Yang-Mills in two dimensions and Chern-Simons in three

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Abstract. Mathematically rigorous approaches to functional integrals of two dimensional Yang-Mills gauge theory and three-dimensional Chern-Simons theory are described.

1. Introduction

Feynman's path integral method connects time evolution under a quantum Hamiltonian to integration over a space of classical paths with a weighting obtained from the classical theory. The power, mystery and beauty of this method endures in countless instances, and especially so in the context of geometric and topological field theories. The most celebrated expression of a topological quantity in terms of a functional integral is given by Witten's formula [**39**] expressing topological invariants of links in three-dimensional space in terms of functional integrals associated to Chern-Simons field theory. Formulating such integrals and establishing the relationship with topological or geometric quantities rigorously is a challenge for mathematicians. We refer to the books of Mazzucchi [**27**] and Johnson and Lapidus [**20**] for wide-ranging accounts of rigorous methods for studying Feynman path integrals. In this article we shall focus on two field theories, Yang-Mills in two dimensions and Chern-Simons in three dimensions, and review progress in rigorously formulating functional integrals for these theories.

2. Functional Integrals: Real and Imagined

The typical functional integral arising in quantum field theory has the form

$$Z^{-1}\int_{\mathcal{A}} f(A)e^{\beta S(A)}\,DA$$

where $S(\cdot)$ is an action functional, β a physical constant (real or complex), f is some function of the field A of interest, DA signifies 'Lebesgue integration' on an infinite-dimensional space $\mathcal{A}$ of field configurations, and Z a 'normalizing constant'. The reason for putting quotes on 'Lebesgue integration' and 'normalizing constant' is that these objects are more imagined than mathematically realizable. In this

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section we will examine the trouble with these notions and how the trouble can be fixed in the special case of Gaussian measures.

2.1. Infinite product Lebesgue measure. 'Lebesgue measure' in infinite dimensions is a natural, yet seriously troublesome notion. To be specific, suppose we wish to work with a measure l_∞ on the countably infinite product space $\mathbb{R}^{\mathbb{N}}$, where $\mathbb{N} = \{1, 2, ..\}$, which assigns measure $\prod_{j=1}^{\infty}(b_j - a_j)$ to the infinite box $\prod_{j\in\mathbb{N}}[a_j, b_j]$, where $a_j < b_j$ are real numbers for each $j \in \{1, 2, ...\}$. Then the measure of the box $[0, a]^{\mathbb{N}}$ is 0 if $a \in [0, 1)$, and ∞ if $a > 1$. It would be very difficult to do any analysis with this kind of background measure. If we were to focus simply on $[0, 1]^{\mathbb{N}}$ the situation would not be as bad, and indeed the product measure on $[0, 1]^{\mathbb{N}}$ is a useful probability measure. Nonetheless, experience suggests that a measure formally given by a density such as $e^{\beta S(A)}$ times a background measure is not even absolutely continuous with respect to the background measure, when both are rigorously meaningful. All of this is simply a reminder that while working formally with an infinite dimensional Lebesgue measure can be very useful, it is a challenge to make the formal work mathematically rigorous.

2.2. Gaussian measure in infinite dimensions. The first truly useful, and rigorously meaningful, measure in ifninite dimensions is Gaussian measure. Nonlinear quantum field theories generally involve more difficult functional integrals, some of which have been put on rigorous mathematical foundations, as explained in Jonathan Weitsman's lecture. Fortunately, in two interesting field theory models, two dimensional Yang-Mills and three-dimensional Chern-Simons, functional integrals can be worked out meaningfully by Gaussian measure techniques. In the case of Chern-Simons, this is still a developing story, in that the conjectured relationships between the mathematically defined functional integrals and topological invariants still remain to be established completely.

Let's now take a look at Gaussian measure in infinite dimensions. To start with, in a formal sense, this is a measure on an infinite dimensional, separable, real Hilbert space $\mathbb{H}$ given formally by the expression

$$d\mu(x) = \frac{1}{Z}e^{-\|x\|^2/2}Dx,$$

where Dx is 'Lebesgue measure' on $\mathbb{H}$ and Z a 'normalizing constant.'

