Pathspace Connections and Categorical Geometry

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Abstract

We develop a new differential geometric structure using category theoretic tools that provides a powerful framework for studying bundles over path spaces. We study a type of connection forms, given by Chen integrals, over path spaces by placing such forms within a category-theoretic framework of principal bundles and connections. A new notion of 'decorated' principal bundles is introduced, along with parallel transport for such bundles, and specific examples in the context of path spaces are developed.

1 Introduction

In this paper we develop a theory of categorical geometry and explore specific examples involving geometry over spaces of paths. Our first objective is to develop a framework that encodes special properties, such as parametrization-invariance, of connection forms on path spaces. For this paper we focus on the case of connections over path spaces given by first-order Chen integrals. We then develop a theory of 'decorated' principal bundles and parallel-transport in such bundles. These constructions all sit naturally inside a framework of categorical connections that we develop. A background motivation is to develop a framework that provides a unified setting for both ordinary gauge theory, governing interactions between point particles, and higher gauge theory, governing the interaction of string-like, or higher-dimensional, extended objects.

There is a considerable current literature (as we cite in a paragraph below) combining geometric ideas and category theoretic structures. However, our work offers several new ideas; these include:

- our formulation of the notion of a *categorical connection*, closer in spirit to the traditional notion of parallel-transport but more general than the formulations that are in use in the existing literature;
- a new notion of *decorated bundles* that provides a natural and rich class of examples of categorical bundles.

Our development of a powerful framework of categorical principal bundles is different from other works, with the action of the categorical structure group playing a more explicit direct role, analogous to the case of classical principal bundles.

Our constructions are not developed for the sake of abstract constructions, but rather as a natural framework that expresses the essential elements of a variety of examples. These examples include connections on bundles over path spaces as well as a new but natural notion of 'decorated' bundles that we introduce. The study of such bundles is motivated by gauge theories in which points in a bundle are replaced by paths, on the bundle, decorated by elements of a group. Our theory of categorical connections then provides a natural framework for formulating the notion of parallel-transport of such decorated paths. Decorated principal bundles provide a framework for generating hierarchies of examples of categorical bundles and connections over higher dimensional (that is, iterations of) path spaces.

We now summarize our results through an overview of the paper. All categories we work with are 'small': the objects and morphisms form sets.

- In section 2 we study a connection form $\omega_{(A,B)}$ on a principal bundle over a path space. This connection form is invariant under reparametrization of the paths. We then describe the connection form $\omega_{(A,B)}$:

$$\omega_{(A,B)}(\tilde{v}) = A\big(\tilde{v}(t_0)\big) + \tau \left[\int_{t_0}^{t_1} B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right) \, dt \right], \tag{1.1}$$

  specifying the meaning of all the terms involved here. Briefly and roughly put, $\omega_{(A,B)}$ is a 1-form on path space obtained by a 'point-evaluation' of a traditional 1-form (the first term on the right) and a first-order Chen integral (the second term on the right)). Next in Proposition 2.2 we prove that this connection form is also invariant under reparametrization of paths, and so induces a connection form on the space of reparametrization-equivalence classes of paths. In Proposition 2.2 we show that $\omega_{(A,B)}$ does have properties analogous to traditional connection forms on bundles.

- In section 3 we consider equivalence classes of paths, identifying paths that differ from each other by erasure of backtracked segments (that is, a composite path $c_2\overline{a}ac_1$, where $\overline{a}$ is the reverse of a, is considered equivalent to c_2c_1). The results of section 3, especially Theorem 3.1, show that $\omega_{(A,B)}$ specifies a connection form on the space of backtrack-erasure equivalence classes of paths. This is the result that links the geometry with category theory: it makes it possible to view $\omega_{(A,B)}$ as specifying a connection form over a category whose morphisms are backtrack-erased paths. In Theorem 3.2 we prove that $\omega_{(A,B)}$ respects another common way of identifying paths: paths γ_1 and γ_2 are said to be 'thin homotopic' if one can obtain γ_2 from γ_1 by means of a homotopy that 'wiggles' γ_1 along itself. Thus, $\omega_{(A,B)}$ specifies a connection also over the space of thin-homotopy equivalence classes of paths.

- In section 4 we study the notion of a *categorical group*; briefly, this is a category whose object set and morphism set are both groups. The main result, Theorem 4.1 establishes the equivalence between categorical groups and *crossed modules* specified by pairs of groups (G, H). These results are known in the literature but we feel it is useful to present this coherent account, as there are many different conventions and definitions used in the literature and our presentation in this section provides us with notation, conventions, and results for use in later sections. We also include several examples here.

- Section 5 introduces the key notion of a *principal categorical bundle* $\mathbf{P} \to \mathbf{B}$, with a categorical group $\mathbf{G}$ as 'structure group ' and with both $\mathbf{P}$ and $\mathbf{B}$ being categories. This general framework does not require $\mathbf{B}$ to have a smooth structure. We give examples, including one that uses backtrack-erased path spaces. We conclude the section by showing that the notion of 'reduction' of a principal bundle carries over to this categorical setting. (In this work we do not explore categorical analogs of local triviality, a topic that is central to most other works in this area.)

- In section 6 we introduce the notion of a *decorated* principal bundle. This gives a useful example of a categorical principal bundle whose structure depends on the action of a given crossed module. Briefly, we start with an ordinary principal bundle $\pi : P \to B$ equipped with a connection $\overline{A}$; we form a categorical principal bundle whose base category $\mathbf{B}$ has object set B and backtrack-erased paths on B as morphisms; the bundle category $\mathbf{P}$ has object set P and morphisms of the form $(\tilde{\gamma}, h)$, with $\tilde{\gamma}$ being any $\overline{A}$-horizontal path on P and h running over a group H. Thus the morphisms are horizontal paths 'decorated' with elements of H. The result is a categorical principal bundle whose structure categorical group is specified by the pair of groups G and H, with G acting on objects of $\mathbf{P}$ and a semi-direct product of G and H acting on the morphisms of $\mathbf{P}$.

- In section 7 we introduce the notion of a *categorical connection* on a categorical principal bundle. We present several examples, and then show, in Theorem 7.1, how to construct a categorical connection on the bundle of decorated paths and then, in Theorem 7.2, in a more abstractly decorated categorical bundle.

- In section 8 we describe, in a precise way, categories whose morphisms are (equivalence classes of) paths. We do not use the popular practice of identifying thin-homotopy equivalent paths and explain how our approach provides a very convenient framework in which to formulate properties of parallel-transport such as invariance under reparametrizations, backtracks and thin-homotopies (Proposition 2.2, Theorem 3.1, Theorem 3.2).

- In section 9 we construct categorical connections at a 'higher' geometric level: here the objects are paths, and the morphisms are paths of paths.

- We present our final and most comprehensive example of a categorical connection in section10, where we develop parallel-transport of 'decorated' paths over a space of paths. Thus the transport of a decorated path $(\tilde{\gamma}, h)$ is specified through the data $(\tilde{\Gamma}, h, k)$, where $\tilde{\Gamma}$ is a path of paths on the bundle, horizontal with respect to a path space connection $\omega_{(A,B)}$, $h \in H$ decorates the initial (or source) path $\tilde{\gamma}_0 = s(\tilde{\Gamma})$ for $\tilde{\Gamma}$, and $k \in K$ encodes the rule for producing the resulting final decorated path $(\tilde{\gamma}_1, h_1)$.

- Section 11 presents a brief account of associated bundles, along with parallel-transport in such bundles, in the categorical framework.

There is a considerable and active literature at the juncture of category theory and geometry. Without attempting to review the existing literature we make some remarks. The relationship between higher categorical structures and quantum theories was explored extensively by Freed [14]. Works broadly related to ours include those of Baez et al. [3, 4, 5, 6, 7], Bartels [10], Martins and Picken [20, 21, 11], Schreiber and Waldorf [24, 25], Abbaspour and Wagemann [1], Viennot [26], and Wockel [28]. We study neither local trivialization of 2-bundles nor the related notion of gerbes, both of which are explored in the other works just cited. Our definition of categorical connection in section 7 appears to be new. Other notions, such as 2-connections [6] may be found in the literature. Roughly speaking, our approach stays much closer to geometry than category theory in comparison to much of the 2-geometry literature; this is because our primary objective is to create a framework for the specific type of connections given by (1.1).

The work of Martins and Picken [20] gives an account of different notions of categorical connections as well as a precise examination of the role of 'thin-homotopy equivalence'. They explain how a traditional connection 1-form A

and a 2-form B (with values in a Lie group H) on a principal G-bundle $\pi : P \to M$, with suitable equivariance properties, and which are related to each other through a 'flatness' condition give rise to a 'categorical holonomy'. Our construction of categorical connections is in terms of parallel-transport rather than holonomies and we explore several specific and new examples of categorical connections, some of which (as explained in the decorated bundle constructions in section 10) involve connection forms A, $\overline{A}$, as well as 1-forms C_1 and C_2 with values in Lie algebras associated to higher crossed modules.

Before proceeding, let us briefly review some language connected with the standard framework of connections on principal bundles for ease of reference when considering the categorical definitions we introduce later.

For a Lie group G, a principal G-bundle $\pi : P \to B$ is a smooth surjection, where P and B are smooth manifolds, along with a free right action of G on P:

$$P \times G \to P : (p, g) \mapsto R_g(p) = pg,$$

with $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$, and there is local triviality: every point of B has a neighborhood U and there is a diffeomorphism

$$\phi : U \times G \to \pi^{-1}(U)$$

such that $\phi(u, g)h = \phi(u, gh)$ and $\pi\phi(u, g) = u$ for all $u \in U$ and $g, h \in G$. In the categorical formulation we will develop in section 5 we will not (in the present paper) impose local triviality.

A vector $v \in T_pP$ is *vertical* if $d\pi_p(v) = 0$. A connection ω is a 1-form on P with values in the Lie algebra $L(G)$, and satisfies the following conditions: (i) $\omega(R'_g(p)X) = \mathrm{Ad}(g^{-1})\omega(X)$ for all $p \in P$, $X \in T_p$ and $g \in G$, and (ii) $\omega(\tilde{Y}(p)) = Y$ for all $p \in P$ and $Y \in L(G)$, where $\tilde{Y}(p) = \frac{d}{dt}|_{t=0} pe^{tY}$. The crucial consequence of this definition is that ω decomposes each T_pP as a direct sum of the vertical subspace $\ker d\pi_p$ and the *horizontal subspace* $\ker \omega_p$ in a manner 'consistent' with the action of G. This leads to the notion of *parallel transport*: if $\gamma : [t_0, t_1] \to B$ is a C^1 path and u_0 is a point on the fiber $\pi^{-1}\big(\gamma(t_0)\big)$ then there is a unique *horizontal lift* $\tilde{\gamma}_\omega : [t_0, t_1] \to P$, that is a C^1 path for which

$$\omega\left(\tilde{\gamma}'_\omega(t)\right) = 0 \qquad \text{for all } t \in [t_0, t_1], \tag{1.2}$$

that initiates at u_0. It is in terms of the notion of parallel transport that we state the definition of a categorical connection on a categorical bundle in section 7. The *curvature* 2-form F^ω, containing information on infinitesimal holonomies, is defined to be

$$F^\omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{1.3}$$

where, for any $L(G)$-valued 1-forms η and ζ, the 2-form $[\eta, \zeta]$ is given by

$$[\eta, \zeta](X, Y) = [\eta(X), \zeta(Y)] - [\eta(Y), \zeta(X)]$$

for all $X, Y \in T_pP$ and all $p \in P$.

2 A connection form over a path space

In this section we prove certain invariance properties of certain connection forms over spaces of paths. Our results are proved for piecewise C^1 paths, but for the sake of consistency with results in later sections we frame the discussions (as opposed to the formal statements and proofs) in terms of C^∞ paths.

For a manifold X the set of all parametrized paths on X is

$$\bigcup_{t_0,t_1\in\mathbb{R},t_0<t_1} C^\infty\left([t_0,t_1];X\right), \tag{2.1}$$

and we denote by $\mathcal{P}X$ the quotient set obtained by identifying paths that differ by a constant time-translation reparametrization. Thus, $\gamma:[t_0,t_1]\to X$ is identified with $\gamma_{+a}:[t_0-a,t_1-a]\to X: t\mapsto \gamma(t+a)$ in $\mathcal{P}X$. We will often not make a notational distinction between γ and its equivalence class $[\gamma]$ of such time-translation reparametrizations, and in using the term 'parametrized path' we will often not distinguish between time-translation reparametrizations of the same path.

We work with a connection $\overline{A}$ on a principal G-bundle

$$\pi: P\to M,$$

where G is a Lie group. We denote by

$$\mathcal{P}_{\overline{A}}P, \tag{2.2}$$

the set of all $\overline{A}$-horizontal C^∞ paths $\gamma:[t_0,t_1]\to P$, again identifying paths that differ by a constant time-translation. (In [12] we used this notation but without any identification procedure.) For a given connection $\overline{A}$ we view $\mathcal{P}_{\overline{A}}P$ in a natural but informal way as a principal G-bundle over $\mathcal{P}M$; the projection $\mathcal{P}_{\overline{A}}P\to\mathcal{P}M$ is the natural one induced by $\pi:P\to M$, and the right action of G on $\mathcal{P}_{\overline{A}}P$ is also given simply from the action of G on P.

Consider a path of $\overline{A}$-horizontal paths, specified through a C^∞map

$$\tilde{\Gamma}:[s_0,s_1]\times[t_0,t_1]\to P:(s,t)\mapsto\tilde{\Gamma}(s,t)=\tilde{\Gamma}_s(t),$$

where $s_0<s_1$ and $t_0<t_1$, such that $\tilde{\Gamma}_s$ is $\overline{A}$-horizontal for every $s\in[s_0,s_1]$. In [12, 2.1] it was shown that

$$\overline{A}(\tilde{v}(t))=\overline{A}(\tilde{v}(t_0))+\int_{t_0}^{t} F^{\overline{A}}\left(\tilde{\gamma}'(s),\tilde{v}(s)\right)\,ds \qquad \text{for all } t\in[t_0,t_1]. \tag{2.3}$$

where $\tilde{v}:[t_0,t_1]\to TP: t\mapsto\partial_s\tilde{\Gamma}(s_0,t)$ is the vector field along $\tilde{\gamma}=\tilde{\Gamma}_{s_0}$ pointing in the variational direction. We refer to (2.3) as a *tangency condition.*

Note that (2.3) is meaningful even if the path $\tilde{\gamma}$ is piecewise C^1 and the vector field $\tilde{v}$ merely continuous. Condition (2.3) is equivalent to the requirement that

$$\frac{\partial\overline{A}(\tilde{v}(t))}{\partial t}=F^{\overline{A}}\left(\tilde{\gamma}'(t),\tilde{v}(t)\right) \tag{2.4}$$

hold at every point where $\tilde{\gamma}'(t)$ exists (which is all $t \in [t_0, t_1]$ except for finitely many points).

In the following result we show that the tangency condition (2.3) is independent of parametrizations, in the sense that if $\tilde{v}$ and $\tilde{\gamma}$ satisfy (2.3) then any reparametrization of $\tilde{\gamma}$ along with the corresponding reparametrization of $\tilde{v}$ also satisfy (2.3). We denote by

$$T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P. \tag{2.5}$$

the vector space of all vector fields $\tilde{v}$ along $\tilde{\gamma}$ satisfying (2.3), with time-translates being identified.

Proposition 2.1 *Let $\overline{A}$ be a connection on a principal G-bundle, and let $\tilde{v}$ be a continuous vector field along an $\overline{A}$-horizontal piecewise C^1 path $\tilde{\gamma} : [t_0, t_1] \to P$ satisfying (2.3). Then:*

(i) *for any strictly increasing piecewise C^1 bijective reparametrization function $\phi : [t'_0, t'_1] \to [t_0, t_1]$, the vector field $\tilde{v} \circ \phi$, along the path $\tilde{\gamma} \circ \phi$ satisfies (2.3) with the domain $[t_0, t_1]$ replaced by $[t'_0, t'_1]$;*

(ii) *if $\pi \circ \tilde{\gamma}$ is constant on an open subinterval of $[t_0, t_1]$ then $\tilde{\gamma}$ is constant on that same subinterval, and if the vector field $v = \pi_* \tilde{v}$ is constant on that subinterval then so is $\tilde{v}$.*

Part (ii) stresses how strongly the behavior of the path $\pi \circ \tilde{\gamma}$ controls the behavior of a vector field $\tilde{v}$ that satisfies the tangency condition (2.3).

Proof. (i) Since ϕ is a strictly increasing bijection $[t'_0, t'_1] \to [t_0, t_1]$ we have, in particular, $\phi(t'_0) = t_0$. From (2.3) we have:

$$\begin{aligned}
\overline{A}\left(\tilde{v} \circ \phi(t')\right) &= \overline{A}\left(\tilde{v}\left(t_0\right)\right) + \int_{t_0}^{\phi(t')} F^{\overline{A}}\left(\tilde{\gamma}'(u), \tilde{v}(u)\right)\, du \\
&= \overline{A}\left(\tilde{v}(t_0)\right) + \int_{t'_0}^{t'} F^{\overline{A}}\left(\tilde{\gamma}'\big(\phi(s)\big), \tilde{v}\big(\phi(s)\big)\right)\, \phi'(s) ds \quad (\text{setting } u = \phi(s)) \\
&= \overline{A}\left(\tilde{v}\big(\phi(t'_0)\big)\right) + \int_{t'_0}^{t'} F^{\overline{A}}\left((\tilde{\gamma} \circ \phi)'(s), \tilde{v} \circ \phi(s)\right)\, ds.
\end{aligned} \tag{2.6}$$

In the second line above the change of variables $u = \phi(s)$ can be done interval by interval on each of which ϕ has a continuous derivative.

(ii) Now suppose $\gamma = \pi \circ \tilde{\gamma}$ is constant on an open interval $(T, T+\delta) \subset [t_0, t_1]$. Then, for any $t \in (T, T+\delta)$ the 'horizontal projection' $\pi_* \tilde{\gamma}'(t)$ is 0 and the 'vertical part' $\overline{A}\big(\tilde{\gamma}'(t)\big)$ is also 0 because $\tilde{\gamma}$ is $\overline{A}$-horizontal. Hence $\tilde{\gamma}'(t) = 0$ for all such t and hence $\tilde{\gamma}$ is constant on this interval. Next suppose $v = \pi_* \tilde{v}$ is constant over $(T, T+\delta)$. Then from (2.3) we have

$$\overline{A}\big(\tilde{v}(t)\big) - \overline{A}\big(\tilde{v}(T)\big) = \int_T^t F^{\overline{A}}\left(\tilde{\gamma}'(s), \tilde{v}(s)\right)\, ds = 0 \tag{2.7}$$

for all $t \in (T, T+\delta)$ because of the constancy of $\tilde{\gamma}$. Thus,

$$\overline{A}\big(\tilde{v}(t)\big) = \overline{A}\big(\tilde{v}(T)\big)$$

for all $t \in (T, T+\delta)$. Thus, since the horizontal parts of $\tilde{v}(t)$ and $\tilde{v}(T)$ are also assumed to be equal (by constancy of v over $(T, T+\delta)$) it follows that $\tilde{v}(t)$ equals $\tilde{v}(T)$ for all $t \in (T, T+\delta)$. $\boxed{\text{QED}}$

We will study connections on the bundle $\mathcal{P}_{\overline{A}}P$, and on related bundles, using crossed modules: a *crossed module* (G, H, α, τ) consist of groups G and H, along with homomorphisms

$$\tau : H \to G \qquad \text{and} \qquad \alpha : G \to \mathrm{Aut}(H). \tag{2.8}$$

satisfying the identities:

$$\begin{aligned} \tau\big(\alpha(g)h\big) &= g\tau(h)g^{-1} \\ \alpha\big(\tau(h)\big)(h') &= hh'h^{-1} \end{aligned} \tag{2.9}$$

for all $g \in G$ and $h \in H$.

The structure of a crossed module was introduced in 1949 by J. H. C. Whitehead [27, sec 2.9], foreshadowed in the dissertation of Peiffer [22] (sometimes misspelled as 'Pfeiffer' in the literature).

When working in the category of Lie groups, we require that τ be smooth, and the map

$$G \times H \to H : (g, h) \mapsto \alpha(g, h) = \alpha(g)h \tag{2.10}$$

be smooth. In this case we call (G, H, α, τ) a *Lie crossed module*.

