 7.2. *There is a weakly continuous function S on $\mathcal{H}'$ which satisfies $0 < S \leq 1$, and S equals 1 exactly on $D_{-1}(R)$.*

Proof. Let
$$s_N = \hat{e}_1^2 + \cdots + \hat{e}_N^2,$$
which is clearly weakly continuous. Then
$$\min\{1, e^{R^2 - s_N}\}$$
is also weakly continuous, lies in $(0,1]$, and is equal to 1 if and only if $s_N \geq R^2$. Then each finite sum
$$\sum_{N=1}^{m} \frac{\min\{1, e^{R^2 - s_N}\}}{2^N}$$
is weakly continuous, has values in $(0,1]$, and has maximum possible value $1 - 2^{-m}$ exactly when s_m is $\leq R^2$. Hence the uniform limit
$$S = \sum_{N=1}^{\infty} \frac{\min\{1, e^{R^2 - s_N}\}}{2^N}$$
is a weakly continuous function, with values in $(0,1]$, and is equal to 1 if and only if all s_N are $\leq R^2$, i.e. on $D_{-1}(R)$. □

For the following recall Fact 4.9. We assume that there are vectors $e_1, e_2, ... \in \mathcal{H}$ which form an orthonormal basis of H_0 and, moreover, $\lambda_n^{-p} e_n$ form an orthonormal basis of H_p, for all $p \in \{0, 1, 2, ...\}$, where
$$\lambda_n = \|e_n\|_1.$$
We require a monotonicity and a Hilbert-Schmidt condition
$$1 \leq \lambda_1 < \lambda_2 < \cdots \qquad \text{and} \qquad \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$$

Fact 7.3. *For the cube C in Fact 4.9, with assumptions as stated in Fact 4.9, the function f given on $\mathcal{H}'$ by*
$$f(x) = \begin{cases} \inf_{n \geq 1}\{\lambda_n^n - |x_n|\} & \text{for } x \in C \\ 0 & \text{for } x \notin C \end{cases} \tag{7.1}$$
is sequentially continuous, where $x_n = \langle x, e_n \rangle$.

Proof. We can consider this as two functions $g : C \to \mathbf{R}$ with $g(x) = \inf \lambda_n^n - |x_n|$ and $h : D \to \mathbf{R}$ with $h(x) = 0$ where
$$D = \{x' \in \mathcal{H}' : |\langle x', e_n \rangle| \geq \lambda_n^n \text{ for all } n \in \{1, 2, 3, ...\}\}.$$
Since g and h agree on the boundary (their intersection) proving the continuity of them separately would give us continuity of f.

Case 1: On the cube "boundary."

Suppose $\{x^{(n)}\} \subset \mathcal{H}'$ and $x^{(n)} \to x$ where x is on the boundary of the cube (i.e. $|x_{k_0}| = \lambda_{k_0}^{k_0}$ for some k_0). Since $x^{(n)} \to x$ in $\mathcal{H}'$ we have $x^{(n)} \to x$ in H_{-p} for some $p \in \{0, 1, 2, ...\}$. Hence $|\langle x^{(n)}, e_k\rangle| \to |\langle x, e_k\rangle| = \lambda_k^k$.

Now

$$\begin{aligned} |f(x^{(n)}) - f(x)| &= |f(x^{(n)}) - 0| \\ &= \inf_k \lambda_k^k - |x_k^{(n)}| \\ &\leq \lambda_{k_0}^{k_0} - |x_{k_0}^{(n)}|, \text{for the } k_0 \text{ mentioned above,} \\ &= \lambda_{k_0}^{k_0} - |\langle x^{(n)}, e_{k_0}\rangle| \end{aligned}$$

and the last term goes to 0 as $n \to \infty$.

Case 2: Outside the Cube.

Take $x \notin C$ and $x^{(n)} \to x$ in $\mathcal{H}'$. Then for some N we have for all $n > N$, $x^{(n)}$ is outside the cube. Thus $f(x^{(n)}) = 0$ for all $n > N$ and hence $f(x^{(n)}) \to f(x) = 0$ as $n \to \infty$.

Case 3: Inside the Cube.

Take $x \in C$ and $x^{(n)} \to x$ in $\mathcal{H}'$. Then $x^{(n)} \to x$ in H_{-p} for some p. In particular,

$$M = \sup_n |x^{(n)}|_{-p}^2 < \infty \tag{7.2}$$

Lemma. There exists an $N > 0$ such that for any $y \in H_{-p}$ with $|y|_{-p}^2 \leq M$ we have

$$\inf_{k\geq 1}\{\lambda_k^k - y_k\} = \min\{\lambda_1 - |y_1|, \lambda_2^2 - |y_2|^2, \ldots, \lambda_N^N - |y_N|^N\}. \tag{7.3}$$

Proof. Since $|y|_{-p}^2 \leq M$ we have,

$$|y|_{-p}^2 = \sum_k \lambda_k^{-2p} y_k^2 \leq M$$

and consequently for each k

$$|y_k| \leq \lambda_k^p M.$$

Therefore

$$\lambda_k^k - |y_k| \geq \lambda_k^k - \lambda_k^p M.$$

Since p and M are fixed, there must be an integer N such that for all $k \geq N$

$$\lambda_k^k - |y_k| \geq \lambda_k^k - \lambda_k^p M \geq \lambda_1,$$

because

$$\lambda_k^k - \lambda_k^p M = \lambda_k^p(\lambda_k^{k-p} - M) \to \infty \qquad \text{as } k \to \infty.$$

Hence, the infimim is realized in the first N terms:

$$\inf_k \{\lambda_k^k - |y_k|\} = \min\{\lambda_1 - |y_1|, \lambda_2^2 - |y_2|, \ldots, \lambda_N^N - |y_N|\}.$$

□

Applying this Lemma to $x^{(n)} \to x$ we have

$$\begin{aligned}\lim_{n\to\infty} f(x^{(n)}) &= \lim_{n\to\infty} \min\{\lambda_1 - |x_1^{(n)}|, \lambda_2^2 - |x_2^{(n)}|, \ldots, \lambda_N^N - |x_N^{(n)}|\} \\ &= \min\{\lambda_1 - |x_1|, \lambda_2^2 - |x_2|, \ldots, \lambda_N^N - |x_N|\} \\ &= f(x) \qquad \text{(by the Lemma.)}\end{aligned} \tag{7.4}$$

Thus, f is sequentially continuous. □

References

[1] Becnel, Jeremy J.:*Equivalence of Topologies and Borel Fields for Countably-Hilbert Spaces*, Proceedings of the American Mathematical Society, 134 (2006), pp. 581-590

[2] Gel'fand, I.M., and Shilov, G. E: Generalized Functions. (Transl. M. D. Friedman, A. Feinstein, C. P. Peltzer) Academic Press, Volume 4 (1964) and Volume 2 (1968)

[3] Rudin, Walter : *Functional Analysis*, New York : McGraw-Hill, c1987.

Department of Mathematics and Statistics, Stephen F. Austin State University, Nacogdoches, Texas 75962-3040, e-mail: *becneljj@sfasu.edu*
URL: `http://www.faculty.sfasu.edu/becneljj`

Ambar N. Sengupta, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail address: `sengupta@math.lsu.edu`
URL: `http://www.math.lsu.edu/∼sengupta`