2.3. A fast construction of Gaussian measure. Choosing an orthonormal basis $u_1, u_2, ...$ in $\mathbb{H}$, and writing x_j for the j-th coordinate $\langle x, u_j\rangle$ of x, the measure μ, viewed on coordinate space $\mathbb{R}^{\mathbb{N}}$, is the product measure

$$\prod_{j\in\mathbb{N}}(2\pi)^{-1/2}e^{-x_j^2/2}dx_j.$$

This is a meaningful measure; for entertainment, let's take a run through one construction. A sequence $(s_n) \in \{0, 1\}^{\mathbb{N}}$ of $0 - 1$ coin tosses encodes a real number $\sum_{n=1}^{\infty} s_n 2^{-n} \in [0, 1]$, and the mapping $\{0, 1\}^{\mathbb{N}} \to [0, 1]$, handled with some care, transfers Lebesgue measure on the unit interval $[0, 1]$ to the coin-tossing product measure $\{0, 1\}^{\mathbb{N}}$ which associates weight 2^{-k} to any subset obtained by specifying the results of k particular tosses, i.e., a subset of the form $\{s \in \{0, 1\}^{\mathbb{N}} : s_{i_1} = j_1, ..., s_{i_k} = j_{i_k}\}$ for any $j_1, ..., j_k \in \{0, 1\}$ and distinct $i_1, ..., i_k \in \mathbb{N}$. In this way, the measure space $\{0, 1\}^{\mathbb{N}}$ is essentially a copy of $[0, 1]$ with Lebesgue measure. The infinite-product $[0, 1]^{\mathbb{N}}$ is viewable as a copy of $\{0, 1\}^{\mathbb{N}\times\mathbb{N}} \simeq \{0, 1\}^{\mathbb{N}}$, by identifying

$\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$. Thus, $[0,1]^{\mathbb{N}} \simeq \{0,1\}^{\mathbb{N}}$, which, again, is viewable as a copy of $[0,1]$ with Lebesgue measure. The measure on $[0,1]^{\mathbb{N}}$, obtained in this way from $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$, is product measure giving weight $\prod_{j\in\mathbb{N}}(b_j - a_j)$ to the box $\prod_{j\in\mathbb{N}}[a_j, b_j]$ if $0 \leq a_j \leq b_j \leq 1$ for all $j \in \mathbb{N}$. Next, we transfer this product measure on $[0,1]^{\mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$, using the Gaussian distribution function $\Phi : \mathbb{R} \to (0,1) : x \mapsto \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-t^2/2}\, dt$ on each coordinate, to obtain the product Gaussian measure μ' on $\mathbb{R}^{\mathbb{N}}$. Each coordinate projection $X_j : \mathbb{R}^{\mathbb{N}} \to \mathbb{R} : (x_n)_{n\in\mathbb{N}} \mapsto x_j$ becomes a Gaussian random variable:

$$\mu'[(x_n) : x_j \in [a,b]] = (2\pi)^{-1/2} \int_a^b e^{-t^2/2}\, dt,$$

with mean 0 and variance 1, i.e., $\int X_j \, d\mu'$ is 0 and $\int X_j^2 \, d\mu'$ is 1. Finally, we transfer the measure to the Hilbert space $\mathbb{H}$, using the map

$$\mathbb{H} \to \mathbb{R}^{\mathbb{N}} : x \mapsto (\langle x, u_j \rangle)_{j\in\mathbb{N}}$$

We face a problem here: the μ' measure of the image of $\mathbb{H}$ is 0. One has to resort to some subterfuge at this point: introduce some sequence of positive reals $c_n \in (0,\infty)$ with $\sum_n c_n^2 < \infty$. Then $\int \sum_n c_n^2 X_n^2 \, d\mu' = \sum_n c_n^2 < \infty$, and so it is sensible to restrict μ' to the subset of $\mathbb{R}^{\mathbb{N}}$ on which $\sum_n c_n^2 X_n^2$ is finite. To this end, consider the new Hilbert space $\mathbb{H}_1$ obtained by completing $\mathbb{H}$ with respect to the norm arising from the inner product which keeps the vectors u_n orthogonal to each other but with new lengths $\|u_n\|_1 = c_n$. An element of $\mathbb{H}_1$ is of the form $\sum_{n=1}^{\infty} x_n u_n$ with $\sum_{n=1}^{\infty} c_n^2 x_n^2 < \infty$. So, in the end, we do have Gaussian measure on an infinite dimensional Hilbert space, except it isn't the original one $\mathbb{H}$ but a slightly larger one $\mathbb{H}_1 \supset \mathbb{H}$.