In what follows we work with a Lie crossed module (G, H, α, τ), connection forms A and $\overline{A}$ on a principal G-bundle $\pi : P \to M$. We denote the right action of G on the bundle space P by

$$P \times G \to P : (p, g) \mapsto pg = R_g p$$

and the corresponding derivative map by

$$T_pP \to T_{pg}P : v \mapsto vg \stackrel{\text{def}}{=} dR_g|_p(v)$$

for all $p \in P$ and $g \in G$. We also work with an $L(H)$-valued 2-form B on P which is α-equivariant, in the sense that

$$B_{pg}(vg, wg) = \alpha(g^{-1})B_p(v, w) \qquad \text{for all } g \in G,\ p \in P, \text{ and } v, w \in T_pP, \tag{2.11}$$

and vanishes on vertical vectors in the sense that

$$B(v, w) = 0 \quad \text{whenever } v, w \in T_p \text{ and } \pi_* v = 0, \tag{2.12}$$

for any $p \in P$.

For a piecewise C^1 $\overline{A}$-horizontal path $\tilde{\gamma} : [t_0, t_1] \to P$ and any continuous vector field $\tilde{v}$ along $\tilde{\gamma}$ we define

$$\omega_{(A,B)}(\tilde{v}) = A\big(\tilde{v}(t_0)\big) + \tau\left[\int_{t_0}^{t_1} B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right) dt\right]. \tag{2.13}$$

(In [12] we used $\tilde{v}(t_1)$ instead of $\tilde{v}(t_0)$ in the definition of $\omega_{(A,B)}$, which is equivalent to changing B to $B - F^{\overline{A}}$ here.) We view $\omega_{(A,B)}$ as a 1-form on a space of paths on P, expressed as

$$\omega_{(A,B)} = \mathrm{ev}_{t_0}^* A + \tau(Z) \tag{2.14}$$

where ev_t is the evaluation map

$$\mathrm{ev}_t : \mathcal{P}_{\overline{A}} P \to P : \tilde{\gamma} \mapsto \tilde{\gamma}(t)$$

and Z is the $L(H)$-valued 1-form on $\mathcal{P}_{\overline{A}} P$ given by the Chen integral

$$Z = \int_{\tilde{\gamma}} B. \tag{2.15}$$

The preceding definitions, but with ev_{t_0} replaced by ev_{t_1}, were introduced in our earlier work [12].

We note that $\omega_{(A,B)}$ is determined by $\tau(B)$, rather than by B directly. Later, in section 7, we will consider a richer structure where B itself, rather than $\tau(B)$, can play a role. (See for example the remark at the end of section 10 and after Theorem 7.1.)

The first observation for $\omega_{(A,B)}$ is that it remains invariant when paths are reparametrized; thus, it can be viewed as a 1-form defined on the space of equivalence classes of paths, with the equivalence relation being reparametrization:

Proposition 2.2 *Let (G, H, α, τ) be a Lie crossed module, and $\overline{A}$ be a connection on a principal G-bundle $\pi : P \to M$. Let $\tilde{\gamma} : [t_0, t_1] \to P$ be a piecewise C^1 $\overline{A}$-horizontal path in P, and $\tilde{v} : [t_0, t_1] \to P$ be a continuous vector field along $\tilde{\gamma}$ that satisfies the constraint (2.3). Let A be a connection form on $\pi : P \to M$, B an $L(H)$-valued α-equivariant 2-form on P vanishing on vertical vectors, and $\omega_{(A,B)}$ be as in (2.13). Then*

$$\omega_{(A,B)}(\tilde{v}) = \omega_{(A,B)}(\tilde{v} \circ \phi) \tag{2.16}$$

for any strictly increasing piecewise C^1 bijective function $\phi : [t_0', t_1'] \to [t_0, t_1]$, where on the left $\tilde{v}$ is along $\tilde{\gamma}$ and on the right $\tilde{v} \circ \phi$ is along $\tilde{\gamma} \circ \phi$.

The significance of this result is that it implies that $\omega_{(A,B)}$ can be viewed as an $L(G)$-valued 1-form on the bundle space $\mathcal{P}_{\overline{A}} P$; in fact, though, for convenience, we defined $\mathcal{P}_{\overline{A}} P$ using only constant-time translations, $\omega_{(A,B)}$ would be defined on a space of paths, with paths that are reparametrizations of each other by increasing piecewise C^1 bijective functions.

Proof. The proof is by change-of-variables along the lines of the proof of Proposition 2.1 (i):

$$\begin{aligned}
\omega_{(A,B)}(\tilde v\circ\phi) &= A\big((\tilde v\circ\phi)(t_0')\big)\big) + \tau\left[\int_{t_0'}^{t_1'} B\left(\big(\tilde\gamma\circ\phi\big)'(u), (\tilde v\circ\phi)(u)\right)\,du\right]\\
&= A\big(\tilde v(t_0)\big) + \tau\left[\int_{\phi(t_0)}^{\phi(t_1)} B\left(\phi'(u)\tilde\gamma'\big(\phi(u)\big), \tilde v\big(\phi(u)\big)\right)\,du\right]\\
&= A\big(\tilde v(t_0)\big) + \tau\left[\int_{\phi(t_0)}^{\phi(t_1)} B\left(\tilde\gamma'\big(\phi(u)\big), \tilde v\big(\phi(u)\big)\right)\,\phi'(u)du\right]\\
&= A\big(\tilde v(t_0)\big) + \tau\left[\int_{t_0}^{t_1} B\left(\tilde\gamma'(t), \tilde v\right)\,dt\right] \quad \text{(switching to } t=\phi(u))\\
&= \omega_{(A,B)}(\tilde v).
\end{aligned} \tag{2.17}$$

QED

Next we check that $\omega_{(A,B)}$ has the properties of a connection form on a principal G-bundle.

Proposition 2.3 *Let (G,H,α,τ) be a Lie crossed module, $\overline{A}$ and A be connection forms on a principal G-bundle $\pi: P\to M$. Let B be an $L(H)$-valued α-equivariant 2-form on P vanishing on vertical vectors, and $\omega_{(A,B)}$ be as in (2.13). Then*

$$\omega_{(A,B)}(\tilde v g) = \mathrm{Ad}(g^{-1})\omega_{(A,B)}(\tilde v) \tag{2.18}$$

for all $g\in G$ and all continuous vector fields $\tilde v$ along any $\overline{A}$-horizontal, piecewise C^1, path $\tilde\gamma:[t_0,t_1]\to P$. Moreover, if Y is any element of the Lie algebra $L(G)$ and $\tilde Y$ is the vector field along $\tilde\gamma$ given by $\tilde Y(t) = \frac{d}{du}\big|_{u=0}\tilde\gamma(t)\exp(uY)$, then

$$\omega_{(A,B)}(\tilde Y) = Y. \tag{2.19}$$

Proof. Both of these statements follow directly from the defining relation

$$\omega_{(A,B)}(\tilde v) = A\big(\tilde v(t_0)\big) + \tau\left[\int_{t_0}^{t_1} B\left(\tilde\gamma'(t), \tilde v(t)\right)\,dt\right]. \tag{2.20}$$

If in this equation we use $\tilde v g$ instead of $\tilde v$ then the first term on the right is conjugated by $\mathrm{Ad}(g^{-1})$ because A is a connection form, whereas in the second term the integrand is conjugated by $\alpha(g^{-1})$ (keeping in mind that $\tilde v g$ is vector field along $\tilde\gamma g$), and then using the relation $\tau\big(\alpha(g^{-1})h\big) = g^{-1}\tau(h)g$ for all $h\in H$, from which we have

$$\tau\big(\alpha(g^{-1})Z\big) = \mathrm{Ad}(g^{-1})Z \qquad \text{for all } Z\in L(H).$$

This shows that the second term is also conjugated by $\mathrm{Ad}(g^{-1})$. Next, if $\tilde v = \tilde Y$, where $Y\in L(G)$ then the integrand in the second term on the right in (2.20) is 0 because B vanishes on vertical vectors and the first term is Y (since A is a connection form). QED

With notation and framework as above, there is a natural notion of horizontal lifts with respect to $\omega_{(A,B)}$. Let

$$\Gamma : [s_0, s_1] \times [t_0, t_1] \to M : (s,t) \mapsto \Gamma_s(t)$$

be a continuous map, viewed as a path of paths in M. By an $\omega_{(A,B)}$-*horizontal lift* of $s \mapsto \Gamma_s$ we mean a continuous mapping

$$\tilde{\Gamma} : [s_0, s_1] \times [t_0, t_1] \to P : (s,t) \mapsto \tilde{\Gamma}_s(t),$$

with $\pi \circ \tilde{\Gamma} = \Gamma$, for which each $\tilde{\Gamma}_s$ is piecewise C^1 and $\overline{A}$-horizontal and, furthermore, the 'flow direction vector field' $\partial_s \tilde{\Gamma}_s(s,\cdot)$ is $\omega_{(A,B)}$-horizontal in the sense that:

$$\omega_{(A,B)}\left(\partial_s \tilde{\Gamma}_s(s,\cdot)\right) = 0. \tag{2.21}$$

The connection $\omega_{(A,B)}$ has been explored in our previous work [12].

The existence and uniqueness of horizontal lifts of C^∞ paths Γ relative to $\omega_{(A,B)}$ follow by using standard results on the existence and uniqueness of solutions of ordinary differential equations in Lie groups. The key observation needed here is that to obtain the $\omega_{(A,B)}$-horizontal lift $\tilde{\Gamma}$ of a given path of paths $\Gamma : [s_0, s_1] \times [t_0, t_1] \to M : (s,t) \mapsto \Gamma_s(t)$, and a given initial C^∞ path $\tilde{\Gamma}_{s_0}$, one need only specify the motion $s \mapsto \tilde{\Gamma}_s(t_0) \in P$ and then each $\overline{A}$-horizontal path $t \mapsto \tilde{\Gamma}_s(t)$ is completely specified through the initial value $\tilde{\Gamma}_s(t_0) \in P$. Needless to say the condition C^∞ can be relaxed to a lower degree of piecewise smoothness.

3 Parallel transport and backtrack equivalence

In this section we prove that path space parallel-transport by the connection $\omega_{(A,B)}$ induces, in a natural way, parallel-transport over the space of backtrack-erased paths, a notion that we will explain drawing on ideas from the careful treatment by Lévy [18].

We say that a map $\gamma : [t_0, t_1] \to X$ into a space X *backtracks over* $[T, T+\delta]$, where $t_0 \le T < T + 2\delta < t_1$, if

$$\gamma(T+u) = \gamma(T + 2\delta - u) \qquad \text{for all } u \in [0,\delta]. \tag{3.1}$$

By *erasure of the backtrack* over $[T, T+\delta]$ from γ we will mean the map

$$[t_0, t_1 - 2\delta] : t \mapsto \begin{cases} \gamma(t) & \text{if } t \in [t_0, T]; \\ \gamma(t - 2\delta) & \text{if } t \in [T + 2\delta, t_1]. \end{cases} \tag{3.2}$$

Clearly, this is continuous if γ is continuous, and is piecewise C^1 if γ is piecewise C^1. (In later sections we will work with C^∞ paths that are constant near the initial and final points. When working with the class of such paths we consider only backtrack erasures that preserve the C^∞ nature.)

In the following we identify a parametrized path c_1 with a parametrized path c_2 if $c_2 = c_1 \circ \phi$, where ϕ is a strictly increasing piecewise C^1 mapping of the domain of c_2 onto the domain of c_1. We say that piecewise C^1 paths c_1 and c_2 on M are *elementary backtrack equivalent* if there are piecewise C^1 parametrized paths a, b, d on M, and a strictly increasing piecewise C^1 function ϕ from some closed interval onto the domain of c_1, and a strictly increasing piecewise C^1 function ψ from some closed interval onto the domain of c_2, such that

$$\{c_1 \circ \phi, c_2 \circ \psi\} = \{add^{-1}b, ab\}, \tag{3.3}$$

where juxtaposition of paths denotes composition of paths, and for any path $d : [s_0, s_1] \to M$ the reverse d^{-1} denotes the path

$$d^{-1} : [s_0, s_1] \to M : s \mapsto d\big(s_1 - (s - s_0)\big). \tag{3.4}$$

Condition (3.3) means that one of the c_i is obtained from the other by erasing the backtracking part dd^{-1}.

We say that piecewise C^1 paths b and c are *backtrack equivalent*, denoted

$$c \simeq_{\rm bt} b, \tag{3.5}$$

if there is a sequence of paths $c = c_0, c_1, \ldots, c_n = b$, where c_i is elementarily backtrack equivalent to c_{i+1} for $i \in \{0, \ldots, n-1\}$.

Backtrack equivalence is an equivalence relation, and if two paths are backtrack equivalent then so are any of their reparametrizations. Furthermore, if

$$a \simeq_{\rm bt} c \quad \text{and} \quad b \simeq_{\rm bt} d$$

and if the composite ab is defined then so is cd and

$$ab \simeq_{\rm bt} cd. \tag{3.6}$$

By a *backtrack-erased path* γ we will mean the backtrack equivalence class containing the specific path γ.

A tangent vector to a path $\gamma : [t_0, t_1] \to M$ is normally taken to be a vector field v, of suitable degree of smoothness, that has the property that $v(t) \in T_{\gamma(t)}M$, that is, it is a vector field *along* γ. Since we wish to identify backtrack equivalent paths we should not allow the vector field v to 'pry open' the backtracked parts of γ; thus we define a tangent vector v to γ in the space of backtrack-equivalence classes of paths, to be a continuous (or suitably smooth) vector field v along $\gamma : [t_0, t_1] \to M$, constant near t_0 and t_1, with the property that if γ backtracks over $[T, T+\delta]$, then v also has the same backtrack:

$$v(T+u) = v(T + 2\delta - u) \qquad \text{for all } u \in [0, \delta]. \tag{3.7}$$

Proposition 3.1 *Let $\overline{A}$ be a connection form on a principal G-bundle $\pi : P \to M$, where G is a Lie group. Consider an $\overline{A}$-horizontal piecewise C^1 path $\tilde{\gamma} : [t_0, t_1] \to P$ and a continuous vector field $\tilde{v} : [t_0, t_1] \to TP$ along $\tilde{\gamma}$ that satisfies the tangency condition (2.4). Suppose that the path $\gamma = \pi \circ \tilde{\gamma}$ backtracks over $[T, T + \delta]$ and suppose the vector field $v = \pi_* \tilde{v}$ along γ also backtracks over $[T, T + \delta]$. Then:*

(i) *The path $\tilde{\gamma}$ backtracks over $[T, T + \delta]$.*

(ii) *The vector field $\tilde{v}$ backtracks over $[T, T + \delta]$.*

(iii) *Erasing the backtrack over $[T, T+\delta]$ from $\tilde{v}$ produces a vector field along the path obtained by erasing the backtrack over $[T, T+\delta]$ from $\tilde{\gamma}$ that continues to satisfy the tangency condition (2.4).*

In the following it is useful to keep in mind that a vector $\tilde{w} \in T_p P$ is completely determined by its projection $\pi_* \tilde{w} \in T_{\pi(p)} M$ and its 'vertical' part $\overline{A}(\tilde{w}) \in L(G)$ (which can be transferred to an actual vertical vector in $T_p P$ by means of the right action of G on P).

Proof. (i) The backtracking in $\tilde{\gamma}$ follows because parallel-transport along the reverse of a path is the reverse of the parallel-transport along the path. Hence

$$\tilde{\gamma}(T + u) = \tilde{\gamma}(T + 2\delta - u) \qquad \text{for all } u \in [0, \delta].$$

Note that as consequence the velocity on the way back is minus the velocity on the way out:

$$\tilde{\gamma}'(T + u) = -\tilde{\gamma}'(T + 2\delta - u) \qquad \text{for all } u \in [0, \delta]. \tag{3.8}$$

(ii) The vector $\tilde{v}(t)$ is completely determined by its projection $v(t) = \pi_* \tilde{v}(t)$ and the 'vertical part' $\overline{A}(\tilde{v}(t)) \in L(G)$. To prove that $\tilde{v}$ backtracks over $[T, T+\delta]$ we need therefore only show that $\overline{a} \circ \tilde{v}$ backtracks over $[T, T + \delta]$.

Recall the tangency condition (2.4) :

$$\overline{A}(\tilde{v}(t)) = \overline{A}(\tilde{v}(t_0)) + \int_{t_0}^{t} F^{\overline{A}}\left(\tilde{\gamma}'(s), \tilde{v}(s)\right) ds \qquad \text{for all } t \in [t_0, t_1]. \tag{3.9}$$

Since $F^{\overline{A}}$ vanishes on vertical vectors, on the right we can replace $\tilde{v}(t)$ by $\tilde{v}_h(t)$, the $\overline{A}$-horizontal lift of $v(t)$ as a vector in $T_{\tilde{\gamma}(t)} P$. Hence

$$\overline{A}\big(\tilde{v}(t)\big) = \overline{A}\big(\tilde{v}(t_0)\big) + \int_{t_0}^{t} F^{\overline{A}}\big(\tilde{\gamma}'(s), \tilde{v}_h(s)\big)\, ds. \tag{3.10}$$

Clearly, the horizontal lift $\tilde{v}_h$ backtracks to reflect the backtracking (3.7) of v:

$$\tilde{v}_h(T + u) = \tilde{v}_h(T + 2\delta - u) \qquad \text{for all } u \in [0, \delta].$$

This, together with (3.8), implies that in the integral on the right in (3.10), the contribution from $s = T + u$ cancels the contribution from $s = T + 2\delta - u$, for all $u \in [0, \delta]$. Hence

$$\overline{A}\big(\tilde{v}(T + u)\big) = \overline{A}\big(\tilde{v}(T + 2\delta - u)\big) \qquad \text{for all } u \in [0, \delta].$$

(iii) When $t \geq T + 2\delta$ the part of the integral on the right in (3.10) over $[T, T + 2\delta]$ is completely erased, and so $\overline{A}\big(\tilde{v}(t)\big)$ retains the same value if the backtracks are erased from γ. From this it follows that the tangency condition (3.10) continues to hold when the backtrack over $[T, T+\delta]$ is erased from $\tilde{v}$ and from $\tilde{\gamma}$. QED

We resume working with a crossed module (G, H, α, τ), connections A, $\overline{A}$, and an $L(H)$-valued 2-form B on a principal G-bundle $\pi : P \to M$, satisfying α-equivariance (2.11) and vanishing on verticals (2.12). The following is a remarkable feature of the connection form $\omega_{(A,B)}$, showing that it specifies parallel-transport over the space of backtrack equivalence classes of paths.

Theorem 3.1 *Let (G, H, α, τ) be a Lie crossed module, $\overline{A}$ and A be connections on a principal G-bundle $\pi : P \to M$, and let $\omega_{(A,B)}$ be the 1-form given by (2.14). Then $\omega_{(A,B)}$ is well-defined as a 1-form on tangent vectors to backtrack equivalence classes of $\overline{A}$-horizontal paths on P in the following sense. Suppose $\gamma : [t_0, t_1] \to M$ is a piecewise C^1 path, and γ_0 a path obtained by erasing a backtrack over $[T, T + \delta]$ from γ. Let $\tilde{\gamma}$ be an $\overline{A}$-horizontal lift of γ and $\tilde{\gamma}_0$ the $\overline{A}$-horizontal lift of γ_0 with the same initial point as $\tilde{\gamma}$. Suppose $\tilde{v}$ is a continuous vector field along $\tilde{\gamma}$ that backtracks over $[T, T + \delta]$. Then*

$$\omega_{(A,B)}(\tilde{v}) = \omega_{(A,B)}(\tilde{v}_0), \tag{3.11}$$

where $\tilde{v}_0$ is the vector field along $\tilde{\gamma}_0$ induced by $\tilde{v}$, through restriction. In particular, $\omega_{(A,B)}(\tilde{v})$ is 0 if and only if $\omega_{(A,B)}(\tilde{v}_0)$ is also 0.

Proof. Consider an $\overline{A}$-horizontal path $\tilde{\gamma} : [t_0, t_1] \to P$. Suppose $\gamma = \pi \circ \tilde{\gamma}$ backtracks over $[T, T+\delta]$. By Proposition 3.1(i), $\tilde{\gamma}$ also backtracks over $[T, T+\delta]$. Hence

$$\tilde{\gamma}'(T + u) = -\tilde{\gamma}'(T + 2\delta - u) \tag{3.12}$$

for all $u \in [0, \delta]$ for which either side exists. From the expression

$$\omega_{(A,B)}(\tilde{v}) = A\left(\tilde{v}(t_0)\right) + \tau\left(\int_{t_0}^{t_1} B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right)\, dt\right) \tag{3.13}$$

we see then that in the second term on the right in the integral $\int_{t_0}^{t_1} \cdot\, dt$ we have a cancellation:

$$\int_{T}^{T+\delta} B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right)\, dt + \int_{T+\delta}^{T+2\delta} B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right)\, dt = 0.$$

Note also that the endpoint of a path is unaffected by backtrack erasure. Hence the first term on the right in (3.13) remains unchanged if the backtrack over $[T, T + \delta]$ in $\tilde{\gamma}$ is erased. Thus the value $\omega_{(A,B)}(\tilde{v})$ remains unchanged if the backtrack of γ over any subinterval of $[t_0, t_1]$ is erased. In particular, $\tilde{v}$ is $\omega_{(A,B)}$-horizontal if and only if the backtrack-erased version of $\tilde{v}$ is $\omega_{(A,B)}$-horizontal. QED

There is a notion, known as 'thin homotopy' equivalence, that is broader than backtrack equivalence. A formalization of this notion using ranks of the derivative of the 'homotopy' functions was first introduced by Caetano and Picken [11, sec. 4] (who attribute it to Machado; an earlier version was introduced by Barrett [8, sec 2.1.1]). Roughly speaking a thin homotopy wriggles a path back and forth along itself with no 'transverse' motion. The following result says essentially that parallel-transport by $\omega_{(A,B)}$ is trivial along a thin homotopy.

Theorem 3.2 *Let (G, H, α, τ) be a Lie crossed module, A and $\overline{A}$ connection forms on a principal G-bundle $\pi : P \to M$, and let $\omega_{(A,B)}$ be as given by (2.14). Consider a C^∞ map*

$$\Gamma : [s_0, s_1] \times [t_0, t_1] \to M : (u, v) \mapsto \Gamma_u(v)$$

for which $\partial_u\Gamma$ and $\partial_v\Gamma$ are linearly dependent at each point of $[s_0, s_1] \times [t_0, t_1]$, and, furthermore, Γ keeps each endpoint of each u-fixed line $\{u\} \times [t_0, t_1]$ constant:

$$\Gamma(u, t_i) = \Gamma(s_0, t_i) \quad \text{for all } u \in [s_0, s_1] \text{ and } i \in \{0, 1\}. \tag{3.14}$$

Next let $\tilde{\Gamma}_0 : [t_0, t_1] \to P$ be any $\overline{A}$-horizontal path that projects to the initial path Γ_0 on M. Then the $\omega_{(A,B)}$-horizontal lift of $u \mapsto \Gamma_u$, with $\tilde{\Gamma}_{s_0} = \tilde{\Gamma}_0$, is the path

$$[s_0, s_1] \to \mathcal{P}_{\overline{A}}P : u \mapsto \tilde{\Gamma}_u \tag{3.15}$$

where $\tilde{\Gamma}_u : [t_0, t_1] \to P$ is the $\overline{A}$-horizontal lift of Γ_u with initial point

$$\tilde{\Gamma}_u(t_0) = \tilde{\Gamma}_0(t_0) \tag{3.16}$$

for all $u \in [s_0, s_1]$. Moreover, $\partial_u\tilde{\Gamma}$ and $\partial_v\tilde{\Gamma}$ are linearly dependent at each point of $[s_0, s_1] \times [t_0, t_1]$, where $\tilde{\Gamma}$ is the map $[s_0, s_1] \times [t_0, t_1] \to P : (u, v) \mapsto \tilde{\Gamma}_u(v)$.

Proof. From the tangency condition (2.3) we have

$$\begin{aligned}\overline{A}(\partial_u\tilde{\Gamma}(u, w)) = \overline{A}&(\partial_u\tilde{\Gamma}(u, t_0)) \\ &+ \int_{t_0}^{w} F^{\overline{A}}\left(\partial_v\tilde{\Gamma}(u, v), \partial_u\tilde{\Gamma}(u, v)\right) dv \quad \text{for all } w \in [t_0, t_1].\end{aligned} \tag{3.17}$$

The first term on the right is 0 because $\tilde{\Gamma}(\cdot, t_0)$ is constant by assumption (3.16). We now show that the second term is also zero. By assumption $\partial_u\Gamma$ and $\partial_v\Gamma$ are linearly dependent; thus there exist $a(u, v), b(u, v) \in \mathbb{R}$, not both zero, such that

$$a(u, v)\partial_v\Gamma(u, v) + b(u, v)\partial_u\Gamma(u, v) = 0, \tag{3.18}$$

for all $(u, v) \in [s_0, s_1] \times [t_0, t_1]$. Observe that

$$\pi_*\partial_v\tilde{\Gamma} = \partial_v(\pi \circ \tilde{\Gamma}) = \partial_v\Gamma$$

and, similarly,

$$\pi_*\partial_u\tilde{\Gamma} = \partial_u\Gamma.$$

Hence

$$a(u,v)\partial_v\tilde{\Gamma}(u,v) + b(u,v)\partial_u\tilde{\Gamma}(u,v) \in \ker \pi_*. \tag{3.19}$$

(We will see shortly that the left side is in fact zero.) Thus, up to addition of a vertical vector (on which $F^{\overline{A}}$ vanishes), the vectors $\partial_v\tilde{\Gamma}(u,v)$ and $\partial_u\tilde{\Gamma}(u,v)$ are linearly dependent and so the 2-form $F^{\overline{A}}$ is 0 when evaluated on this pair of vectors. Hence

$$\overline{A}(\partial_u\tilde{\Gamma}(u,v)) = 0, \tag{3.20}$$

for all $(u,v) \in [a_0,a_1] \times [t_0,t_1]$. Using this and the $\overline{A}$-horizontality of $\tilde{\Gamma}_u$ we have

$$\overline{A}\left(a(u,v)\partial_v\tilde{\Gamma}(u,v) + b(u,v)\partial_u\tilde{\Gamma}(u,v)\right) = 0, \tag{3.21}$$

which, in combination with (3.19), implies that

$$a(u,v)\partial_v\tilde{\Gamma}(u,v) + b(u,v)\partial_u\tilde{\Gamma}(u,v) = 0. \tag{3.22}$$

This proves that $\partial_u\tilde{\Gamma}$ and $\partial_v\tilde{\Gamma}$ are linearly dependent.

From the definition of $\omega_{(A,B)}$ in (2.13) we have

$$\omega_{(A,B)}(\partial_u\tilde{\Gamma}) = A\big(\partial_u\tilde{\Gamma}(u,t_0)\big) + \tau\left[\int_{t_0}^{t_1} B\left(\partial_v\tilde{\Gamma}(u,v), \partial_u\tilde{\Gamma}(u,v)\right)\, dv\right]. \tag{3.23}$$

The second term vanishes because of the linear dependence (3.21). The first term on the right (3.23) is also 0 because $\tilde{\Gamma}_u(t_0)$ constant in $u \in [s_0,s_1]$ by assumption (3.14). Hence the path (3.15) on $\mathcal{P}_{\overline{A}}P$ is $\omega_{(A,B)}$-horizontal. QED

4 2-groups by many names

In this section we organize known notions, results and examples in a manner that is useful for our purposes in later sections. These ideas and related topics on categorical groups are discussed in the works [2, 5, 6, 9, 10, 11, 13, 15, 16, 19, 20, 23, 27]. There is considerable diversity of terminology and specific definitions prevalent in the literature. We present a self-contained account of the topic to save the reader the trouble of searching through the literature to make sure our proofs are consistent with definitions available elsewhere.

A category $\mathbf{K}$ along with a bifunctor

$$\otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K} \tag{4.1}$$

forms a *categorical group* if both $\mathrm{Obj}(\mathbf{K})$ and $\mathrm{Mor}(\mathbf{K})$ are groups under the operation $\otimes$ (on objects and on morphisms, respectively). Sometimes it will be more convenient to write ab instead of $a \otimes b$. Being a functor, $\otimes$ carries the identity morphism $(1_a, 1_b)$, where $1_x : x \to x$ denotes the identity arrows on x, to the identity morphism 1_{ab}:

$$1_a \otimes 1_b = 1_{a\otimes b}.$$

and so, taking a to be the identity element e in $\mathrm{Obj}(\mathbf{K})$, it follows that 1_e is the identity element in the group $\mathrm{Mor}(\mathbf{K})$.

The functoriality of $\otimes$ implies the 'exchange law':

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g') \tag{4.2}$$

for all $f, f', g, g' \in \mathrm{Mor}(\mathbf{K})$ for which the composites $f \circ f'$ and $g \circ g'$ are defined.

The following is a curious but useful consequence of the definition of a categorical group:

Proposition 4.1 *Let $\mathbf{K}$ be a categorical group, with the group operation written as juxtaposition. Then for any morphisms $f : a \to b$ and $h : b \to c$ in $\mathbf{K}$, the composition $h \circ f$ can be expressed in terms of the product operation in $\mathbf{K}$:*

$$h \circ f = f 1_{b^{-1}} h = h 1_{b^{-1}} f. \tag{4.3}$$

In particular,

$$hk = h \circ k = kh \qquad \text{if } t(k) = s(h) = e. \tag{4.4}$$

Proof. For $f : a \to b$ and $h : b \to c$ we have, on using the exchange law (4.2),

$$h \circ f = (1_e h) \circ (f 1_{b^{-1}} 1_b) = (1_e \circ f 1_{b^{-1}})(h \circ 1_b) = f 1_{b^{-1}} h. \tag{4.5}$$

Interchanging the order of the multiplication we also have

$$h \circ f = (h 1_e) \circ (1_b 1_{b^{-1}} f) = (h \circ 1_b)(1_e \circ 1_{b^{-1}} f) = h 1_{b^{-1}} f \tag{4.6}$$

QED

Here is an alternative formulation of the definition of a categorical group:

Proposition 4.2 *If $\mathbf{K}$, with operation $\otimes$, is a categorical group then the source and target maps*

$$s, t : \mathrm{Mor}(\mathbf{K}) \to \mathrm{Obj}(\mathbf{K})$$

are group homomorphisms, and so is the identity-assigning map

$$\mathrm{Obj}(\mathbf{K}) \to \mathrm{Mor}(\mathbf{K}) : x \mapsto 1_x.$$

Conversely, if $\mathbf{K}$ is a category for which both $\mathrm{Obj}(\mathbf{K})$ and $\mathrm{Mor}(\mathbf{K})$ are groups, s, t, and $x \mapsto 1_x$ are homomorphisms, and the exchange law (4.2) holds then $\mathbf{K}$ is a categorical group.

Proof. Suppose $\mathbf{K}$ is a categorical group. Consider any morphisms $f : a \to b$ and $g : c \to d$ in $\mathrm{Mor}(\mathbf{K})$. Then, by definition of the product category $\mathbf{K} \times \mathbf{K}$, we have the morphism $(f, g) \in \mathrm{Mor}(\mathbf{K} \times \mathbf{K})$ running from the domain (a, c) to the codomain (b, d):

$$(f, g) : (a, c) \to (b, d) \quad \text{in } \mathrm{Mor}(\mathbf{K} \times \mathbf{K}).$$

Then, since $\otimes$ is a functor, $f \otimes g$ runs from domain $a \otimes c$ to target $b \otimes d$. Thus,

$$s(f \otimes g) = s(f) \otimes s(g) \quad \text{and} \quad t(f \otimes g) = t(f) \otimes t(g).$$

Thus s and t are homomorphisms.

Next, for any objects $x, y \in \mathrm{Obj}(\mathbf{K})$, the identity morphism

$$(1_x, 1_y) : (x, y) \to (x, y)$$

in $\mathrm{Mor}(\mathbf{K} \times \mathbf{K})$ is mapped by the functor $\otimes$ to the identity morphism

$$1_{x \otimes y} : x \otimes y \to x \otimes y,$$

and this just means that

$$1_x \otimes 1_y = 1_{x \otimes y}.$$

Thus $x \mapsto 1_x$ is a group homomorphism.

Conversely, suppose s, t, $x \mapsto 1_x$ are group homomorphisms and the exchange law (4.2) holds (both sides of (4.2) are meaningful because s and t are homomorphisms). Then (4.2) says that $\otimes$ maps composites to composites, while $x \mapsto 1_x$ being a homomorphism implies that $1_{x \otimes y} = 1_x \otimes 1_y$, and so $\otimes$ is indeed a functor, preserving composition and mapping identities to identities. QED

See [13, 17] for more on 2-groups, related notions, along with more references (the relation (4.3), proved above, is mentioned in [13].)

The preceding result suggests that when working with groups with more structure, for example with Lie groups, it would be more natural to continue to require that the source, targets, and identity-assigning maps respect the additional structure. Thus by a *categorical Lie group* we mean a category $\mathbf{K}$ along with a functor $\otimes$ as above, such that $\mathrm{Mor}(\mathbf{K})$ and $\mathrm{Obj}(\mathbf{K})$ are Lie groups, and the maps s, t, and $x \mapsto 1_x$ are smooth homomorphisms.

Example CG1. For any group K we can construct a categorical group $\mathbf{K}_0$ by taking K as the object set and requiring there be a unique morphism $a \to b$ for every $a, b \in K$. At the other extreme we have the discrete categorical group $\mathbf{K}_d$ whose object set is K but whose morphisms are only the identity morphisms. □

Example CG2. Let

$$\pi : \hat{K} \to K \tag{4.7}$$

be a surjective homomorphism of groups. We think of $\hat{K}$ as a 'covering group', or a principal bundle over K, with each fiber $\pi^{-1}(k)$ standing 'above' the point k. The structure group is

$$Z = \ker \pi = \pi^{-1}(e),$$

with e being the identity element of K. For the category $\mathbf{K}$ we take the object set to be K itself. A morphism $a \to b$ is to be thought of as an arrow $\hat{a} \to \hat{b}$, with $\pi(\hat{a}) = a$ and $\pi(\hat{b}) = b$ and such that $\hat{a} \to \hat{b}$ is identified with $\hat{a}\hat{k} \to \hat{b}\hat{k}$, for every $\hat{k} \in \ker \pi$. Thus the discrete categorical group $\mathbf{Z}_d$ acts on the right on the category $\hat{\mathbf{K}}_0$ (notation as in CG1) as follows: an object $z \in Z$ acts on the right on an object $\hat{a} \in \hat{K}$ to produce the product $\hat{a}z$; a morphism $z \to z$ in $\mathrm{Mor}(\mathbf{Z}_d)$ acts on a morphism $\hat{a} \to \hat{b}$ in $\mathrm{Mor}(\hat{\mathbf{K}}_0)$ to produce the morphism $\hat{a}z \to \hat{b}z$. (See (5.1) for a precise definition of right action in this context.) We construct a category $\mathbf{K}$ which we view as the quotient $\hat{\mathbf{K}}/\mathbf{Z}_d$. The object set of $\mathbf{K}$ is K. A morphism $f : a \to b$ for $\mathbf{K}$ is obtained as follows: we choose some $\hat{a} \in \pi^{-1}(a)$, $\hat{b} \in \pi^{-1}(b)$ and take f to be the arrow $\hat{a} \to \hat{b}$, identifying this with $\hat{a}z \to \hat{b}z$ for all $z \in Z$. More compactly, $\mathrm{Mor}(\mathbf{K})$ is the quotient $\mathrm{Mor}(\hat{\mathbf{K}})/\mathrm{Mor}(\mathbf{Z}_d)$ where the action of the group $\mathrm{Mor}(\mathbf{Z}_d)$ on $\mathrm{Mor}(\hat{\mathbf{K}})$ is given by: $(\hat{a} \to \hat{b})(z \to z) = \hat{a}z \to \hat{b}z$. In order to make the category $\mathbf{K}$ into a categorical group we define a multiplication on $\mathrm{Mor}(\mathbf{K})$ by using the multiplication structure on $\mathrm{Mor}(\hat{\mathbf{K}})$: for a morphism $f : a \to b$ given by $\hat{a} \to \hat{b}$ and $g : c \to d$ given by $\hat{c} \to \hat{d}$ we define fg to be

$$fg = (\hat{a}\hat{c} \to \hat{b}\hat{d}). \tag{4.8}$$

For this to be well-defined, the right side should not depend on the choices $\hat{x} \in \pi^{-1}(x)$; if we assume that the subgroup Z is *central* in $\hat{K}$, then:

$$\hat{a}z\hat{c}w = \hat{a}\hat{c}zw$$

holds for all $\hat{a}, \hat{c} \in \hat{K}$ and $z, w \in Z$, and so the right side in (4.8) is determined entirely by f and g. □

A more specific class of examples of categorical groups is provided by taking K to be any compact Lie group and $\hat{K}$ its universal covering group.

A group K gives rise to a category $\mathbf{G}(K)$ in a natural way: $\mathbf{G}(K)$ has just one object O, the morphisms of $\mathbf{G}(K)$ are the elements of K, all having O as both source and target, with composition of morphisms being given by the group operation in K.

Next we discuss a concrete form of a categorical group. For this recall the notion of a crossed module (G, H, α, τ) from (2.9).

Theorem 4.1 *Suppose* $\mathbf{G}$ *is a categorical group, with the group operation written as juxtaposition:* $a \otimes b = ab$, *and with* $s, t : \mathrm{Mor}(\mathbf{G}) \to \mathrm{Obj}(\mathbf{G})$ *denoting the source and target maps. Let* $H = \ker s$. *Let* $\tau : H \to G$ *and* $\alpha : G \to \mathrm{Aut}(H)$ *be the maps specified by*

$$\begin{aligned}\tau(\theta) &= t(\theta),\\ \alpha(g)\theta &= 1_g\theta 1_{g^{-1}}\end{aligned} \tag{4.9}$$

for all $g \in G$ *and* $\theta \in H$. *Then* (G, H, α, τ) *is a crossed module.*

Conversely, suppose (G, H, α, τ) *is a crossed module. Then there is a category* $\mathbf{G}$ *whose object set is* G *and for which a morphism* $h : a \to b$ *is an ordered pair*

$$(h, a),$$

where $h \in H$ *satisfies*

$$\tau(h)a = b,$$

with composition being given by

$$(h_2, b) \circ (h_1, a) = (h_2 h_1, a). \tag{4.10}$$

Moreover, **G** *is a categorical group, with group operation on* Obj(**G**) *being the one on* G*, and the group operation on* Mor(**G**) *being*

$$(h, a)(k, c) = (h\alpha(a)(k), ac)\,. \tag{4.11}$$

Stated in technical terms, we have an equivalence between the category of crossed modules and the category of categorical groups, defined in the obvious way.

Let (G, H, α, τ) be the crossed module constructed as above from a categorical group **G**, as above. Let **K** be the categorical group constructed from (G, H, α, τ) according to the prescription in Theorem 4.1. The objects of **K** are just the elements of G. A morphism in Mor(**K**) is a pair

$$\big(s(\phi), \phi 1_{s(\phi)^{-1}}\big)\,, \tag{4.12}$$

where $\phi \in \mathrm{Mor}(\mathbf{G})$, so that

$$\phi 1_{s(\phi)^{-1}} \in H = \ker s. \tag{4.13}$$

<u>Proof</u>. Assuming that **G** is a categorical group we will show that (G, H, α, τ) is a crossed module. Let τ be the target map t restricted to the subgroup H of Mor(**G**)

$$\tau : H \to G : h \mapsto \tau(h) = t(h). \tag{4.14}$$

Next for $g \in G$ let $\alpha(g) : H \to H$ be given on any $h \in H$ by

$$\alpha(g)h = 1_g h 1_{g^{-1}}. \tag{4.15}$$

Note that the source of $\alpha(g)h$ is

$$s\big(\alpha(g)h\big) = geg^{-1} = e,$$

where e is the identity element of Obj(**G**). Thus $\alpha(g)h$ is indeed an element of H. Moreover, it is readily checked that $\alpha(g)$ is an automorphism of H.

The target of $\alpha(g)h$ is

$$\tau\big(\alpha(g)h\big) = t\big(\alpha(g)h\big) = g\tau(h)g^{-1}. \tag{4.16}$$

Next, consider $h, h' \in H$, and suppose $\tau(h) = g$ and $\tau(h') = g'$. Then

$$\alpha\big(\tau(h)\big)(h') = 1_g h' 1_{g^{-1}} = hh'h^{-1}, \tag{4.17}$$

the last equality of which can be verified by using Proposition 4.1 (specifically (4.4)), which implies that the element $h^{-1}1_g : g \mapsto e$ commutes with $h' : e \to$

g'. Equations (4.16) and (4.17) are exactly the conditions (2.9) that make (G, H, α, τ) a crossed module.