2.4. Frameworks for measures in infinite dimensions. We have gone through a quick and dirty construction of Gaussian measure in infinite dimensions. There are two very convenient settings for doing analysis in infinite dimensions with a Gaussian background measure: (i) the Abstract Wiener Space formalism of Gross [**14**], and (ii) the setting of nuclear spaces. The construction of non-Gaussian measures in the setting of nuclear spaces is heavily dependent on a result of Minlos which guarantees the existence of a probability measure on such spaces with specified Fourier transform.

3. The Yang-Mills Functional Integral for Two Dimensions

The functional integral for quantum Yang-Mills on the plane $\mathbb{R}^2$ is realized mathematically by means of Gaussian measure. Consider gauge theory over $\mathbb{R}^2$ with gauge group a compact matrix group $G \subset U(N)$, whose Lie algebra $L(G)$ is equipped with the Ad-invariant inner product specified by $\langle a, b\rangle = -\mathrm{Tr}(ab)$. Formally, the functional measure is given by

$$\frac{1}{Z} e^{-\frac{1}{2g^2}\|F^A\|_{L^2}^2} DA$$

where F^A is the curvature of a generic connection $A = A_x dx + A_y dy$, the integration element DA is a formal Lebesgue measure on the infinite dimensional space $\mathcal{A}$ of connections, Z is a formal normalizing constant, and g a physical constant. (See section 5 for a compendium of relevant geometric notions.) More precisely, this measure should be considered on the space of connections modulo gauge transformations. For convenience, it is best to work with $\mathcal{G}_o$, the infinite dimensional group of all gauge transformations which are the identity over the fixed basepoint $o = (0,0) \in \mathbb{R}^2$, and with the quotient $\mathcal{A}/\mathcal{G}_o$. For notational convenience, we take the gauge group G to be a compact matrix group.

For the base manifold $\mathbb{R}^2$, the quotient $\mathcal{A}/\mathcal{G}_0$ can be identified with the infinite dimensional *linear space* $\mathcal{A}_0$ of all connections $A = A_x dx + A_y dy$ for which $A_y = 0$, A_x is 0 along the x-axis, and A vanishes at o; the point is that every connection is gauge transformable to such a special connection in $\mathcal{A}_0$. Arguing formally, the 'Lebesgue measure' DA goes over to the formal Lebesgue measure DA on $\mathcal{A}_0$. The best feature of the setting $\mathcal{A}_0$ is that on $\mathcal{A}_0$ the curvature F^A depends linearly on A:

$$F^A = dA + A \wedge A = -\underbrace{\partial_y A_x}_{f^A}\, dx \wedge dy.$$

The function A_x can be recovered from f^A here by integration, and so f^A itself can be used as a coordinate on $\mathcal{A}_0$. Consequently, the functional measure takes the form

$$\frac{1}{Z} e^{-\frac{1}{2g^2}\|f^A\|_{L^2}}\, df^A.$$

This is, formally, a Gaussian measure, plain and simple. Thus, the functional measure for Yang-Mills on the plane is rigorously meaningful as Gaussian measure for the Hilbert space of $L(G)$-valued L^2-functions f on $\mathbb{R}^2$. From the probabilistic point of view, this is a white-noise with values in the Lie algebra $L(G)$. Planar Yang-Mills theory was developed rigorously in [**15**] and by Driver [**10**], to which we refer for much more as well as bibliography on this topic.

Next, still staying in two dimensions, it is possible to construct the functional measure for the Yang-Mills field over compact surfaces. One approach [**29, 30, 31**] is to take the Gaussian measure described above and *condition* it to satisfy necessary topological constraints. Fine [**11, 12**] developed the Yang-Mills functional integral for compact surfaces. Lévy [**24, 25**] developed the full probabilistic theory for the measure. Once the measure is constructed, the Yang-Mills action itself appears somehow far removed, and one may well ask what precisely is the relationship between the measure and the Yang-Mills action; this is answered by Lévy and Norris [**26**] who show that the large deviation principle for the Yang-Mills measure for compact surfaces is the Yang-Mills action. This parallels a more classical result for Brownian motion, where the large deviation principle is the action/energy of the path. For more insights into the two dimensional Yang-Mills measure we refer to Witten [**37, 38**] and Singer [**36**].