Before proceeding to the proof of the converse part, we observe that a morphism $f \in \mathrm{Mor}(\mathbf{G})$ is completely specified by its source $a = s(f)$ and by the morphism

$$h = f1_{a^{-1}} \in \ker s.$$

Now suppose (G, H, α, τ) is a crossed module. We construct a category $\mathbf{K}$ with object set G. If $f : a \to b$ is to be a morphism then $f1_{a^{-1}}$ would be a morphism $e \to ba^{-1}$. Thus, for $a, b \in G$ we take a morphism $a \to b$ to be specified by the source a along with an element $h \in H$ for which $\tau(h) = ba^{-1}$. So we define a morphism $f \in \mathrm{Mor}(\mathbf{K})$ to be an ordered pair $(h, a) \in H \times G$, with source and target given by

$$s(f) = a \qquad \text{and} \qquad t(f) = \tau(h)a. \tag{4.18}$$

Thus the source and target maps are

$$s(h, a) = a \quad \text{and} \quad t(h, a) = \tau(h)a. \tag{4.19}$$

To understand what the composition law should be, recall from (4.3) that for any morphisms $f : a \to b$ and $g : b \to c$ we have

$$(g \circ f)1_{a^{-1}} = g1_{b^{-1}} f1_{a^{-1}}. \tag{4.20}$$

Thus we can define the composition of morphisms $(h_1, a), (h_2, b) \in H \times G$, with $b = \tau(h_1)a$, by

$$(h_2, b) \circ (h_1, a) = (h_2h_1, a). \tag{4.21}$$

The identity morphism 1_a is then (e, a) because

$$(h_2, a) \circ (e, a) = (h_2, a) \quad \text{and} \quad (e, c) \circ (h_1, b) = (h_1, b),$$

where $c = \tau(h_1)b$. Associativity of composition is clearly valid. Thus $\mathbf{K}$ is indeed a category. It remains to define a product on $\mathrm{Mor}(\mathbf{K})$ and prove that this product is functorial.

For $f : a \to b$ and $g : c \to d$ we have

$$(fg)1_{(ac)^{-1}} = f1_{a^{-1}}1_a g1_{c^{-1}}1_{a^{-1}}$$

which motivates us to define the product on $\mathrm{Mor}(\mathbf{K})$ by

$$(h, a)(k, c) = (h\alpha(a)(k), ac)\,. \tag{4.22}$$

If we identify H and G with subsets of $\mathrm{Mor}(\mathbf{K})$ through the injections

$$\begin{aligned} H \to \mathrm{Mor}(\mathbf{K}) &: h \mapsto (h, e) \\ G \to \mathrm{Mor}(\mathbf{K}) &: a \mapsto (e, a) \end{aligned} \tag{4.23}$$

then, using (4.22), the product ha corresponds to the element

$$ha = (h, e)(e, a) = (h, a). \tag{4.24}$$

Hence the multiplication operation (4.22) takes the form

$$hakc = h\alpha(a)(k)\, ac, \tag{4.25}$$

which means the commutation relation

$$ah = \alpha(a)(h)\, a \quad \text{for all } a \in G \text{ and } h \in H, \tag{4.26}$$

or, equivalently:

$$\alpha(a)(h) = aha^{-1} \quad \text{for all } a \in G \text{ and } h \in H. \tag{4.27}$$

The product (4.22) is a standard semi-direct product of groups, and it is a straightforward, if lengthy, calculation to verify that $\mathrm{Mor}(\mathbf{K})$ is indeed a group under the operation:

$$\mathrm{Mor}(\mathbf{K}) = H \rtimes_{\alpha} G. \tag{4.28}$$

It is readily checked that the source and target maps s and t given by (4.19) are homomorphisms, and so is the map

$$\mathrm{Obj}(\mathbf{K}) \to \mathrm{Mor}(\mathbf{K}) : a \mapsto 1_a = (e, a). \tag{4.29}$$

Thus $\mathbf{K}$ is a categorical group.

It is straightforward to verify the exchange law (4.2) holds. Finally, by construction, $G = \mathrm{Obj}(\mathbf{K})$, and H is identifiable as the subgroup of $\mathrm{Mor}(\mathbf{K})$ given by $\ker s$. QED

Example CG3. Let L_o be the set of all backtrack erased piecewise C^1 loops based at a point o in a manifold B; under composition, this is clearly a group. Now the method of example CG2 can be used, with $K = L_o$, to form categorical groups with object group L_o. □

By a 2-*category* $\mathbf{C}_2$ over a category $\mathbf{C}_1$ we mean a category $\mathbf{C}_2$ whose objects are the arrows of $\mathbf{C}_1$:

$$\mathrm{Obj}(\mathbf{C}_2) = \mathrm{Mor}(\mathbf{C}_1).$$

Let $\mathbf{K}$ be a categorical group, and K the group formed by $\mathrm{Obj}(\mathbf{K})$. Then $\mathbf{K}$ is a 2-category over $\mathbf{G}(K)$, which is the categorical group whose object group has just one element and whose morphism group is K. (For an extensive account of the theory of 2-categories see Kelly and Street [17].)

Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be categorical groups such that $\mathrm{Obj}(\mathbf{G}_2) = \mathrm{Mor}(\mathbf{G}_1)$. Let (G, H, α_1, τ_1) be the crossed module associated with $\mathbf{G}_1$, and $(H \rtimes_{\alpha_1} G, K, \alpha_2, \tau_2)$ the crossed module associated with $\mathbf{G}_2$. We identify H and G naturally with subgroups of $H \rtimes_{\alpha_1} G$, so that each element of this semi-direct product can be expressed as hg, with $h \in H$ and $g \in G$. The following computation will be useful later.

Lemma 4.2 *With notation as above,*

$$\alpha_2\Big(\alpha_1(g_1^{-1})(h_1^{-1}h)\Big)\Big(\alpha_2(g_1^{-1})(k)\Big) = \alpha_2(g_1^{-1}h_1^{-1}h)(k), \tag{4.30}$$

for all $g_1 \in G_1$, $h_1, h \in H$, *and* $k \in K$.

Proof. Recall from (4.26) the commutation rule:

$$gh = \alpha_1(g)(h)\, g \qquad \text{for all } g \in G \text{ and } h \in H. \tag{4.31}$$

Then we have

$$\begin{aligned}\alpha_2\Big(\alpha_1(g_1^{-1})(h_1^{-1}h)\Big)\Big(\alpha_2(g_1^{-1})(k)\Big) &= \alpha_2\Big(\alpha_1(g_1^{-1})(h_1^{-1}h)g_1^{-1}\Big)(k)\\ &= \alpha_2(g_1^{-1}h_1^{-1}h)(k) \quad \text{(using (4.31) with } g = g_1^{-1}\text{)},\end{aligned} \tag{4.32}$$

which proves (4.30). QED

5 Principal categorical bundles

In the traditional topological description, a principal bundle $\pi : P \to B$ is a smooth surjection of manifolds, along with a Lie group G acting freely on P on the right, the action being transitive on each fiber $\pi^{-1}(b)$ (we do not consider local triviality here). In this section we formulate a categorical description that includes a wider family of geometric objects than is included in the traditional notion of principal bundles. In particular, the framework of categorical bundles includes bundles over path spaces with a pair of groups serving as structure groups, one at the level of objects and one at the level of morphisms. Categorical bundles, in different formalizations, have been studied in the literature (for example in Baez et al. [3, 4, 5, 6, 7], Bartels [10], Schreiber and Waldorf [24, 25], Abbaspour and Wagemann [1], Viennot [26], and Wockel [28]). Our exploration is distinct and our focus is more on the geometric side than on category theoretic exploration. However, unlike many other works in the area (such as the analysis by Wockel [28] of the classification 2-bundles in terms of Čech cohomology), we do not explore any notions of local triviality nor do we impose any smooth structure on the base categories in the formal definition.

Let $\mathbf{P}$ be a category and $\mathbf{Z}$ a categorical group. Denote by P the set of objects of $\mathbf{P}$ and by Z the set of objects of $\mathbf{Z}$:

$$P = \mathrm{Obj}(\mathbf{P}) \quad \text{and} \quad Z = \mathrm{Obj}(\mathbf{Z}).$$

By a *right action* of $\mathbf{Z}$ on $\mathbf{P}$ we mean a functor

$$\mathbf{P} \times \mathbf{Z} \to \mathbf{P} : (x, g) \mapsto \rho(g)x = \rho(x, g) \tag{5.1}$$

that is a right action both at the level of objects and at the level of morphisms; thus, both

$$\begin{aligned} \mathrm{Obj}(\mathbf{P}) \times \mathrm{Obj}(\mathbf{Z}) \to \mathrm{Obj}(\mathbf{P}) : (A, g) \mapsto \rho(A, g) \\ \mathrm{Mor}(\mathbf{P}) \times \mathrm{Mor}(\mathbf{Z}) \to \mathrm{Mor}(\mathbf{P}) : (F, \phi) \mapsto \rho(F, \phi) \end{aligned} \tag{5.2}$$

are right actions. We assume, moreover, that both these actions are *free*.

Note that functoriality of ρ implies, in particular, that

$$\begin{aligned} s\big(\rho(F, \phi)\big) = \rho\big(s(F), s(\phi)\big) \\ t\big(\rho(F, \phi)\big) = \rho\big(t(F), t(\phi)\big) \end{aligned} \tag{5.3}$$

By analogy with principal bundles we define a *principal categorical bundle* to be a functor $\pi : \mathbf{P} \to \mathbf{B}$ along with a right action of a categorical group $\mathbf{Z}$ on $\mathbf{P}$ satisfying the following conditions:

(i) π is surjective both at the level of objects and at the level of morphisms;

(ii) the action of $\mathbf{Z}$ on $\mathbf{P}$ is free on both objects and morphisms;

(iii) the action of $\mathrm{Obj}(\mathbf{Z})$ on the fiber $\pi^{-1}(b)$ is transitive for each object $b \in \mathrm{Obj}(\mathbf{B})$, and the action of $\mathrm{Mor}(\mathbf{Z})$ on the fiber $\pi^{-1}(\phi)$ is transitive for each morphism $\phi \in \mathrm{Mor}(\mathbf{B})$.

Notice, however, that we are not imposing any form of local triviality.

There is a consistency property of compositions in $\mathbf{P}$ with respect to the action of $\mathbf{Z}$.

Lemma 5.1 *Suppose ρ is a right action of a categorical group $\mathbf{Z}$ on a category $\mathbf{P}$, the action being free on both objects and morphisms. Let F and H be morphisms in $\mathrm{Mor}(\mathbf{P})$, with the composition $H \circ F$ defined, and let F' and H' be morphisms in $\mathrm{Mor}(\mathbf{P})$, with F' being in the $\mathrm{Mor}(\mathbf{Z})$-orbit of F and H' in the $\mathrm{Mor}(\mathbf{Z})$-orbit of H, and the composite $H' \circ F'$ also defined; then the composition $H' \circ F'$ is in the $\mathrm{Mor}(\mathbf{Z})$-orbit of $H \circ F$.*

Proof. Suppose F' is in the $\mathrm{Mor}(\mathbf{Z})$-orbit of F, and H' is in the $\mathrm{Mor}(\mathbf{Z})$-orbit of H. Then

$$F' = F\rho(\phi) \qquad \text{and} \qquad H' = H\rho(\psi) \tag{5.4}$$

for some $\phi, \psi \in \mathrm{Mor}(\mathbf{Z})$. The composability $H' \circ F'$ implies that the target of $F\rho(\phi) = \rho(F, \phi)$ is the source of $H\rho(\psi)$, and so, by (5.3),

$$\rho\big(t(F), t(\phi)\big) = t\left(\rho(F, \phi)\right) = s\left(\rho(H, \psi)\right) = \rho\big(s(H), s(\psi)\big). \tag{5.5}$$

Now $t(F) = s(H)$ when $H \circ F$ is defined; hence by (5.5) and by freeness of the action ρ on $\mathrm{Mor}(\mathbf{P})$ we conclude that

$$t(\phi) = s(\psi),$$

and so the composite $\psi \circ \phi$ is defined. Then, using the functoriality of the categorical group action ρ, we have

$$H' \circ F' = \rho(H, \psi) \circ \rho(F, \phi) = \rho(H \circ F, \psi \circ \phi) \tag{5.6}$$

which shows that $H' \circ F'$ is in the $\mathrm{Mor}(\mathbf{Z})$-orbit of $H \circ F$. Hence composition is well-defined on the quotient $\mathrm{Mor}(\mathbf{B})$ as specified in (5.7). $\boxed{\text{QED}}$

The consistency property of composition allows us to form a 'quotient' category $\mathbf{P}/\mathbf{Z}$.

Theorem 5.2 *Suppose ρ is a right action of a categorical group $\mathbf{Z}$ on a category $\mathbf{P}$, the action being free on both objects and morphisms.*

Let $\mathbf{B}$ be the category whose object set is $B = \mathrm{Obj}(\mathbf{P})/\mathrm{Obj}(\mathbf{Z})$, and whose morphisms are $\mathrm{Mor}(\mathbf{Z})$-orbits of morphisms in $\mathrm{Mor}(\mathbf{P})$, with composition defined by

$$[H \circ F] = [H] \circ [F], \tag{5.7}$$

where $[\cdots]$ denotes the $\mathrm{Mor}(\mathbf{Z})$-orbit. Then

$$\pi : \mathbf{P} \to \mathbf{B}$$

taking every object $X \in \mathrm{Obj}(\mathbf{P})$ to the $\mathrm{Obj}(\mathbf{Z})$-orbit of X, and every morphism F to its $\mathrm{Mor}(\mathbf{Z})$-orbit is a functor, and, along with the right action of $\mathbf{Z}$ on $\mathbf{P}$, specifies a principal categorical bundle.

Proof. Let

$$B = \mathrm{Obj}(\mathbf{P})/\mathrm{Obj}(\mathbf{Z}),$$

and

$$\pi_P : P \to B$$

the quotient map. For $x \in B$ the identity morphism 1_x is the orbit of 1_X, for any $X \in P$ whose orbit is x. Taking B as the set of objects, and morphisms as just described, we have a category $\mathbf{B}$.

It is clear from the construction of morphisms and compositions in $\mathbf{B}$ that the quotient association

$$\pi : \mathbf{P} \to \mathbf{B}$$

is a functor. Moreover, again from the construction of $\mathbf{B}$ in terms of orbits of the action on $\mathbf{Z}$, the right action of $\mathbf{Z}$ on $\mathbf{P}$ is transitive on fibers. By hypothesis, the action is also free. Hence π, along with the action, specifies a principal categorical bundle. $\boxed{\text{QED}}$

We can revisit Example CG2 in light of the preceding theorem. For this we take $\mathbf{P}$ to be the category $\hat{\mathbf{K}}_0$ whose objects are the elements of a given group $\hat{K}$, and whose morphisms are all ordered pairs of elements of $\hat{K}$; let Z be a subgroup of $\hat{K} = \mathrm{Obj}(\hat{\mathbf{K}})$, acting by right multiplication on $\hat{K}$, and $\mathbf{Z}_d$ the categorical group in which the only morphisms are the identities $z \to z$ with z running over Z. Then, as in Example CG2, there is a category $\mathbf{K} = \hat{\mathbf{K}}/\mathbf{Z}_d$.

This provides a categorical principal bundle $\hat{\mathbf{K}} \to \mathbf{K}$ with structure categorical group $\mathbf{Z}_d$ (for this there is no need to assume Z is central).

Example P1. A traditional principal G-bundle $\pi : P \to B$ generates a categorical principal bundle in the following way. Take $\mathbf{P}$ to be the discrete category with object set P (in a discrete category the only morphisms are the identity morphisms), $\mathbf{B}$ to be the discrete category with object set B, and $\mathbf{G}_d$ to be the categorical group whose object set is G and whose only morphisms are the identity morphisms. Then we have a principal categorical bundle.

Example P2. A more interesting example is obtained again from a principal G-bundle $\pi : P \to B$, but with the categorical group (Example CG1) being $\mathbf{G}_0$, whose object set is G and for which there is a unique morphism $g_1 \to g_2$ for every $g_1, g_2 \in G$; we denote this morphism by (g_1, g_2). Take $\mathbf{B}$ to be the category with object set B and with morphisms being backtrack-erased paths on B. For $\mathbf{P}$ take the category whose object set is P and for which a morphism $p \to q$ is a triple

$$(p, q; \gamma),$$

where γ is any backtrack-erased path on B from $\pi(p)$ to $\pi(q)$. Define composition of morphisms in the obvious way, and define a right action of $\mathbf{G}$ on $\mathbf{P}$ by taking it to be the usual right action of G on P at the level of objects and by setting

$$(p, q; \gamma)(g_1, g_2) = (pg_1, qg_2; \gamma), \tag{5.8}$$

for all $p, q \in P$, all backtrack-erased paths γ on B from $\pi(p)$ to $\pi(q)$ and all morphisms (g_1, g_2) in $\mathbf{G}_0$. Defining the projection $\mathbf{P} \to \mathbf{B}$ to be π at the level of objects and $(p, q; \gamma) \mapsto \gamma$ at the level of morphisms yields then a principal categorical bundle.

Example P3. Next consider a connection $\overline{A}$ on a principal G-bundle $\pi : P \to B$. Let $\mathbf{B}$ have object set B, with arrows being backtrack-erased paths in B. Let $\mathbb{P}^{\mathrm{bt}}(P)^0_{\overline{A}}$ have objects the points of P, with arrows being backtrack erased $\overline{A}$-horizontal paths. There is then naturally the projection functor $\pi : \mathbf{P} \to \mathbf{B}$. The categorical group $\mathbf{G}_d$ whose objects form the group G, and whose morphisms are all the identity morphisms, has an obvious right action on $\mathbb{P}^{\mathrm{bt}}(P)^0_{\overline{A}}$, and thus we have a categorical principal bundle.

In the following section we will explore a more substantive example.

First, however, we explore the notion of reduction of a bundle. The basic example of a principal bundle is that of the frame bundle Fr_M of a manifold M: a typical point $p \in \mathrm{Fr}_M$ is a basis $(u_1, \ldots, u_n)$ of a tangent space T_mM to M at some point m, and $\pi : P \to M$ is defined by $\pi(p) = m$. The group $GL(\mathbb{R}^n)$ acts on the right on Fr_M by transformations of frames:

$$(u_1, \ldots, u_n) \cdot g = [u_1, \ldots, u_n] \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}.$$

With this structure $\pi : \mathrm{Fr}_M \to M$ is a principal $GL(\mathbb{R}^n)$-bundle. If M is equipped with a metric, say Riemannian, then we can specialize to orthonormal bases; two such bases are related by an orthogonal matrix. Let $O(n)$ be the subgroup consisting of orthogonal matrices inside $GL(\mathbb{R}^n)$. Then we have the bundle $\mathrm{OFr}_M \to M$ of orthonormal frames, and this is a principal $O(n)$-bundle. There is the natural 'inclusion' map $\mathrm{OFr}_M \to \mathrm{Fr}_M$, which preserves the principal bundle structures in the obvious way. The general notion here is that of 'reduction' of a principal bundle. Let $\pi : P \to M$ be a principal G-bundle and $\beta : G_o \to G$ be a homomorphism of Lie groups; then a *reduction* of π by β to G_o is a principal G_0-bundle $\pi_o : P_o \to M$ along with a smooth map

$$f : P_o \to P$$

that maps each fiber $\pi_o^{-1}(m)$ into the fiber $\pi^{-1}(m)$ and $f(pg) = f(p)\beta(g)$, for all $p \in P_o$ and $g \in G_o$.

We can define an analogous notion for principal categorical bundles. Suppose $\pi : \mathbf{P} \to \mathbf{B}$ is a principal categorical bundle with structure group $\mathbb{G}$, a categorical group. Next suppose $\mathbf{G}_o$ is a categorical group and

$$\beta : \mathbf{G}_o \to \mathbf{G}$$

is a functor that is a homomorphism on objects as well as morphisms. Then by a *reduction* of $\pi : \mathbf{P} \to \mathbf{B}$ by β we mean a principal categorical bundle $\pi_o : \mathbf{P}_o \to \mathbf{B}$, with structure group $\mathbf{G}_o$, along with a functor

$$f : \mathbf{P}_o \to \mathbf{P}$$

that is fiber preserving both on objects and morphisms and satisfies

$$f(pg) = f(p)\beta(g)$$

both when $(p, g) \in \mathrm{Obj}(\mathbf{P}_0) \times \mathrm{Obj}(\mathbf{G}_0)$ and when $(p, g) \in \mathrm{Mor}(\mathbf{P}_0) \times \mathrm{Mor}(\mathbf{G}_0)$. We will see an example of such a reduction in Proposition 6.1 in the next section.

6 A decorated principal categorical bundle

In this section we introduce a new notion, that of a decorating a principal categorical bundle. This construction will eventually (in section 10, specifically (10.10)) make it possible to start with an ordinary principal G-bundle $\pi : P \to M$ with connection and a categorical group $\mathbf{K}$, with object group G, and obtain a principal categorical bundle, over a space of paths on M, with structure categorical group $\mathbf{G}_2$ *whose object group* is $\mathrm{Mor}(\mathbf{K})$. We will continue to explore this construction in later sections, in the context of path spaces and in the abstract.

Let $\mathbf{K}$ be a categorical group and (G, H, α, τ) the corresponding crossed module. Recall that each morphism $\phi \in \mathrm{Mor}(\mathbf{K})$ can be identified with the ordered pair $(h, g) \in H \rtimes_\alpha G$ where $g = s(\phi)$ and $h = \phi 1_{s(\phi)^{-1}} \in H$, the subgroup of Mor($\mathbf{K}$) consisting of morphisms that have source the identity element e in $G = \text{Obj}(\mathbf{K})$. The composition law for morphisms translates to the product in H, as explained in (4.21).

Let $\mathbb{P}^{\text{bt}}(M)$ be the category whose objects are the points of M and whose morphisms are the backtrack erased piecewise C^1 paths on M. (Note that by 'backtrack-erased path' we mean an equivalence class of paths that are backtrack equivalent to each other.) Instead of working with piecewise C^1 paths we could also work with C^∞ paths that are constant near the initial and final points; the latter condition ensures that composites of such paths are C^∞.

We construct a category $\mathbb{P}^{\text{bt}}(P)^{\text{dec}}_{\overline{A}}$ by decorating the morphisms of the category in Example P3 in section 5 with elements of the group H. Specifically, we define the category $\mathbb{P}^{\text{bt}}(P)_{\overline{A}}$ as follows: (i) the object set is P; (ii) a morphism f is a pair $(\tilde{\gamma}, h)$, where $\tilde{\gamma}$ is a piecewise C^1 backtrack-erased $\overline{A}$-horizontal path on P, and $h \in H$ (corresponding to a morphism $\phi \in \text{Mor}(\mathbf{K})$ with $s(\phi) = e$), source and targets being specified by

$$s(\tilde{\gamma}, h) = s(\tilde{\gamma}) \quad \text{and} \quad t(\tilde{\gamma}, h) = t(\tilde{\gamma})\tau(h^{-1}). \tag{6.1}$$

Define composition of morphisms by

$$(\tilde{\gamma}_2, h_2) \circ (\tilde{\gamma}_1, h_1) = (\tilde{\gamma}_3, h_2 h_1), \tag{6.2}$$

where $\tilde{\gamma}_3$ is the composite of the path $\tilde{\gamma}_1$ with the right translate $\tilde{\gamma}_2\tau(h_1)$ (so that the final and initial points match correctly):

$$\tilde{\gamma}_3 = \tilde{\gamma}_2\tau(h_1) \circ \tilde{\gamma}_1. \tag{6.3}$$

Note that

$$t(\tilde{\gamma}_3, h_2 h_1) = t(\tilde{\gamma}_2)\tau(h_1)\tau(h_2 h_1)^{-1} = t(\tilde{\gamma}_2, h_2), \tag{6.4}$$

and the corresponding result for the sources is clear.

Then $\mathbb{P}^{\text{bt}}(P)^{\text{dec}}_{\overline{A}}$ is a category, and there is the functor

$$\pi : \mathbb{P}^{\text{bt}}(P)^{\text{dec}}_{\overline{A}} \to \mathbb{P}^{\text{bt}}(M) \tag{6.5}$$

that associates to each object $p \in P$ the object $\pi(p) \in M$, and to each morphism $(\tilde{\gamma}, h)$ associates the backtrack erased path $\pi \circ \tilde{\gamma}$.

Now we define a right action ρ of the categorical group $\mathbf{K}$ on $\mathbb{P}^{\text{bt}}(P)^{\text{dec}}_{\overline{A}}$. At the level of objects this is simply the usual right action of G on P. For morphisms we define the action as follows. Recall that a morphism in $\mathbf{K}$ has the form (h_1, g_1), where

$$s(h_1, g_1) = g_1, \quad t(h_1, g_1) = \tau(h_1)g_1.$$

Then for $(\tilde{\gamma}, h), \in \text{Mor}(\mathbb{P}^{\text{bt}}(P)_{\overline{A}})$ and $(h_1, g_1) \in \text{Mor}(\mathbf{K})$, we define

$$\rho\left((\tilde{\gamma}, h), (h_1, g_1)\right) = (\tilde{\gamma}, h) \cdot (g_1, h_1) = \left(\tilde{\gamma} g_1, \alpha(g_1^{-1})(h_1^{-1} h)\right), \tag{6.6}$$

where, on the right, $\tilde{\gamma} g_1$ is the usual right-translate of $\tilde{\gamma}$ by g_1.

The following computation shows that (6.6) does define a right action:

$$
\begin{aligned}
[(\tilde{\gamma}, h)\cdot(h_1,g_1)]\cdot(h_2,g_2) &= \big(\tilde{\gamma}g_1, \alpha(g_1^{-1})(h_1^{-1}h)\big)\cdot(h_2,g_2)\\
&= \big(\tilde{\gamma}g_1g_2,\ \alpha(g_2^{-1})\left(h_2^{-1}\alpha(g_1^{-1})(h_1^{-1}h)\right)\big)\\
&= \big(\tilde{\gamma}g_1g_2, \alpha((g_1g_2)^{-1})(\alpha(g_1)(h_2^{-1})h_1^{-1}h)\big)\\
&= (\tilde{\gamma}, h)\cdot(h_1\alpha(g_1)(h_2), g_1g_2)\\
&= (\tilde{\gamma}, h)\cdot[(h_1,g_1)\cdot(h_2,g_2)] \qquad \text{(using (4.22))}.
\end{aligned}
\tag{6.7}
$$

From the definition (6.6) we see directly that $(\tilde{\gamma}, h)\cdot(h_1, g_1)$ is equal to $(\tilde{\gamma}, h)$ if and only if $g_1 = e$ and $h_1 = e$, and so the action is free. It is also readily checked that the action is transitive on fibers: suppose $(\tilde{\gamma}_0, h_0)$ and $(\tilde{\gamma}, h)$ project to γ; then $\tilde{\gamma}_0 = \tilde{\gamma}g_1$ for a unique $g_1 \in G$, and then on taking

$$h_1 = h\alpha(g_1)(h_0^{-1}),$$

we have

$$(\tilde{\gamma}_0, h_0) = (\tilde{\gamma}, h)\cdot(h_1, g_1).$$

For computational purposes we switch back to the regular morphism notation: every morphism in $\mathbb{P}^{\mathrm{bt}}(P)^{\mathrm{dec}}_{\overline{A}}$ is of the form

$$(\tilde{\gamma}, \theta) \in \mathrm{Mor}(\mathbb{P}^{\mathrm{bt}}(P)_{\overline{A}}) \times \ker s \subset \mathrm{Mor}(\mathbb{P}^{\mathrm{bt}}(P)_{\overline{A}}) \times \mathrm{Mor}(\mathbf{K})$$

where $s(\theta) = e$, the identity in $\mathrm{Obj}(\mathbf{K})$. The source and targets are

$$s(\tilde{\gamma}, \theta) = s(\tilde{\gamma}) \quad \text{and} \quad t(\tilde{\gamma}, \theta) = t(\tilde{\gamma})t(\theta)^{-1}. \tag{6.8}$$

We will now check that the expression for the action (6.6) in the morphism notation is

$$(\tilde{\gamma}, \theta)\cdot\phi = (\tilde{\gamma}s(\phi), \phi^{-1}\theta 1_{s(\phi)}) \tag{6.9}$$

where $\phi^{-1} : a^{-1} \to b^{-1}$ is the multiplicative inverse of $\phi : a \to b \in \mathrm{Mor}(\mathbf{K})$ (we rarely need to use the compositional inverse $b \to a$ and avoid introducing a new notation relying on the context to make the intended meaning clear). To make the comparison we take

$$\theta = h,$$

and we recall from (4.12) that a morphism ϕ of $\mathrm{Mor}(\mathbf{K})$ is of the form

$$(h_1, g_1) = \Big(\phi 1_{s(\phi)^{-1}}, s(\phi)\Big). \tag{6.10}$$

It is clear that the first component on the right side in (6.9) corresponds to the first component on the right side in (6.6). For the second component we have

$$
\begin{aligned}
\alpha(g_1)^{-1}(h_1^{-1}h) &= \alpha(s(\phi)^{-1})\Big(1_{s(\phi)}\phi^{-1}\theta\Big)\\
&= 1_{s(\phi)^{-1}}\Big(1_{s(\phi)}\phi^{-1}\theta\Big)1_{s(\phi)} \quad \text{(using (4.9))}\\
&= \phi^{-1}\theta 1_{s(\phi)},
\end{aligned}
\tag{6.11}
$$

which is exactly the first component on the right in (6.9). Thus the right action given by (6.9) is the same as that given by (6.6).

The source and target assignments behave functorially:

$$\begin{aligned} s\left((\tilde\gamma,\theta)\cdot\phi\right) &= s\left(\tilde\gamma s(\phi), \phi^{-1}\theta 1_{s(\phi)},\right) = s(\tilde\gamma)s(\phi) = s(\tilde\gamma,\theta)s(\phi) \\ t\left((\tilde\gamma,\theta)\cdot\phi\right) &= t\left(\tilde\gamma s(\phi), \phi^{-1}\theta 1_{s(\phi)}\right) \\ &= t(\tilde\gamma)s(\phi)t\big(\phi^{-1}\theta 1_{s(\phi)}\big)^{-1} = t(\tilde\gamma)t(\theta)^{-1}t(\phi) \\ &= t(\tilde\gamma,\theta)t(\phi). \end{aligned} \tag{6.12}$$

Next we check that (6.9) does specify a right action:

$$\begin{aligned} \big((\tilde\gamma,\theta)\cdot\phi_1\big)\cdot\phi_2 &= \big(\tilde\gamma s(\phi_1), \phi_1^{-1}\theta 1_{s(\phi_1)}\big)\cdot\phi_2 \\ &= \big(\tilde\gamma s(\phi_1)s(\phi_2), \phi_2^{-1}\phi_1^{-1}\theta 1_{s(\phi_1)}1_{s(\phi_2)}\big) \\ &= (\tilde\gamma,\theta)\cdot(\phi_1\phi_2). \end{aligned} \tag{6.13}$$

(We have already proved this in (6.7) in terms of the crossed module (G, H, α, t).)

The composition law (6.2) reads

$$(\tilde\gamma_2,\theta_2)\circ(\tilde\gamma_1,\theta_1) = (\tilde\gamma_2 t(\theta_1)\circ\tilde\gamma_1, \theta_2\theta_1), \tag{6.14}$$

whose target is clearly the same as the target of $(\tilde\gamma_2,\theta_2)$ (note that the composite on the right in (6.14) is defined). We check associativity:

$$\begin{aligned} (\tilde\gamma_3,\theta_3)\circ\left((\tilde\gamma_2,\theta_2)\circ(\tilde\gamma_1,\theta_1)\right) &= (\tilde\gamma_3,\theta_3)\circ(\tilde\gamma_2 t(\theta_1)\circ\tilde\gamma_1, \theta_2\theta_1) \\ &= (\tilde\gamma_3 t(\theta_2\theta_1)\circ\tilde\gamma_2 t(\theta_1)\circ\tilde\gamma_1\,,\, \theta_3\theta_2\theta_1), \end{aligned} \tag{6.15}$$

while

$$\begin{aligned} \left((\tilde\gamma_3,\theta_3)\circ(\tilde\gamma_2,\theta_2)\right)\circ(\tilde\gamma_1,\theta_1) &= (\tilde\gamma_3 t(\theta_2)\circ\tilde\gamma_2, \theta_3\theta_2)\circ(\tilde\gamma_1,\theta_1) \\ &= \big(\big(\tilde\gamma_3 t(\theta_2)\circ\tilde\gamma_2\big)t(\theta_1)\circ\tilde\gamma_1\,,\, \theta_3\theta_2\theta_1\big), \end{aligned} \tag{6.16}$$

in agreement with the last expression in (6.15). The existence of identity morphisms is readily verified.

We can now check functoriality of the right action by examining

$$\left((\tilde\gamma_2,\theta_2)\circ(\tilde\gamma_1,\theta_1)\right)\cdot(\phi_2\circ\phi_1),$$

wherein note that the composability of ϕ_2 with ϕ_1 means that

$$s(\phi_2) = t(\phi_1). \tag{6.17}$$

Composing and then acting produces:

$$\begin{aligned} \left((\tilde\gamma_2,\theta_2)\circ(\tilde\gamma_1,\theta_1)\right)\cdot(\phi_2\circ\phi_1) &= (\tilde\gamma_2 t(\theta_1)\circ\tilde\gamma_1, \theta_2\theta_1)\cdot(\phi_2\circ\phi_1) \\ &= \big(\big(\tilde\gamma_2 t(\theta_1)\circ\tilde\gamma_1\big)s(\phi_1), (\phi_2\circ\phi_1)^{-1}\theta_2\theta_1 1_{s(\phi_1)}\big), \end{aligned} \tag{6.18}$$

while first acting and then composing produces:

$$
\begin{aligned}
&(\tilde{\gamma}_2,\theta_2)\cdot\phi_2 \;\circ\; (\tilde{\gamma}_1,\theta_1)\cdot\phi_1 = \\
&\big(\tilde{\gamma}_2 s(\phi_2), \phi_2^{-1}\theta_2 1_{s(\phi_2)}\big)\circ\big(\tilde{\gamma}_1 s(\phi_1), \phi_1^{-1}\theta_1 1_{s(\phi_1)}\big) \\
&= \left(\tilde{\gamma}_2 \underbrace{s(\phi_2)[t(\phi_1)^{-1}t(\theta_1)s(\phi_1)]}_{t(\theta_1)s(\phi_1)} \circ \tilde{\gamma}_1 s(\phi_1)\,,\; \phi_2^{-1}\theta_2 1_{s(\phi_2)}\phi_1^{-1}\theta_1 1_{s(\phi_1)}\right)
\end{aligned}
\tag{6.19}
$$

(where in the last step we used $s(\phi_2) = t(\phi_1)$) which clearly agrees, in the second entry, with the right side of (6.18); agreement in the first entry follows using the identities:

$$
\theta_2^{-1}(\phi_2\circ\phi_1) \stackrel{(4.3)}{=} \theta_2^{-1}\phi_1 1_{t(\phi_1)^{-1}}\,\phi_2 \stackrel{(4.4)}{=} \phi_1 1_{t(\phi_1)^{-1}}\theta_2^{-1}\phi_2. \tag{6.20}
$$

We have thus proved the following.

Theorem 6.1 *Let (G,H,α,τ) be a Lie crossed module corresponding to a categorical group $\mathbf{K}$, $\overline{A}$ a connection form on a principal G-bundle $\pi: P\to M$. Let $\mathbb{P}^{\rm bt}(M)$ and $\mathbb{P}^{\rm bt}(P)^{\rm dec}_{\overline{A}}$ be the categories constructed above, and $\pi:\mathbb{P}^{\rm bt}(P)^{\rm dec}_{\overline{A}}\to\mathbb{P}^{\rm bt}(M)$ the functor given in (6.5). Then $\pi:\mathbb{P}^{\rm bt}(P)^{\rm dec}_{\overline{A}}\to\mathbb{P}^{\rm bt}(M)$, along with the right action of $\mathbf{K}$ on $\mathbf{P}$ defined in (6.6) is a principal categorical bundle.*

In section 7 (equation (7.27)) we will construct a categorical principal bundle

$$\pi:\mathbf{P}^{\rm dec}_{\overline{\mathbf{A}}}\to\mathbf{B}$$

starting from a principal bundle $\mathbf{P}\to\mathbf{B}$ and some additional data. This will generalize the specific construction of Theorem 6.1 to provide a 'decorated' version of a given categorical principal bundle $\mathbf{P}\to\mathbf{B}$.

We note that the decoration process is not the same as an associated bundle construction (which we consider in section 11), where a quotient is taken of the product of a principal bundle and a space on which the structure group acts.

Now recall that from the original principal bundle $\pi: P\to M$ and the connection $\overline{A}$ we also have an undecorated principal categorical bundle $\mathbb{P}^{\rm bt}(P)^{0}_{\overline{A}}$ for which the categorical structure group $\mathbf{G}_d$ has object set G and all the morphisms are the identity morphisms; the morphisms of $\mathbb{P}^{\rm bt}(P)^{0}_{\overline{A}}$ are simply the $\overline{A}$-horizontal backtrack-erased paths in P. The following result, whose proof is quite clear, is worth noting.

Proposition 6.1 *Let $\mathbf{G}$ be a categorical group corresponding to a Lie crossed module (G,H,α,τ), and $\mathbf{G}_d$ be the categorical group whose object set is G and all of whose morphisms are the identity morphisms. Let*

$$R_d:\mathbf{G}_d\to\mathbf{G}$$

be the identity map on objects and the inclusion map on morphisms. Let $\overline{A}$ be a connection on a principal G-bundle $\pi : P \to M$, and let $\mathbb{P}^{\rm bt}(P)^{\rm dec}_{\overline{A}}$ and $\mathbb{P}^{\rm bt}(P)^{0}_{\overline{A}}$ be the principal categorical bundles described above. Consider the association

$$R : \mathbb{P}^{\rm bt}(P)^{0}_{\overline{A}} \to \mathbb{P}^{\rm bt}(P)^{\rm dec}_{\overline{A}} \tag{6.21}$$

that, at the level of objects, is the identity map on P and for morphisms is given by

$$R(\tilde{\gamma}) = (e_H, \tilde{\gamma}),$$

where e_H is the identity element in H. Then R is a reduction by R_d, in the sense that R maps each fiber into itself, both on objects and on morphisms, and

$$R(\tilde{\gamma}\phi) = R(\tilde{\gamma})R_d(\phi)$$

for all $\tilde{\gamma} \in {\rm Mor}\left(\mathbb{P}^{\rm bt}(P)^{0}_{\overline{A}}\right)$ and $\phi \in {\rm Mor}(\mathbf{G}_d)$.

7 Categorical connections

By a *connection* $\mathbf{A}$ on a principal categorical bundle $\pi : \mathbf{P} \to \mathbf{B}$, with structure categorical group $\mathbf{K}$, we mean a prescription for lifting morphisms in $\mathbf{B}$ to morphisms in $\mathbf{P}$. More specifically, for each $p \in {\rm Obj}(\mathbf{P})$ and morphism $\gamma \in {\rm Mor}(\mathbf{B})$, with source $\pi(p)$, a connection specifies a morphism $\gamma_p^{\rm hor} : p \to q$, for some q with $\pi(q) = t(\gamma)$; we call the lift $\gamma_p^{\rm hor}$ the *lift* of γ *through* p. Of course, we require

$$\pi\left(\gamma_p^{\rm hor}\right) = \gamma,$$

and the lifting should be functorial: if ζ is also a morphism with source $\pi(q)$ then the lift of $\zeta \circ \gamma$ through p is

$$\zeta_q^{\rm hor} \circ \gamma_p^{\rm hor}. \tag{7.1}$$

Furthermore, we require that a 'rigid vertical motion' of a horizontal morphism produces a horizontal morphism:

$$\gamma_p^{\rm hor} 1_g = \gamma_{pg}^{\rm hor} \tag{7.2}$$

for all $g \in {\rm Obj}(\mathbf{G})$. Identifying morphisms of $\mathbf{K}$ with pairs $(h, g_1) \in H \rtimes_\alpha G$, this condition reads

$$\gamma_{pg}^{\rm hor} = \gamma_p^{\rm hor} \cdot (e, g) \tag{7.3}$$

The morphism $\gamma_p^{\rm hor}$ will be called the $\mathbf{A}$-*horizontal lift* of γ through p. Such morphisms will be called $\mathbf{A}$-*horizontal morphisms.*

In the definition of a connection, as given above, we have imposed a minimal set of rules, to reflect just the situation for traditional bundles and connections. (A general morphism $\phi \in {\rm Mor}(\mathbf{K})$ would act on a morphism $\gamma_p^{\rm hor} : p \to q$ and produce a morphism $pg_1 \to qg_2$, where $g_1 = s(\phi)$ and $g_2 = t(\phi)$. There is no reason to require that this be horizontal or special in any way.) At first it seems to be too minimal, not involving much role of the morphisms of the structure categorical group. To bring out the richness and flexibility in the definition we will develop several examples.