In yet another direction, it is fruitful to inquire into the large-N limit of planar $U(N)$ Yang-Mills functional integrals, specifically Wilson loop expectations. The remarkable fact is that large-N limits, with suitable scaling of the coupling constant g^2, do exist and correspond to a theory which can be described in terms of free probabilty theory. For more on the underlying ideas see Singer [**36**] and the papers [**33, 35**] for more bibliography on this topic.

Let us note very briefly that quantum Yang-Mills theory in three dimensions remains an outstanding challenge. The works by Karabali, Nair et al. [**21, 22**] and Rajeev [**28**] provide ideas for establishing some important properties, such as the existence of a mass gap, in this theory.

4. The Chern-Simons functional integral

We shall work first with Chern-Simons theory over $\mathbb{R}^3$, with gauge group a compact matrix group G whose Lie algebra is denoted $L(G)$. The formal Chern-Simons functional integral has the form

$$\frac{1}{Z}\int_{\mathcal{A}} f(A)e^{i\mathrm{CS}(A)}\,DA$$

where f is a function of interest on the linear space $\mathcal{A}$ of all $L(G)$-valued 1-forms A on $\mathbb{R}^3$, and $\mathrm{CS}(\cdot)$ is the Chern-Simons action given by

$$\mathrm{CS}(A) = \frac{\kappa}{4\pi}\int_{\mathbb{R}^3} \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \tag{4.1}$$

involving a parameter κ. Working on the flat space $\mathbb{R}^3$ allows us to ignore topological issues at this point and focus on the analytic problem of constructing the integral; indeed, the factor of $\kappa/(4\pi)$ could be dropped for this purpose, but we will hold on to it. We will describe a non-perturbative method to the functional integral (perturbative methods, starting with Axelrod and Singer's work [**5, 6**], have been enormously successful).

The trick used for two dimensional Yang-Mills is also effective for the Chern-Simons action: we choose a gauge in which one component of $A = a_0dx_1 + a_1dx_1 + a_2dx_2$ vanishes, say $a_2 = 0$. This makes the triple wedge term $A\wedge A\wedge A$ disappear, and we end up with a quadratic expression

$$\mathrm{CS}(A) = \frac{\kappa}{4\pi}\int_{\mathbb{R}^3} \mathrm{Tr}(A\wedge dA) \qquad \text{for } A = a_0dx_0 + a_1dx_1. \tag{4.2}$$

Then the functional integral has the form

$$\frac{1}{Z}\int_{\mathcal{A}_0} \phi(A)e^{i\frac{\kappa}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}(A\wedge dA)}\,DA \tag{4.3}$$

where $\mathcal{A}_0$ consists of all A for which $a_2 = 0$. (Note that the functions ϕ on $\mathcal{A}$ of ultimate interest are gauge invariant.) As in the two dimensional case, the integration element remains DA after gauge-fixing. Having a quadratic in the exponent in the integrand (4.3) makes us happy, but the factor i throws a spanner in the works: we no longer have a measure here (not even a complex measure).

All is not lost, however. The map

$$\phi \mapsto \langle\phi\rangle_{\mathrm{CS}} = \frac{1}{Z}\int_{\mathcal{A}_0} \phi(A)e^{i\frac{\kappa}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}(A\wedge dA)}\,DA, \tag{4.4}$$

whatever it is rigorously, would be a linear functional on a space of functions ϕ over $\mathcal{A}_o$. As it turns out, it is possible to give rigorous meaning to this functional as a *distribution* on a space of test functions ϕ in infinite dimensions.

Now for $A = a_0dx_0 + a_1dx_1 \in \mathcal{A}_0$, decaying fast enough at infinity, we have, on integrating by parts,

$$\mathrm{CS}(a_0dx_0 + a_1dx_1) = -\frac{\kappa}{2\pi}\int_{\mathbb{R}^3} \mathrm{Tr}(a_0f_1)\,dx_0dx_1dx_2 \tag{4.5}$$

where

$$f_1 = \partial_2 a_1. \tag{4.6}$$

So now the original functional integral is reformulated as an integral of the form

$$\langle\phi\rangle_{CS} = \frac{1}{Z}\int e^{i\frac{\kappa}{2\pi}\langle a_0,f_1\rangle}\phi(a_0,f_1)\,Da_0Df_1 \tag{4.7}$$

where

$$\langle a,f\rangle = -\int_{\mathbb{R}^3}\mathrm{Tr}(af)\,d\mathrm{vol}.$$

and Z always denotes the relevant formal normalizing constant.