The simplest example arises from an ordinary connection on an ordinary principal bundle $\pi : P \to M$ with structure group G; this is explained in Example CC1 below. To really see the richness of the categorical notion of connection, and how the morphism group $\mathrm{Mor}(\mathbf{G})$ of a categorical group $\mathbf{G}$ plays a role, we need to pass to decorated bundles obtained from $\pi : P \to M$ over the path space of M. We summarize the essence of the idea in a compact way here. We start with a principal categorical bundle $\pi : \mathbf{P} \to \mathbf{B}$, with a structure categorical group $\mathbf{G}$, associated to a crossed module (G, H, α, τ), and some additional structure. We construct a decorated bundle, whose objects are morphisms of $\mathbf{P}$ decorated with elements of H. Next we will construct a principal categorical bundle $\mathbf{P}_2 \to \mathbf{B}_2$ whose structure categorical group $\mathbf{G}_2$ has as objects the morphisms of $\mathbf{G}$:

$$\mathrm{Obj}(\mathbf{G}_2) = \mathrm{Mor}(\mathbf{G}). \tag{7.4}$$

The relation (7.4) means that now we have a principal categorical bundle whose structure categorical group $\mathbf{G}_2$ has as objects the morphisms of the original structure categorical group $\mathbf{G}$. *The objects of* $\mathbf{P}_2$ *are the morphisms of* $\mathbf{P}_1$*, and so it is* $\mathrm{Mor}(\mathbf{G})$ *that now plays the role that* $\mathrm{Obj}(\mathbf{G})$ *did for* $\mathbf{P}$. Thus, *a connection on this new bundle will have behavior controlled by the morphisms of* $\mathbf{G}$. Details of this idea are developed through Theorem 7.1, section 10 (for example, in the context of equation (10.8)). Thus the morphism group of a categorical group can play a significant role in the setting of bundles over path spaces.

Example CC1. The simplest example is the usual connection on a principal bundle. For this, let $\overline{A}$ be a connection on a principal bundle $\pi : P \to B$. Let $\mathbf{P}_0$ be the category whose object set is P and for which a morphism $p \to q$ is a triple $(p, q; \gamma)$, where γ is any backtrack-erased path from $\pi(p)$ to $\pi(q)$ on B. We take $\mathbf{B}$ to be the category with object set B and morphisms being backtrack-erased paths. Let $\mathbf{G}_0$ have object set G, and have a unique morphism $g_1 \to g_2$, denoted (g_1, g_2), for every $g_1, g_2 \in G$. The action of $\mathbf{G}_0$ on $\mathbf{P}_0$ is as described in (5.8), and then we have a principal categorical bundle $\pi : \mathbf{P}_0 \to \mathbf{B}$. For any $p \in P$ and any parametrized piecewise C^1 path γ on B, corresponding to a morphism of $\mathbf{B}$ with source $\pi(p)$, let $\tilde{\gamma}_p$ be the morphism of $\mathbf{P}_0$ specified by the path γ, the source point p and the target is the point of P obtained by $\overline{A}$-parallel-transporting p along γ to its end. Note that this target, and hence $\tilde{\gamma}_p$, is determined by the backtrack-erased form of γ along with p. Thus, we have a connection $\overline{\mathbf{A}_0}$ on the principal categorical bundle $\mathbf{P}_0 \to \mathbf{B}$.

Example CC2. Let $\overline{A}$ be a connection form on a principal G-bundle $\pi : P \to M$, and $\mathbf{K}$ a categorical Lie group with Lie crossed module (G, H, α, τ). We now describe a categorical connection on the principal categorical bundle $\pi : \mathbb{P}^{\mathrm{bt}}(P)^{\mathrm{dec}}_{\overline{A}} \to \mathbb{P}^{\mathrm{bt}}(M)$. Let $p \in P$ and γ a parametrized piecewise C^1 path on M with initial point $\pi(p)$. Define the lift of the morphism γ, with source p, to be

$$(e_H, \tilde{\gamma}_p),$$

where $\tilde{\gamma}_p$ is the backtrack-erased form of the $\overline{A}$-horizontal lift of γ initiating at p, and e_H is the identity element in H.

We have the following more general construction of a connection on $\mathbb{P}^{\mathrm{bt}}(P)^{\mathrm{dec}}_{\overline{A}}$. Suppose (G, H, α, τ) is a Lie crossed module corresponding to a categorical Lie group $\mathbf{K}$, let $\overline{A}$ be a connection on a principal G-bundle $P \to M$, and let C be an $L(H)$-valued C^∞ 1-form on P that is α-equivariant in the sense that

$$C_{pg}(vg) = \alpha(g^{-1})C_p(v)$$

for all $p \in P$, $g \in G$, and $v \in T_pP$, where $pg = R_g(p)$ is from the right action of G on P, and $vg = R'_g(p)v$.

Theorem 7.1 *We use notation and hypotheses as above. For any backtrack-erased path $\gamma : [t_0, t_1] \to M$, and any point u on the fiber over $s(\gamma) = \gamma(t_0)$, define*

$$\gamma_u^{\mathrm{hor}} = (\tilde{\gamma}_u, h_u(\gamma)), \tag{7.5}$$

where $\tilde{\gamma}_u$ is the $\overline{A}$-horizontal lift of γ through u and $h_u(\gamma)$ is the final point of the path $[t_0, t_1] \to H : t \mapsto h(t)$ satisfying the differential equation

$$h(t)^{-1}h'(t) = -C\big(\tilde{\gamma}'_u(t)\big) \tag{7.6}$$

at all $t \in [t_0, t_1]$ where $\gamma'(t)$ exists, with initial point $h(t_0) = e$, the identity in H. Then this lifting process defines a connection on the categorical principal bundle $\pi : \mathbb{P}^{\mathrm{bt}}(P)^{\mathrm{dec}}_{\overline{A}} \to \mathbb{P}^{\mathrm{bt}}(M)$.

The connection $\omega_{(A,B)}$ over path space that we have discussed before in (2.14) was determined by an $L(G)$-valued 1-form A and by $\tau(B)$, where B is an $L(H)$-valued 2-form. We see now that $L(H)$-valued forms such as C, not intermediated by τ, play a role in connections on decorated bundles.

Proof. Replacing u by ug in (7.6), for any $g \in G$, is equal to applying $\alpha(g^{-1})$, and so, by the uniqueness of a solution of the differential equation, the solution $h(t)$ gets replaced by $\alpha(g^{-1})h(t)$ (recall that $\alpha(g)$ is an automorphism of H and $(g, h) \mapsto \alpha(g^{-1}, h)$ is smooth). Hence

$$h_{ug}(\gamma) = \alpha(g^{-1})h_u(\gamma), \tag{7.7}$$

and so

$$(\tilde{\gamma}_u, h_u(\gamma)) \cdot (e, g) = \big(\tilde{\gamma}_u g, \alpha(g^{-1})h_u(\gamma)\big) = (\tilde{\gamma}_{ug}, h_{ug}(\gamma)), \tag{7.8}$$

confirming the property (7.3).

Now consider backtrack-erased paths γ_1 and γ_2 on M for which the composite $\gamma_2 \circ \gamma_1$ exists, and let u be in the fiber above $s(\gamma_1)$. Then

$$\gamma_u^{\mathrm{hor}} = (\tilde{\gamma}_{1,u}, h_1),$$

where

$$h_1 = h_u(\gamma_1), \tag{7.9}$$

has target

$$v' = t\left(\gamma_{1,u}^{\rm hor}\right) = v\tau(h_1)^{-1}, \tag{7.10}$$

on using (6.1), where

$$v \stackrel{\rm def}{=} t\left(\tilde{\gamma}_{1,u}\right) \tag{7.11}$$

is the endpoint of $\tilde{\gamma}_{1,u}$. The $\overline{A}$-horizontal lift of γ_2 with initial point v' is

$$\tilde{\gamma}_{2,v'} = \tilde{\gamma}_{2,v}\tau(h_1)^{-1}.$$

Next note that

$$\gamma_{2,v}^{\rm hor} = (\tilde{\gamma}_{2,v}, h_2) \tag{7.12}$$

where

$$h_2 = h_v(\gamma_2). \tag{7.13}$$

Then using (7.10) , (7.8) and the formula for the right action given in (6.6), we have

$$\begin{aligned} \gamma_{2,v'}^{\rm hor} &= \gamma_{2,v}^{\rm hor} \cdot (e, \tau(h_1)^{-1}) \\ &= \left(\tilde{\gamma}_{2,v}\tau(h_1)^{-1}\,,\, h_2'\right), \end{aligned} \tag{7.14}$$

where

$$h_2' = \alpha(\tau(h_1))h_2. \tag{7.15}$$

Composing with the lift of γ_1 we have

$$\gamma_{2,v'}^{\rm hor} \circ \gamma_{1,u}^{\rm hor} = \left(\tilde{\gamma}_{2,v} \circ \tilde{\gamma}_{1,u}, h_2'h_1\right). \tag{7.16}$$

The second term here is clearly

$$\tilde{\gamma}_{2,v} \circ \tilde{\gamma}_{1,u} = \widetilde{(\gamma_2 \circ \gamma_1)}_u.$$

The first term is

$$h_2'h_1 = h_1h_2h_1^{-1}\,h_1 = h_1h_2.$$

From the differential equation (7.6) we know that any solution when left multiplied by a constant term in H is again a solution, just with a new initial condition. Let γ_1 have parameter domain $[t_0, t_1]$ and γ_2 have $[t_1, t_2]$. Hence h_1h_2 is the terminal value $a(t_2)$ of the solution $t \mapsto a(t) \in H$ of

$$a(t)^{-1}a'(t) = -C\left(\tilde{\gamma}_{2,v}'(t)\right) \quad \text{for } t \in [t_1, t_2],$$

with initial value h_1. But h_1 itself is the terminal value of the solution of

$$a(t)^{-1}a'(t) = -C\left(\tilde{\gamma}_{1,u}'(t)\right) \quad \text{for } t \in [t_0, t_1],$$

with initial value $a(t_0) = e \in H$. Thus, fitting these two differential equations into one, we see that h_1h_2 is the terminal value $a(t_2)$ of the solution $a(\cdot)$ of the equation

$$a(t)^{-1}a'(t) = -C\big(\tilde{\gamma}'_u(t)\big) \quad \text{for } t \in [t_0, t_2],$$

where $\gamma = \gamma_2 \circ \gamma_1$. Here γ, its $\overline{A}$-horizontal lift $\tilde{\gamma}$, and $a(\cdot)$ are piecewise C^1.

Hence

$$h'_2h_1 = h_1h_2 = a(t_2) = h_u(\gamma_2 \circ \gamma_1). \tag{7.17}$$

Recalling the definitions of h_1 and h_2 from (7.9) and (7.13), let us write this explicitly (for future reference and use) as

$$h_u(\gamma_2 \circ \gamma_1) = h_u(\gamma_1)h_v(\gamma_2). \tag{7.18}$$

(We will prove a general version of this observation in Proposition 8.1.) Returning to (7.16) we conclude that

$$\gamma^{\rm hor}_{2,v'} \circ \gamma^{\rm hor}_{1,u} = (\tilde{\gamma}_u, h_u(\gamma)) = \gamma^{\rm hor}_u, \tag{7.19}$$

where again $\gamma = \gamma_2 \circ \gamma_1$.

Thus $\gamma \mapsto \gamma^{\rm hor}_u$ takes composites to composites. QED

A special case of the preceding construction is obtained by taking C of the type

$$(d\Phi)\Phi^{-1},$$

for some smooth α-equivariant function $\Phi : P \to H$. In this case the horizontal lift is given by

$$h^{\rm hor}_u(\gamma) = \big(\tilde{\gamma}_u, \Phi(v)\Phi(u)^{-1}\big), \tag{7.20}$$

with usual notation, writing v for the endpoint of $\tilde{\gamma}_u$.

Let $\mathbf{A}_0$ be a connection on a principal categorical bundle $\pi : \mathbf{P} \to \mathbf{B}$ with structure categorical group $\mathbf{G}_0$. We will now describe an abstract form of the construction used for $\mathbb{P}^{\rm bt}(P)^{\rm dec}_{\overline{A}}$ in Theorem 7.1.

Consider the category $\mathbf{P}_{\mathbf{A}_0}$ whose object set is $\mathrm{Obj}(\mathbf{P})$ and morphisms are $\mathbf{A}_0$-horizontal morphisms. The discrete categorical group $\mathbf{G}_{0d}$, whose object group is $G = \mathrm{Obj}(\mathbf{G}_0)$, acts on the right on $\mathbf{P}_{\mathbf{A}_0}$ by restriction of the original action of $\mathbf{G}_0$ on $\mathbf{P}$ as stated in (7.2). Thus, restricting the projection functor π in the obvious way, we see that

$$\pi : \mathbf{P}_{\mathbf{A}_0} \to \mathbf{B} \tag{7.21}$$

is a principal categorical bundle with structure group $\mathbf{G}_{0d}$. Now consider a categorical Lie group $\mathbf{G}_1$ whose object group is G. We will construct a principal categorical bundle with structure categorical group $\mathbf{G}_1$ by suitably decorating the morphisms of $\mathbf{P}_{\mathbf{A}_0}$. Let (G, K, α, τ) be the crossed module corresponding to $\mathbf{G}_1$. For the decorated version of $\mathbf{P}_{\mathbf{A}_0}$ we take as objects just the objects of $\mathbf{P}$, and as morphisms

$$(\tilde{\gamma}, h) \in \mathrm{Mor}(\mathbf{P}_{\mathbf{A}_0}) \times K,$$

where $K = \ker s_1$, with s_1 being the source map on the morphisms of $\mathbf{G}_2$. We define source and targets by

$$s(\tilde{\gamma}, h) = s(\tilde{\gamma}) \quad \text{and} \quad t(\tilde{\gamma}, h) = t(\tilde{\gamma})t_1(h)^{-1}, \tag{7.22}$$

and composition by

$$(\tilde{\gamma}_2, h_2) \circ (\tilde{\gamma}_1, h_1) = \left(\tilde{\gamma}_2 1_{\tau(h_1)} \circ \tilde{\gamma}_1, h_2 h_1\right). \tag{7.23}$$

Let

$$\mathbf{P}^{\mathrm{dec}}_{\mathbf{A}_0} \tag{7.24}$$

be the category thus defined. Next we define an action of $\mathbf{G}_1$ on $\mathbf{P}^{\mathrm{dec}}_{\mathbf{A}_0}$ by

$$(\tilde{\gamma}, h) \cdot (h_1, g_1) = \left(\tilde{\gamma} 1_{g_1}, \alpha(g_1^{-1})(h_1^{-1} h)\right), \tag{7.25}$$

for all $(h_1, g_1) \in K \times G$, or, equivalently,

$$(\tilde{\gamma}, h) \cdot \phi = \left(\tilde{\gamma} 1_{s(\phi)}, \phi^{-1} h 1_{s(\phi)}\right). \tag{7.26}$$

Then

$$\pi : \mathbf{P}^{\mathrm{dec}}_{\mathbf{A}_0} \to \mathbf{B} \tag{7.27}$$

is a principal categorical bundle with structure group $\mathbf{G}_1$. The proof is the same as for Theorem 6.1. Thus we have constructed a 'decorated' version of a given principal categorical bundle with a given categorical connection.

Let $\tilde{\gamma}_u$ denote, as usual, the $\mathbf{A}_0$-horizontal lift of $\gamma \in \mathrm{Mor}(\mathbf{B})$ with source u. Now suppose k^* is a map from $\mathrm{Mor}(\mathbf{P}_{\mathbf{A}_1})$ to K satisfying:

$$\begin{aligned} k^*(\tilde{\gamma}_u 1_{g_1}) &= \alpha(g_1^{-1}) k^*(\tilde{\gamma}_u) \\ k^*\left(\tilde{\delta}_v \circ \tilde{\gamma}_u\right) &= k^*\left(\tilde{\gamma}_u\right) k^*\left(\tilde{\delta}_v\right) \end{aligned} \tag{7.28}$$

for all $\mathbf{A}_0$-horizontal lift $\tilde{\gamma}_u, \tilde{\delta}_v$ for which $\tilde{\gamma}_v \circ \tilde{\delta}_u$ is defined and for all $g_1 \in G$. (The notation k^* is not meant to suggest any type of 'pullback'.) The first equality corresponds to (7.7). The second equality above corresponds to the equality (7.18) noted earlier.

Define the horizontal lift of $\gamma \in \mathbf{B}$ with initial point u on the fiber over $s(\gamma)$ to be

$$h_u^{\mathrm{hor}}(\gamma) = \left(\tilde{\gamma}_u, k^*(\tilde{\gamma}_u)\right), \tag{7.29}$$

where $\tilde{\gamma}_u$ is the $\mathbf{A}_0$-horizontal lift of γ with source u.

We summarize this construction in the following.

Theorem 7.2 *Let $\mathbf{A}_0$ be a connection on a principal categorical bundle $\pi : \mathbf{P} \to \mathbf{B}$ with structure categorical group $\mathbf{G}_0$, and let*

$$\mathbf{P}_{\mathbf{A}_0} \to \mathbf{B}$$

be the corresponding categorical bundle where the morphisms of $\mathbf{P}_{\mathbf{A}_0}$ are the $\mathbf{A}_0$-horizontal morphisms of $\mathbf{P}$. Let (G, K, α, τ) be the crossed module corresponding *to a categorical group $\mathbf{G}_1$, where the object group of $\mathbf{G}_1$ is the same as the object group G of $\mathbf{G}_0$. Then the construction described for (7.27) produces a categorical principal bundle*

$$\pi : \mathbf{P}^{\mathrm{dec}}_{\mathbf{A}_0} \to \mathbf{B}, \tag{7.30}$$

with structure group $\mathbf{G}_1$.

Let $k^* : \mathrm{Mor}(\mathbf{P}_{\mathbf{A}_0}) \to K$ satisfy the conditions (7.28). Then $h^{\mathrm{hor}} : \mathrm{Mor}(\mathbf{B}) \to \mathrm{Mor}(\mathbf{P}_{\mathbf{A}_0})$ defined by (7.29) specifies a categorical connection $\mathbf{A}_1$ on the categorical principal bundle (7.30).

We omit the proof, which is a straightforward abstract reformulation of the first part of the proof of Theorem 7.1.

8 Path categories

In this section we set up some definitions and conventions for categories whose morphisms are paths in some spaces. For the sake of avoiding technicalities that obscure the main ideas we will focus only on paths that are C^∞ and constant near the endpoints. The latter condition makes it possible to compose two paths without losing smoothness at the interface of composition between the paths.

Our first focus is on a generalization of path spaces. Let X be a smooth manifold. Consider a 'box'

$$I = \prod_{k=1}^{N} [a_k, b_k] \subset \mathbb{R}^N,$$

where N is some positive integer, and $a_k, b_k \in \mathbb{R}$ with $a_k < b_k$. We denote by

$$C_c^\infty(I; X)$$

the set of all C^∞ maps $I \to X$ with the property that there is some $\epsilon > 0$ such that for any $u_i \in [a_i, b_i]$ for $i \in \{1, \dots, N\} - \{k\}$, the function $u_k \mapsto f(u_1, \dots, u_N)$ is constant when $|u_k - a_k| < \epsilon$ and when $|u_k - b_k| < \epsilon$.

Our interest in the following discussion is in the cases $N = 1$ and $N = 2$.

Let I_0 be the 'lower face' of I, and I_1 the 'upper face' :

$$I_0 = \left(\prod_{k=1}^{N-1} [a_k, b_k]\right) \times \{a_N\} \quad \text{and} \quad I_1 = \left(\prod_{k=1}^{N-1} [a_k, b_k]\right) \times \{b_N\}. \tag{8.1}$$

Note that if $f \in C_c^\infty(I; X)$ then the restrictions $f|I_0$ and $f|I_1$ specify elements in $C_c^\infty(\prod_{k=1}^{N-1}[a_k, b_k]; X)$, and we think of these elements as the 'initial' (source) and 'final' (target) values of f. Thus, we think of $f \in C_c^\infty(I; X)$ as a 'morphism' from its source $s(f)$ to its target $t(f)$:

$$s(f)(x) = f(x, a_N) \quad \text{and} \quad t(f)(x) = f(x, b_N). \tag{8.2}$$

Now consider a second box $J \subset \mathbb{R}^N$, whose lower face J_0 is the upper face I_1 of I:

$$J_0 = I_1.$$

In particular, the largest N-th coordinate for points in I is equal to the smallest N-th coordinate for J and $I \cup J$ is a box in $\mathbb{R}^N$. Then we compose $f \in C_c^\infty(I; X)$ with $g \in C_c^\infty(J; X)$ if $t(f) = s(g)$, defining the composite $g \circ_{\mathrm{v}} f$ to be the element of $C_c^\infty(I \cup J; X)$ given by

$$g \circ_{\mathrm{v}} f(u) = \begin{cases} f(u) & \text{if } u \in I; \\ g(u) & \text{if } v \in J. \end{cases} \tag{8.3}$$

Clearly, the composition operation is associative.

Since one should be able to compose a morphism with another if the target of the first is the source of the second, regardless of the exact domains of the morphisms when taken as maps,

> *we identify $f \in C_c^\infty(I; X)$ and $h \in C_c^\infty(K; X)$ if there is some $d \in \mathbb{R}^N$ such that $I = K + d$ and $f(u) = h(u + d)$ for all $u \in K$.*

We denote the resulting quotient set by

$$\mathrm{Map}_N(X) \tag{8.4}$$

if the domains of the functions are boxes in $\mathbb{R}^N$. Then there are well-defined source and target maps

$$s, t : \mathrm{Map}_N(X) \to \mathrm{Map}_{N-1}(X) \tag{8.5}$$

for all positive integers N. Clearly the composition $g \circ_{\mathrm{v}} f$ is meaningful for $f, g \in \mathrm{Map}_N(X)$ if $t(f) = s(g)$. In order to get a category we must have identity morphisms and so we quotient one more time, by requiring that

$$f \circ_{\mathrm{v}} i = f \qquad \text{and} \qquad i \circ_{\mathrm{v}} g = g \tag{8.6}$$

whenever the compositions $f \circ_{\mathrm{v}} i$ and $i \circ_{\mathrm{v}} g$ are meaningful and $i \in C_c^\infty(I; X)$ is independent of the N-th coordinate direction. Let

$$\mathrm{Mor}_N(X) \tag{8.7}$$

be the set of all equivalence classes of $f \in \mathrm{Map}_N(X)$, with the equivalence being defined by the requirements (8.6). Then the source and target maps s and t in (8.5) descend to well-defined maps

$$s_{\mathrm{v}}, t_{\mathrm{v}} : \mathrm{Mor}_N(X) \to \mathrm{Mor}_{N-1}(X). \tag{8.8}$$

Again, composition $g \circ_{\mathrm{v}} f$ is meaningful whenever $t(f) = s(g)$, and the operation of composition is associative. For any $a \in \mathrm{Mor}_{N-1}(X)$ (here $N \geq 1$) let $1_a \in \mathrm{Mor}_N(X)$ be the element with a as both source and target, and 1_a, viewed as a mapping on an N-box, is constant along the N-th coordinate direction (thus corresponding to the i in (8.6)). Then $f \circ_{\mathrm{v}} 1_a = f$ and $g = 1_a \circ_{\mathrm{v}} g$ whenever $s(f) = a$ and $t(g) = a$.

Thus, in this way we obtain, for every positive integer N, a category

$$\mathbf{P}_N(X) \tag{8.9}$$

whose object set is $\mathrm{Mor}_{N-1}(X)$ and whose morphism set is $\mathrm{Mor}_N(X)$.

Our main interest is in the notion of parallel-transport. Parallel-transport has certain invariance properties: for example, parallel-transport is invariant under reparametrization of paths and backtrack-erasure. One way of expressing these properties is to say that parallel-transport is well-defined on categories whose morphisms are obtained by identification of certain classes of morphisms in $\mathbf{P}_N(X)$, such as those that are 'thin homotopy' equivalent (as in Theorem 3.2). However, instead of passing to such quotient categories we have and will state the relevant invariance properties as they arise. The only quotients/identification we work with are the bare minimum necessary ones to ensure that a category is formed by the maps of interest.

In the following we use terminology introduced in the context of (8.1). Let X be a manifold, $N \geq 2$ a positive integer, $I = \prod_{j=1}^{N}[a_j, b_j]$ a box in $\mathbb{R}^N$, and $f : I \to X$ a C^∞ map. Let C be a C^∞ $(N-1)$-form on X with values in the Lie algebra $L(H)$ of a Lie group H. Consider the solution

$$w_f : [a_N, b_N] \to H : u \mapsto w(u)$$

to the differential equation

$$w_f(u)^{-1}w_f'(u) = -\int_{\prod_{j=1}^{N-1}[a_j,b_j]} C\big(\partial_1 f(t,u), \ldots, \partial_{N-1} f(t,u)\big)\, dt, \tag{8.10}$$

with initial condition

$$w(a_N) = e.$$

We define

$$w_C(f) = w_f(b_N). \tag{8.11}$$

With this notation we have the following result on composites.

Proposition 8.1 *Let X be a manifold, $N \geq 2$ a positive integer, I and J boxes in $\mathbb{R}^N$ such that the lower face of J is the upper face of I. Let $f \in C_c^\infty(I; X)$ and $g \in C_c^\infty(J; X)$ be such that the composite $g \circ_{\mathrm{v}} f$, given by (8.6), is defined. Let C be a C^∞ $(N-1)$-form on X with values in the Lie algebra $L(H)$ of a Lie group H. Then*

$$w_C(g \circ_{\mathrm{v}} f) = w_C(f)w_C(g). \tag{8.12}$$

The property (8.12) makes it possible to construct examples of categorical connections over path spaces; we have used it in (7.18) and will make use of it again in a more complex setting in section 9.

Proof. Let I be the box $\prod_{j=1}^{N}[a_j, b_j]$ and J the box $\prod_{j=1}^{N}[c_j, d_j]$; since the upper face of I is the lower face of J, we have $b_N = c_N$ and $[a_i, b_i] = [c_i, d_i]$ for $i \in \{1, \ldots, N-1\}$. The function of u on the right in (8.10) is C^∞ and so the differential equation (8.10) has a C^∞ solution that is uniquely determined by the initial value $w_f(a_N)$. Furthermore, for any $h \in H$ the left-translate hw_f is the solution of (8.10) with initial value h. Hence $w_f(b_N)w_g$ is the solution of the differential equation

$$w(u)^{-1}w'(u) = -\int_{\prod_{j=1}^{N-1}[c_j,d_j]} C\big(\partial_1 g(t,u), \ldots, \partial_{N-1}g(t,u)\big)\,dt,$$

with initial condition $w(c_N) = w_f(b_N)$. Thus the composite

$$\big(w_f(b_N)w_g\big) \circ w_f : [a_N, d_N] \to H : u \mapsto \begin{cases} w_f(u) & \text{if } u \in [a_N, b_N]; \\ w_f(b_N)w_g(u) & \text{if } u \in [c_N, d_N] \end{cases} \tag{8.13}$$

is continuous (with value $w_f(b_N)$ at $u = b_N$), has initial value $w_f(a_N) = e$ and is a solution of the differential equation

$$w(u)^{-1}w'(u) = -\int_{\prod_{i=1}^{N-1}[a_i,b_i]} C\big(\partial_1(g \circ_{\rm v} f)(t,u), \ldots, \partial_{N-1}(g \circ_{\rm v} f)(t,u)\big)\,dt, \tag{8.14}$$

at all $u \in [a_N, d_N]$ except possibly at $u = b_N$. Thus $w_f(b_N)w_g$ agrees with the solution $w_{g\circ_{\rm v} f}$ of (8.14), with initial value e, at all points $u \in [a_N, d_N]$ except possibly at b_N; continuity of both $w_f(b_N)w_g$ and $w_{g\circ_{\rm v} f}$ ensures then that these are equal also at $u = b_N$. Thus

$$w_{g\circ_{\rm v} f} = w_f(b_N)w_g(u) \qquad \text{for all } u \in [a_N, d_N].$$

Taking $u = d_N$ gives

$$w_{g\circ_{\rm v} f}(d_N) = w_f(b_N)w_g(d_N),$$

which is just the identity (8.12). QED

We can modify w_C to another example of a function with a property similar to (8.12):

Proposition 8.2 *Let X be a manifold, $N \geq 2$ a positive integer. Let C be a C^∞ $(N-1)$-form on X with values in the Lie algebra $L(H)$ of a Lie group H, and w_0 any H-valued function on* $\mathrm{Mor}_{N-1}(X)$*:*

$$w_0 : \mathrm{Mor}_{N-1}(X) \to H. \tag{8.15}$$

Define

$$w_{C,0}(f) = w_0\big(s(f)\big)w_C(f)w_0\big(t(f)\big)^{-1} \tag{8.16}$$

for all $f \in C_c^\infty(I; X)$, where I is any box in $\mathbb{R}^N$. Then

$$w_{C,0}(g \circ_{\rm v} f) = w_{C,0}(f)w_{C,0}(g). \tag{8.17}$$

for all $f \in C_c^\infty(I; X)$ and $g \in C_c^\infty(J; X)$ for which $g \circ_{\rm v} f$ is defined, with I and J being boxes in $\mathbb{R}^N$.

Note that in particular we could take for w_0 the function w_D obtained from an $(N-2)$-form D on X with values in H.
Proof. The composite $g \circ_{\mathrm{v}} f$ has source $s(f)$ and target $t(g)$; hence

$$\begin{aligned}
w_{C,0}(g \circ_{\mathrm{v}} f) &= w_0\big(s(g \circ_{\mathrm{v}} f)\big) w_C(g \circ_{\mathrm{v}} f) w_0\big(t(g \circ_{\mathrm{v}} f)\big)^{-1} \\
&= w_0\big(s(f)\big) w_C(f) w_C(g) w_0\big(t(g)\big)^{-1} \\
&= w_0\big(s(f)\big) w_C(f) w_D\big(t(f)\big)^{-1} w_0\big(t(f)\big) w_C(g) w_0\big(t(g)\big)^{-1} \\
&\qquad\qquad (\text{using } t(f) = s(g).) \\
&= w_0\big(s(f)\big) w_C(f) w_0\big(t(f)\big)^{-1} w_0\big(s(g)\big) w_C(g) w_0\big(t(g)\big)^{-1} \\
&= w_{C,0}(f) w_{C,0}(g).
\end{aligned} \tag{8.18}$$

Thus we have determined the behavior of $w_{C,0}$ with respect to vertical composition. QED

9 Categorical connections over path space

We will construct a categorical connection over a path space using the 1-form $\omega_{(A,B)}$ given in (2.13). For this purpose it will be more convenient to work with C^∞ paths that are constant near their endpoints, and define paths of paths as in section 8. As before we work with a Lie crossed module (G, H, α, τ), connection forms A and $\overline{A}$ on a principal G-bundle $\pi : P \to M$, and an $L(H)$-valued α-equivariant 2-form B on P with values in $L(H)$ that vanishes on (v, w) whenever v or w is a vertical vector in P.

Let $\mathbf{P}_1(X)$ and $\mathbf{P}_2(X)$ be the categories described in section 8. We will work with the case where X is either M or P.

In Theorem 7.1 we constructed a categorical connection on a categorical bundle whose objects are points and whose morphisms are paths. We now construct an example that is an analog of this, but one dimension higher in the sense that the objects are now paths and the morphisms are paths of paths. We focus on the subcategory $\mathbf{P}_1^{\overline{A}}(P)$ of $\mathbf{P}_1(P)$ in which the morphisms arise from $\overline{A}$-horizontal paths on P, and the subcategory $\mathbf{P}_2^{\omega_{(A,B)}}(P)$ of $\mathbf{P}_2(P)$ where the morphisms come from $\omega_{(A,B)}$-horizontal paths of $\overline{A}$-horizontal paths on P. We have then the categorical principal bundle

$$\pi : \mathbf{P}_2^{\omega_{(A,B)}}(P) \to \mathbf{P}_2(M), \tag{9.1}$$

whose structure categorical group is the categorical group $\mathbf{G}_d$ (object set is G and morphisms are all the identity morphisms). This is the analog of Example P3 in section 5, one dimension higher, with paths replaced by paths of paths. The action of any object $g \in G$ on any object $\tilde{\gamma}$ of $\mathbf{P}_2^{\omega_{(A,B)}}(P)$ produces $\tilde{\gamma}g$, which is again an $\overline{A}$-horizontal path on P. The action of $1_g : g \to g$ on $\tilde{\Gamma} \in \mathrm{Mor}\left(\mathbf{P}_2^{\omega_{(A,B)}}(P)\right)$ produces $\tilde{\Gamma}g$:

$$\tilde{\Gamma}1_g \stackrel{\text{def}}{=} \tilde{\Gamma}g,$$

which arises from the map $[s_0, s_1] \times [t_0, t_1] \to P : (s,t) \mapsto \tilde{\Gamma}(s,t)g$, where we use a representative map $\tilde{\Gamma}$. Note that if $\tilde{\Gamma}_s = \tilde{\Gamma}_{s_0}$ for all $s \in [s_0, s_1]$ then $\tilde{\Gamma}_s g = \tilde{\Gamma}_{s_0} g$ for all $s \in [s_0, s_1]$.

In order to produce a categorical connection on the bundle (9.1) we have to provide the rule for horizontal lifts of morphisms in $\mathbf{P}_2(M)$ to morphisms in $\mathbf{P}_2^{\omega_{(A,B)}}(P)$. A morphism in $\mathbf{P}_2(M)$ arises from some $\Gamma : [s_0, s_1] \times [t_0, t_1] \to M : (s,t) \mapsto \Gamma_s(t)$; let $\tilde{\Gamma}^{\rm h} : [s_0, s_1] \times [t_0, t_1] \to P : (s,t) \mapsto \tilde{\Gamma}^{\rm h}_s(t)$ be its $\omega_{(A,B)}$-horizontal lift, with a given choice of initial path $\tilde{\Gamma}^{\rm h}_{s_0}$. It is readily checked that there is an $\epsilon > 0$ such that each path $\tilde{\Gamma}^{\rm h}_s : [t_0, t_1] \to P$ is constant within distance ϵ from t_0 and from t_1 (because the same is true for the projected path Γ_s of which $\tilde{\Gamma}^{\rm h}_s$ is an $\overline{A}$-horizontal lift). Moreover, $\tilde{\Gamma}^{\rm h}_s = \tilde{\Gamma}^{\rm h}_{s_0}$ for s near s_0, and $\tilde{\Gamma}^{\rm h}_s = \tilde{\Gamma}^{\rm h}_{s_1}$ for s near s_1. Furthermore, of course, $\tilde{\Gamma}^{\rm h}$ is C^∞. Hence $\tilde{\Gamma}^{\rm h} \in {\rm Mor}\left(\mathbf{P}_2^{\omega_{(A,B)}}(P)\right)$. We label this with the given initial path $\tilde{\gamma} = \tilde{\Gamma}^{\rm h}_{s_0}$:

$$\tilde{\Gamma}^{\rm h}_{\tilde{\gamma}}.$$

It is clear that the assignment

$$\Gamma \mapsto \tilde{\Gamma}^{\rm h}_{\tilde{\gamma}}$$

is functorial in the sense that composites of paths are lifted to composites of the lifted paths; this is the analog of (7.1), and says that the horizontal lift of $\Delta \circ_{\rm v} \Gamma$ with initial path $\tilde{\gamma}$ is

$$\left(\widetilde{\Delta \circ_{\rm v} \Gamma}\right)^{\rm h}_{\tilde{\gamma}} = \tilde{\Delta}^{\rm h}_{\tilde{\delta}} \circ_{\rm v} \tilde{\Gamma}^{\rm h}_{\tilde{\gamma}}, \tag{9.2}$$

where

$$\tilde{\delta} = s\left(\tilde{\Delta}^{\rm h}_{\tilde{\delta}}\right) = t\left(\tilde{\Gamma}^{\rm h}_{\tilde{\gamma}}\right).$$

(see (8.3) for the definition of 'vertical' composition $\circ_{\rm v}$). The condition of 'rigid motion' (7.2) is also clearly valid since $s \mapsto \tilde{\Gamma}_s g$ is $\omega_{(A,B)}$-horizontal if $s \mapsto \tilde{\Gamma}_s$ is $\omega_{(A,B)}$-horizontal (by (2.18)).

10 Parallel-transport for decorated paths

We turn now to our final example of a categorical connection. This will provide a connection on a decorated bundle over space of paths. Before turning to the technical details let us summarize the essence of the construction. As input we have a connection $\overline{A}$ on a principal G-bundle

$$\pi : P \to M.$$

Next let $\mathbf{G}_1$ be a categorical group with associated crossed module

$$(G, H, \alpha_1, \tau_1).$$

Using a connection A and an equivariant 2-form B on P with values in $L(H)$, we have the 1-form

$$\omega_{(A,B)}$$

as specified earlier in (2.13). Now consider a categorical group $\mathbf{G}_2$ with

$$\mathrm{Obj}(\mathbf{G}_2) = \mathrm{Mor}(\mathbf{G}_1),$$

with associated crossed module

$$(H \rtimes_{\alpha_1} G, K, \alpha_2, \tau_2).$$

We will use this to construct a doubly-decorated category $\mathbf{P}_2^{\mathrm{dec}}(P)^{\mathrm{dec}}_{\omega_{(A,B)}}$ whose objects are of the form

$$(\tilde{\gamma}, h),$$

where $\tilde{\gamma}$ is any $\overline{A}$-horizontal path on P and $h \in H$, and whose morphisms are of the form

$$(\tilde{\Gamma}, h, k),$$

where $\tilde{\Gamma}$ is any $\omega_{(A,B)}$-horizontal path of paths on P. We will show that there is a principal categorical bundle

$$\mathbf{P}_2^{\mathrm{dec}}(P)^{\mathrm{dec}}_{\omega_{(A,B)}} \to \mathbf{P}_2(M),$$

with structure categorical group $\mathbf{G}_2$, and then construct categorical connections on this bundle. Briefly put, our method provides a way of constructing parallel-transport of decorated paths

$$(\tilde{\gamma}, h)$$

along doubly decorated paths of paths

$$(\tilde{\Gamma}, h, k),$$

where $\tilde{\gamma}$ is any $\overline{A}$-horizontal path on the original bundle and $\tilde{\Gamma}$ is an $\omega_{(A,B)}$-horizontal path on the bundle of $\overline{A}$-horizontal paths over the path space of M.

Let us recall the construction provided by Theorem 7.2. Consider a categorical connection $\mathbf{A}_0$ on a categorical principal bundle $\mathbf{P} \to \mathbf{B}$, with structure categorical group $\mathbf{G}_0$. Let $\mathbf{P}_{\mathbf{A}_0} \to \mathbf{B}$ be the categorical bundle obtained by working only with $\mathbf{A}_0$-horizontal morphisms of $\mathbf{P}$. Next consider a map

$$k^* : \mathrm{Mor}\Big(\mathbf{P}_{\mathbf{A}_0}\Big) \to K,$$

where K is a group, satisfying

$$\begin{aligned} k^*(\tilde{\gamma}_u 1_{g_0}) &= \alpha({g_0}^{-1}) k^*(\tilde{\gamma}_u) \\ k^*\left(\tilde{\delta}_v \circ \tilde{\gamma}_u\right) &= k^*\left(\tilde{\gamma}_u\right) k^*\left(\tilde{\delta}_v\right) \end{aligned} \tag{10.1}$$

for all $g_0 \in \mathrm{Obj}(\mathbf{G}_0)$ and all $\tilde{\delta}_v, \tilde{\gamma}_u \in \mathrm{Mor}\Big(\mathbf{P}_{\mathbf{A}_0}\Big)$ that are composable. Now let

$$\mathbf{G}_1$$

be a categorical group whose object group is the same as the object group of $\mathbf{G}_0$:

$$\mathrm{Obj}(\mathbf{G}_1) = \mathrm{Obj}(\mathbf{G}_0) = G. \tag{10.2}$$

Let (G, H, α_1, τ_1) be the crossed module associated to $\mathbf{G}_1$. Theorem 7.2 then provides a 'decorated' categorical principal bundle $\mathbf{P}^{\mathrm{dec}}_{\mathbf{A}_0} \to \mathbf{B}$, with structure group $\mathbf{G}_1$ (whose objects are the objects of $\mathbf{G}_0$) along with a categorical connection $\mathbf{A}_1$ on this bundle. Thus, the objects of $\mathbf{P}^{\mathrm{dec}}_{\mathbf{A}_0}$ are the objects $\tilde{\gamma}$ of $\mathbf{P}_{\mathbf{A}_0}$, but the morphisms are decorated morphisms of $\mathbf{P}_{\mathbf{A}_0}$:

$$(\tilde{\gamma}, h) \in \mathrm{Mor}(\mathbf{P}_{\mathbf{A}_0}) \times H.$$

We now apply this procedure with $\mathbf{P} \to \mathbf{B}$ being a categorical bundle

$$\mathbf{P}_2^{\mathrm{dec}}(P) \to \mathbf{P}_2(M),$$

which we now describe.