Taking ϕ to be of the special form

$$\phi_j(a_0,f_1) = e^{ia_0(j_0)+if_1(j_1)} \tag{4.8}$$

where j_0 and j_1 are, say, rapidly decreasing $L(G)$-valued smooth functions on $\mathbb{R}^3$, we find, from a formal calculation,

$$\langle\phi_j\rangle_{CS} = e^{-i\frac{2\pi}{\kappa}\langle j_0,j_1\rangle}. \tag{4.9}$$

This finally brings us to a point where a rigorous framework can be installed.

Let

$$\mathcal{E} = \{(a,f) : \text{ all smooth rapidly decreasing } a,f:\mathbb{R}^3\to L(G)\} \tag{4.10}$$

with the Schwartz topology and let $\mathcal{E}'$ be the topological dual.

For $z\in\mathcal{E}_{\mathbb{C}}$ (the complexification of $\mathcal{E}$), let

$$(\cdot,z):\mathcal{E}'\to\mathbb{C}:\xi\mapsto\xi(z) \tag{4.11}$$

where we have taken $\xi(z)$ to mean $\xi(x)+i\xi(y)$ if $z=x+iy$ with $x,y\in\mathcal{E}$.

Consider

$$\mathcal{P} = \text{linear span of the functions } e^{(\cdot,z)} \text{ as } z \text{ runs over } \mathcal{E}_{\mathbb{C}}, \tag{4.12}$$

which is an algebra under pointwise operations,

If $\psi\in\mathcal{P}$ is of the form $\psi_z = e^{(\cdot,z)}$ where $z\in\mathcal{E}_{\mathbb{C}}$ is (z_0,z_1), we set

$$\langle e^{(\cdot,z)}\rangle_{CS} = e^{i\frac{2\pi}{\kappa}(z_0,z_1)_0} \tag{4.13}$$

where

$$(z_0,z_1)_0 = \int_{\mathbb{R}^3}\mathrm{Tr}(z_0z_1)\,d\mathrm{vol} \tag{4.14}$$

Then by linearity, and independence of exponentials, $\langle\psi\rangle_{CS}$ is meaningful for all $\psi\in\mathcal{P}$.

So now we have a well-defined linear functional

$$\mathcal{P}\to\mathbb{C}:\psi\mapsto\langle\psi\rangle_{CS} \tag{4.15}$$

It would be a very minimal theory if all we had was the value of $\langle\psi\rangle_{CS}$ for $\psi\in\mathcal{P}$. A continuity property of $\langle\cdot\rangle_{CS}$ needs to be established to extend its scope to a larger class of functions of interest. And to have any notion of continuity we need a topology on a suitable space of functions on the space $\mathcal{E}'$.

Topologies on spaces of 'test functions' on the infinite dimensional space $\mathcal{E}'$ are used in white noise analysis [**23**]. We state here a rough and rapid summary. Let $\mathcal{E}_0$ be the real Hilbert space of $L(G)\oplus L(G)$-valued square-integrable functions on $\mathbb{R}^3$. Sobolev norms are obtained using a suitable differential operator T for which T^{-1} is a Hilbert-Schmidt operator on $\mathcal{E}_0$. Let $\mathcal{E}_p$ be the range $\mathrm{Im}(T^{-p})$, and $\langle u,v\rangle_p = \langle T^pu,T^pv\rangle$. Identifying $\mathcal{E}_0$ with its dual $\mathcal{E}_0^*$, there is a chain of inclusions:

$$\mathcal{E} = \cap_p\mathcal{E}_p\subset\cdots\subset\mathcal{E}_2\subset\mathcal{E}_1\subset\mathcal{E}_0\simeq\mathcal{E}_0^*\subset\mathcal{E}_{-1}\subset\mathcal{E}_{-2}\cdots\subset\mathcal{E}^*\stackrel{\mathrm{def}}{=}\cup_p\mathcal{E}_p \tag{4.16}$$

where $\mathcal{E}_{-p} = \mathcal{E}_p^*$, for $p \in \{0, 1, 2, ...\}$. The operator T is chosen so that the Schwartz space $\mathcal{E}$ is the intersection of the completions $\mathcal{E}_p$. White noise analysis provides inner-products $\langle\langle\cdot,\cdot\rangle\rangle_p$ on $\mathcal{P}$ obtained from the Schwartz/Sobolev inner products $\langle\cdot,\cdot\rangle_p$ on $\mathcal{E}$. These inner-products are obtained quite naturally through identification of $L^2(\mathcal{E}', \mu)$, where μ is standard Gaussian measure for $\mathcal{E}_0$ realized on $\mathcal{E}'$, with the symmetric Fock space over $\mathcal{E}_0$. Leaving details aside, the inner-product $\langle\langle\cdot,\cdot\rangle\rangle_p$ on $\mathcal{P}$ works out to

$$\langle\langle e^{(\cdot,z_1)}, e^{(\cdot,z_2)}\rangle\rangle_p = e^{\langle z_1,z_2\rangle_p + \frac{1}{2}(z_1,z_1)_0 + \frac{1}{2}(\overline{z}_2,\overline{z}_2)_0} \tag{4.17}$$

where $(\cdot,\cdot)_0$ is the complex bilinear extension of the L^2-inner-product on $\mathcal{E} = \mathcal{S}(\mathbb{R}^3, L(G)) \oplus \mathcal{S}(\mathbb{R}^3, L(G))$. The completion $[\mathcal{E}]_p$ of $\mathcal{P}$ under the norm $\|\cdot\|_p$ is a space of functions on $\mathcal{E}'$, and the intersection $[\mathcal{E}] = \cap_{p\geq 0}[\mathcal{E}]_p$ provides a good test function space over $\mathcal{E}'$. There is then also the dual space $[\mathcal{E}]^*$ of all continuous linear functionals on $[\mathcal{E}]$; these are *distributions* over the infinite dimensional space $\mathcal{E}'$.

In [**3**] it is shown that there is a continuous linear functional

$$[\mathcal{E}] \to \mathbb{C} : \psi \mapsto \langle\psi\rangle_{\mathrm{CS}} \tag{4.18}$$

which agrees with $\langle\psi\rangle_{CS}$, as defined in (4.13), for $\psi \in \mathcal{P}$. Thus, (4.18) provides a rigorous realization for the Chern-Simons functional integral with axial gauge fixing.

In the case of abelian gauge groups the functional integral was developed rigorously and studied in detail by Albeverio and Schäfer [**1, 2**].

What is disappointing about (4.18) is that it appears hardly likely that holonomy (Wilson loop) variables are in the test function space $[\mathcal{E}]$. A smearing process was developed by Hahn [**4, 17**] to define the functional integral on Wilson loop variables. The smearing of a loop into a thickened tube requires making certain directional choices of the smearing; this encodes a *framing* of the loop.

Everything we have done depends very crucially on the possibility of a gauge choice which made the Chern-Simons action become a quadratic expression in the connection. For manifolds such as $\Sigma \times S^1$, where Σ is a compact surface, no such gauge choice is possible. However, instead of setting one directional component of the connection form A to 0, we can choose this component to fall in the Lie algebra of a maximal torus in the gauge group. This leads to a considerable simplification in the functional integral. This method of torus-gauge fixing has been developed by Hahn [**18, 19**] and appears promising for a rigorous formulation of the Chern-Simons functional integral for compact surfaces. The method of torus-gauge fixing goes back to the work of Blau and Thompson [**7**].

The work of Freed [**13**] describes the geometry of the Chern-Simons functional as well as the functional integral in relation to other topological and geometric field theories, including two dimensional Yang-Mills theory. For this see also [**34**]. The book of Guadagnini [**16**] gives an overview of many aspects of the Chern-Simons functional integral and its relationship with topology.

5. A summary of geometric notions

For ease of consultation, we summarize some standard notions and notation from differential geometry here (for more see Bleecker [**8**]).

Let M be a smooth manifold of dimension $n \geq 1$, and G a compact Lie group with Lie algebra $L(G)$. A principal G-bundle over M is a smooth manifold P, on which there is a smooth right action $P \times G \to P : (p, g) \mapsto R_g p = pg$ of G, along with a smooth surjection $\pi : P \to M$ with the local triviality property: every $m \in M$ has a neighborhood U such that there is a diffeomorphism $\phi : U \times G \to \pi^{-1}(U)$ satisfying $\pi\phi(u, g) = u$ and $\phi(u, g)h = \phi(u, gh)$ for all $u \in U$ and $g, h \in G$. The subset $\pi^{-1}(m)$ is the *fiber* over the point m. A *gauge transformation* is a diffeomorphism $f : P \to P$ which maps each fiber into itself, i.e. $\pi \circ f = \pi$, and is equivariant under the action of G on P. For $v \in T_pP$ and $g \in G$ we denote by vg the vector $(R_g)_* v \in T_{pg}P$. The subspace $V_p = \ker d\pi_p \subset T_pP$ is called the vertical subspace at p, and a vector in V_p is called *vertical.* The map

$$L(G) \to T_pP : H \mapsto pH = \frac{d}{dt}\Big|_{t=0} p \exp(tH)$$

is a linear isomorphism of $L(G)$ onto V_p.

A *connection* on the principal bundle $\pi : P \to M$ is an $L(G)$-valued 1-form ω on P for which: (i) $\omega(vg) = \mathrm{Ad}(g^{-1})\omega(v)$ for all $g \in G$ and $v \in TP$, and (ii) $\omega(pH) = H$ for all $p \in P$ and $H \in L(G)$.

A k-form η on P with values in a vector space W on which there is a left-action ρ of G is said to be *ρ-equivariant* if

$$\eta(v_1 g, ..., v_k g) = \rho(g^{-1})\eta(v_1, ..., v_k)$$

for all $v_1, ..., v_k \in T_pP$, all $p \in P$, and all $g \in G$ (when $k = 0$ read this as $\eta(pg) = \rho(g^{-1})\eta(p)$ for all $p \in P$ and $g \in G$). For example, a connection form is Ad-equivariant. Many forms of interest are equivariant and also vanish on vertical vectors. The *curvature* of a connection ω is the LG-valued 2-form Ω^ω given by

$$\Omega^\omega(v, w) = d\omega(v, w) + [\omega(v), \omega(w)] \tag{5.1}$$

for all $v, w \in T_pP$ and $p \in P$. This form is Ad-equivariant and vanishes on (v, w) if v or w is vertical.

The subspace

$$H_p^\omega = \ker \omega_p \tag{5.2}$$

is called the *horizontal* subspace for the connection ω at p, and a vector in H_p^ω is called *ω-horizontal.*

A piecewise differentiable path in P is said to be *horizontal* if its tangent vector, wherever defined, is ω-horizontal. A horizontal path $\tilde{c} : [a, b] \to P$ is said to be a horizontal *lift* of a C^1 path $c : [a, b] \to M$ if $c = \pi \circ \tilde{c}$; a unique C^1 horizontal lift exists for each choice of initial point $\tilde{c}(a)$ on the fiber over $c(a)$. If c is a loop, and $\tilde{c}_u$ the horizontal lift with initial point u, then the unique $h \in G$ for which the end point of $\tilde{c}$ is uh is called the *holonomy* $h_u(c; \omega)$ of ω around c with initial point u. Horizontal paths remain horizontal under the right action of G, and this implies that $h_{ug}(c; \omega)$ is $g^{-1}h_u(c; \omega)g$. Now suppose M has a metric, and suppose also that the Lie algebra $L(G)$ has a metric $\langle \cdot, \cdot \rangle_{L(G)}$ which is invariant under the adjoint action of G on $L(G)$. Then the function $\langle \Omega^\omega, \Omega^\omega \rangle$ is the function on M whose value at any point $m \in M$ is $\sum_{1 \leq j < k \leq n} \langle \Omega^\omega(e_j, e_k), \Omega^\omega(e_j, e_k) \rangle_{LG}$ where $e_1, ..., e_n \in T_pP$, with $p \in \pi^{-1}(m)$, project to an orthonormal basis of T_mM; this value is determined by ω, the point m, and the metrics on T_mM and $L(G)$, and is independent of the choice of p in the fiber $\pi^{-1}(m)$ and the basis $\{e_i\}$. The *Yang-Mills action* of a connection ω is given by

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2g^2}\int_M \langle \Omega^\omega, \Omega^\omega\rangle \, d\mathrm{vol} \tag{5.3}$$

where the integration is with respect to the volume measure induced by the metric on M, and the parameter g is a physical coupling constant. A *section* of the bundle $\pi : P \to M$ is a smooth map $s : M \to P$ such that $\pi(s(m)) = m$ for all $m \in M$. Such an s exists if $M = \mathbb{R}^2$, and then it is easier to work with $A = s^*\omega$, which an LG-valued 1-form on $\mathbb{R}^2$. The curvature pulls down to the LG-valued 2-form $F^A = s^*\Omega^\omega$ whose relation to A is conveniently expressed, with G a group of matrices, by $F^A = dA + A \wedge A$

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