Let $\pi : P \to M$ a principal G-bundle equipped with connection $\overline{A}$. Then we have a categorical principal bundle

$$\mathbf{P}_1^{\overline{A}}(P) \to \mathbf{P}_1(M),$$

where the object set of $\mathbf{P}_1^{\overline{A}}(P)$ is P and the morphisms arise from $\overline{A}$-horizontal paths $\tilde{\gamma}$ on P. The structure categorical group $\mathbf{G}_0$ is discrete, with object group being G. There is a categorical connection $\mathbf{A}_0$ on this bundle: it associates to any $\gamma \in \mathrm{Mor}\big(\mathbf{P}_1(M)\big)$ the $\overline{A}$-horizontal lift $\tilde{\gamma}_u$ through any given initial point $u \in P$ in the fiber over $s(\gamma)$.

Now let $\mathbf{G}_1$ be a categorical Lie group with crossed module (G, H, α_1, τ_1). Then by Theorem 7.2 we have the decorated construction, yielding a categorical principal bundle

$$\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}} \to \mathbf{P}_1(M),$$

whose object set is P and whose morphisms are of the form

$$(\tilde{\gamma}, h),$$

where $\tilde{\gamma}$ arises from an $\overline{A}$-horizontal path on P, and $h \in H$. The structure categorical group is $\mathbf{G}_1$. We define the category $\mathbf{P}_2^{\mathrm{dec}}(P)_{\omega_{(A,B)}}$ as follows. Its objects are the morphisms $(\tilde{\gamma}, h)$ of $\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}$. A morphism of $\mathbf{P}_2^{\mathrm{dec}}(P)_{\omega_{(A,B)}}$ is of the form

$$(\tilde{\Gamma}, h) \in \mathrm{Mor}\big(\mathbf{P}_2^{\omega_{(A,B)}}(P)\big) \times H, \tag{10.3}$$

where $\tilde{\Gamma}$ is $\omega_{(A,B)}$-horizontal, with source and target being

$$s(\tilde{\Gamma}, h) = \big(s(\tilde{\Gamma}), h\big), \quad \text{and} \quad t(\tilde{\Gamma}, h) = \big(t(\tilde{\Gamma}), h\big), \tag{10.4}$$

where $\mathbf{P}_2^{\omega_{(A,B)}}(P)$ is as we discussed in (9.1). Composition in $\mathbf{P}_2^{\mathrm{dec}}(P)_{\omega_{(A,B)}}$ is specified by

$$(\tilde{\Gamma}_2, h) \circ (\tilde{\Gamma}_1, h) = (\tilde{\Gamma}_2 \circ_{\mathrm{v}} \tilde{\Gamma}_1, h). \tag{10.5}$$

(We will develop a 'twisted' decorated version of this below in (10.12).) Then

$$\mathbf{P}_2^{\rm dec}(P)_{\omega_{(A,B)}} \to \mathbf{P}_2(M)$$

is a categorical principal bundle with structure categorical group $\mathbf{G}_0$ being discrete, with object group $H \rtimes_{\alpha_1} G$:

$$\mathrm{Obj}(\mathbf{G}_0) = \mathrm{Mor}(\mathbf{G}_1) = H \rtimes_{\alpha_1} G.$$

(We use the notation $\mathbf{G}_0$ again in order to make the application of Theorem 7.2 clearer.) The action of an object $(h_1, g_1) \in \mathrm{Obj}(\mathbf{G}_0)$ on an object $(\tilde{\gamma}, h)$ of $\mathbf{P}_2^{\rm dec}(P)_{\omega_{(A,B)}}$ is given by

$$(\tilde{\gamma}, h) \cdot (h_1, g_1) = \Big(\tilde{\gamma} g_1, \alpha(g_1^{-1})(h_1^{-1} h)\Big). \tag{10.6}$$

The action of the identity morphism $1_{(h_1,g_1)} \in \mathrm{Mor}(\mathbf{G}_0)$ on $(\tilde{\Gamma}, h)$ is given by

$$(\tilde{\Gamma}, h) 1_{(h_1,g_1)} = \Big(\tilde{\Gamma} g_1, \alpha(g_1^{-1})(h_1^{-1} h)\Big). \tag{10.7}$$

For the categorical connection $\mathbf{A}_0$ we take the lifting of Γ through $(\tilde{\gamma}, h)$ to be

$$(\tilde{\Gamma}, h)$$

where $\tilde{\Gamma}$ is $\omega_{(A,B)}$-horizontal and has initial path $s(\tilde{\Gamma}) = \tilde{\gamma}$.

Now let $\mathbf{G}_2$ be a categorical group for which

$$\mathrm{Obj}(\mathbf{G}_2) = \mathrm{Obj}(\mathbf{G}_0) = \mathrm{Mor}(\mathbf{G}_1) = H \rtimes_{\alpha_1} G, \tag{10.8}$$

with Lie crossed module

$$(H \rtimes_{\alpha_1} G, K, \alpha_2, \tau_2). \tag{10.9}$$

Then by Theorem 7.2 we obtain a doubly decorated categorical bundle

$$\mathbf{P}_2^{\rm dec}(P)^{\rm dec}_{\omega_{(A,B)}} \to \mathbf{P}_2(M), \tag{10.10}$$

with structure categorical group $\mathbf{G}_2$, whose object group is $H \rtimes_{\alpha_1} G = \mathrm{Obj}(\mathbf{G}_0)$ and whose morphism group is $K \rtimes_{\alpha_2} (H \rtimes_{\alpha_1} G)$.

A morphism of $\mathbf{P}_2^{\rm dec}(P)^{\rm dec}_{\omega_{(A,B)}}$ is of the form

$$(\tilde{\Gamma}, h, k),$$

where $\tilde{\Gamma}$ is $\omega_{(A,B)}$-horizontal, $h \in H$ and $k \in K$; its source and target are (obtained from (6.2)):

$$s(\tilde{\Gamma}, h, k) = \big(s(\tilde{\Gamma}), h\big) \qquad \text{and} \qquad t(\tilde{\Gamma}, h, k) = \big(t(\tilde{\Gamma}), h\big)\tau_2(k^{-1}), \tag{10.11}$$

where in the second term on the right note that $\tau_2(k^{-1}) \in H \times_{\alpha_1} G$ acts on the right on $\mathrm{Mor}(\mathbf{P}_1^{\rm dec}(P))$.

The composition of morphisms in this decorated bundle is given (again from (6.2)) by

$$(\tilde{\Delta}, h, k) \circ_{\mathrm{v}} (\tilde{\Gamma}, h', k') = \big((\tilde{\Delta}, h)\tau_2(k') \circ_{\mathrm{v}} (\tilde{\Gamma}, h'), kk'\big) \tag{10.12}$$

(This is the 'twisted' decorated version of the composition law (10.5).)

The right action of $\mathbf{G}_2$ on $\mathbf{P}_2^{\mathrm{dec}}(P)^{\mathrm{dec}}_{\omega_{(A,B)}}$ is given as follows. On objects the action of $(h_1, g_1) \in H \rtimes_{\alpha_1} G$ on $(\tilde{\gamma}, h)$ is

$$(\tilde{\gamma}, h)(h_1, g_1) = \Big(\tilde{\gamma}g_1, \alpha_1(g_1^{-1})\big(h_1^{-1}h\big)\Big), \tag{10.13}$$

just as seen before in (7.25). On morphisms, the right action of $(k_1, h_1, g_1) \in K \rtimes_{\alpha_2} (H \rtimes_{\alpha_1} G)$ on $(\tilde{\Gamma}, h, k)$ is given by

$$(\tilde{\Gamma}, h, k)(k_1, h_1, g_1) = \Big(\tilde{\Gamma}g_1, \alpha_1(g_1^{-1})(h_1^{-1}h), \alpha_2\big((h_1g_1)^{-1}\big)(k_1^{-1}k)\Big) \tag{10.14}$$

Now we will construct a categorical connection on the categorical principal bundle (10.10). To this end assume that we are given a map from the morphisms of the bundle $\mathbf{P}_2^{\mathrm{dec}}(P)_{\omega_{(A,B)}}$ (before the K-decoration) to the group K:

$$k^* : \mathrm{Mor}\Big(\mathbf{P}_2^{\mathrm{dec}}(P)_{\omega_{(A,B)}}\Big) \to K,$$

satisfying the conditions (10.1):

$$\begin{aligned} k^*(\tilde{\Gamma}1_{g_1}) &= \alpha_2(g_1^{-1})k^*(\tilde{\Gamma}) \\ k^*(\tilde{\Delta} \circ_{\mathrm{v}} \tilde{\Gamma}) &= k^*(\tilde{\Gamma})k^*(\tilde{\Delta}) \end{aligned} \tag{10.15}$$

for all $g_1 \in G$, and all $\omega_{(A,B)}$-horizontal $\tilde{\Gamma}$ and $\tilde{\Delta}$ for which the composite $\tilde{\Delta} \circ_{\mathrm{v}} \tilde{\Gamma}$ is defined. We have seen in Proposition 8.1 how such k^* may be constructed by using 'doubly path-ordered' integrals of forms with values in the Lie algebra of K as well as, in Proposition 8.2, how to obtain additional examples by including a 'boundary term' to such integrals. In more detail, suppose C_1 is an $L(K)$-valued 1-form on P, and C_2 an $L(K)$-valued 2-form on P such that they are both 0 when contracted on a vertical vector and are α_2-equivariant in the sense that

$$\begin{aligned} C_1((R_g)_*v_1) &= \alpha_2(g)^{-1}C_1(v_1) \\ C_2\big((R_g)_*v_1, (R_g)_*v_2\big) &= \alpha_2(g^{-1})C_2(v_1, v_2) \end{aligned} \tag{10.16}$$

for all $g \in G$, $v_1, v_2 \in T_pP$ with p running over P. For $\tilde{\Gamma} : [s_0, s_1] \times [t_0, t_1] \to P$ we define $w^*_{C_2}(\tilde{\Gamma})$ to be $w(t_1)$, where $[t_0, t_1] \to G : t \mapsto w(t)$ solves

$$w(t)^{-1}w'(t) = -\int_{s_0}^{s_1} C_2\Big(\partial_u\tilde{\Gamma}(u, t), \partial_v\tilde{\Gamma}(u, t)\Big)\, du$$

with $w(t_0) = e \in K$. (See equation (8.10).) By α_2-equivariance of C_2 we then have

$$k_2^*(\tilde{\Gamma}g_1) = \alpha_2(g_1^{-1})k_2^*(\tilde{\Gamma}) \tag{10.17}$$

for all $g_1 \in G$. Moreover, k_2^* satisfies the second property of k^* in (10.15) by Proposition 8.1. Now define $w_1(\tilde{\gamma})$, for any path $\tilde{\gamma} : [s_0, s_1] \to P$, to be $w_1(s_1)$, where w_1 solves

$$w_1(u)^{-1} w_1'(u) = -C_1\big(\tilde{\gamma}'(u)\big),$$

with $w_1(s_0) = e \in K$. Then by equivariance of C_1 we have

$$w_1(\tilde{\gamma} g_1) = \alpha_2(g_1^{-1}) w_1(\tilde{\gamma})$$

for all $g_1 \in G$. Finally, set

$$k^*(\tilde{\Gamma}) = w_1\big(s(\tilde{\Gamma})\big) w_{C_2}(\tilde{\Gamma}) w_1\big(t(\tilde{\Gamma})\big)^{-1}. \tag{10.18}$$

Then k^* clearly satisfies the first condition in (10.15) because both w_1 and w_{C_2} satisfy this condition; moreover, k^* also satisfies the second condition in (10.15) by Proposition 8.2.

To construct a connection on the decorated bundle we need to obtain a mapping κ^* similar to k^* but for the decorated bundle:

$$\kappa^* : \mathrm{Mor}\Big(\mathbf{P}_2^{\mathrm{dec}}(P)^{\mathrm{dec}}_{\omega_{(A,B)}}\Big) \to K,$$

satisfying the conditions (10.1):

$$\begin{aligned} \kappa^*\big((\tilde{\Gamma}, h)1_{(h_1,g_1)}\big) &= \alpha_2((h_1 g_1)^{-1})\kappa^*\big(\tilde{\Gamma}, h\big) \\ \kappa^*\big((\tilde{\Delta}, h) \circ_{\mathrm{v}} (\tilde{\Gamma}, h)\big) &= \kappa^*\big(\tilde{\Gamma}, h\big)\kappa^*\big(\tilde{\Delta}, h\big), \end{aligned} \tag{10.19}$$

where, in the second relation, note that for the composition on the left side to exist, the h-component must be the same for $\tilde{\Delta}$ and $\tilde{\Gamma}$.

Lemma 10.1 *Suppose*

$$k^* : \mathrm{Mor}\Big(\mathbf{P}_2^{\mathrm{dec}}(P)_{\omega_{(A,B)}}\Big) \to K,$$

satisfies the conditions (10.15). Then the mapping

$$\kappa^* : \mathrm{Mor}\Big(\mathbf{P}_2^{\mathrm{dec}}(P)^{\mathrm{dec}}_{\omega_{(A,B)}}\Big) \to K,$$

specified by

$$\kappa^*(\tilde{\Gamma}, h) = \alpha_2(h)\big(k^*(\tilde{\Gamma})\big) \tag{10.20}$$

satisfies (10.19).

Proof. Using the formula for the right action for decorated bundles given in (6.6), we have

$$(\tilde{\Gamma}, h)1_{(h_1,g_1)} = \big(\tilde{\Gamma} g_1, \alpha_1(g_1^{-1})(h_1^{-1}h)\big). \tag{10.21}$$

Applying κ^* we have

$$
\begin{aligned}
\kappa^*\big((\tilde{\Gamma},h)1_{(h_1,g_1)}\big) &= \kappa^*\big(\tilde{\Gamma}g_1, \alpha_1(g_1^{-1})(h_1^{-1}h)\big) \\
&= \alpha_2\big(\alpha_1(g_1^{-1})(h_1^{-1}h)\big)\big(k^*(\tilde{\Gamma}g_1)\big) \\
&= \alpha_2\big(\alpha_1(g_1^{-1})(h_1^{-1}h)\big)\big(\alpha_2(g_1^{-1})k^*(\tilde{\Gamma})\big) \quad \text{(using (10.15))} \\
&= \alpha_2(g_1^{-1}h_1^{-1}h)\big(k^*(\tilde{\Gamma})\big) \quad \text{(using Lemma 4.2)} \\
&= \alpha_2\big((h_1g_1)^{-1}\big)\big(\alpha_2(h)\big(k^*(\tilde{\Gamma})\big)\big) \\
&= \alpha_2((h_1g_1)^{-1})\kappa^*\big(\tilde{\Gamma},h\big),
\end{aligned}
\tag{10.22}
$$

which establishes the first of the relations (10.19).

For the second relation we have

$$
\begin{aligned}
\kappa^*\big((\tilde{\Delta},h)\circ_{\mathrm{v}}(\tilde{\Gamma},h)\big) &= \kappa^*\Big(\tilde{\Delta}\circ_{\mathrm{v}}\tilde{\Gamma},h\Big) \qquad \text{(using (10.5))} \\
&= \alpha_2(h)\big(k^*(\tilde{\Delta}\circ_{\mathrm{v}}\tilde{\Gamma})\big) \quad \text{(using (10.20))} \\
&= \alpha_2(h)\Big(k^*(\tilde{\Gamma})k^*(\tilde{\Delta})\Big) \quad \text{(using (10.15))} \\
&= \alpha_2(h)\Big(k^*(\tilde{\Gamma})\Big)\alpha_2(h)\Big(k^*(\tilde{\Delta})\Big) \\
&= \kappa^*\big(\tilde{\Gamma},h\big)\kappa^*\big(\tilde{\Delta},h\big). \qquad \boxed{\text{QED}}
\end{aligned}
\tag{10.23}
$$

Combining all of this we obtain a categorical connection $\mathbf{A}_2$ on the doubly decorated principal bundle $\mathbf{P}_2^{\mathrm{dec}}(P)^{\mathrm{dec}}_{\omega_{(A,B)}}$ with structure group $\mathbf{G}_2$ whose objects are the morphisms of an initially given categorical group $\mathbf{G}_1$, whose object group G in turn is the structure group of the original principal bundle $\pi : P \to M$. Specifically, given a path of paths Γ on M, and an initial $\overline{A}$-horizontal path $\tilde{\gamma}$ on P lying above $s(\tilde{\Gamma})$, and an element $h \in H$, the result of parallel-transport of $(\tilde{\gamma}, h)$ by the connection $\mathbf{A}_2$ along Γ is the target

$$t(\tilde{\Gamma}, h, k),$$

where $\tilde{\Gamma}$ is $\omega_{(A,B)}$-horizontal with $s(\tilde{\Gamma}) = \tilde{\gamma}$, and $k = \kappa^*(\tilde{\Gamma}, h)$.

Previously we have considered a connection $\omega_{(A,B)}$ over path space determined by an $L(G)$-valued 1-form A and an $L(H)$-valued 2-form B. In that setting $\omega_{(A,B)}$ was determined by $\tau(B)$ rather than by B itself. The discussions in this section show that $L(H)$-valued forms can play a direct role in connections on decorated bundles.

11 Associated bundles

In the traditional theory of bundles, one constructs bundles that are associated to a principal bundle through representations of the structure group. In gauge theory the associated bundles are used to describe matter fields (these are sections of bundles associated to an underlying principal bundle whose structure group is the gauge group). In this section we briefly explore an analog for categorical principal bundles.

We define a categorical vector space to be a category $\mathbf{V}$ analogously to a categorical group. Both $\mathrm{Obj}(\mathbf{V})$ and $\mathrm{Mor}(\mathbf{V})$ are vector spaces, over some given field $\mathbb{F}$. The addition operation

$$\mathbf{V} \times \mathbf{V} \to \mathbf{V}$$

is a functor, as is the scalar multiplication

$$\mathbf{F} \times \mathbf{V} \to \mathbf{V},$$

where $\mathbf{F}$ is the category with object set $\mathbb{F}$ and morphisms just the identity morphisms for each $a \in \mathbb{F}$. (The notion of categorical vector space as defined here is very restrictive and it may well be worthwhile to consider other structures.)

If $\mathbf{K}$ is a categorical group and $\mathbf{V}$ a categorical vector space then a *representation* ρ of $\mathbf{K}$ on $\mathbf{V}$ is a functor

$$\rho : \mathbf{K} \times \mathbf{V} \to \mathbf{V} : (k, v) \mapsto \rho(k)v$$

such that $\rho(k) : \mathrm{Obj}(\mathbf{V}) \to \mathrm{Obj}(\mathbf{V})$, for each $k \in \mathrm{Obj}(\mathbf{K})$ and $\rho(\phi) : \mathrm{Mor}(\mathbf{V}) \to \mathrm{Mor}(\mathbf{V})$, for all $\phi \in \mathrm{Mor}(\mathbf{K})$, are linear, and ρ gives a representation of the group $\mathrm{Obj}(\mathbf{K})$ on the vector space $\mathrm{Obj}(\mathbf{V})$ as well as a representation of $\mathrm{Mor}(\mathbf{K})$ on $\mathrm{Mor}(\mathbf{V})$.

Let $\pi : \mathbf{P} \to \mathbf{B}$ be a principal categorical bundle with group $\mathbf{K}$, and ρ a representation of $\mathbf{K}$ on a categorical vector space $\mathbf{V}$. Then we construct a twisted product

$$\mathbf{P} \times_\rho \mathbf{V}$$

An object of this category is an equivalence class $[p, v]$ of pairs (p, v), with two such pairs (p', v') and (p, v) considered equivalent if $p' = p\rho(k)$ and $v' = \rho(k^{-1})v$ for some $k \in \mathrm{Obj}(\mathbf{K})$. A morphism of $\mathbf{P} \times_\rho \mathbf{V}$ is an equivalence class $[F, f]$ of pairs (F, f), with (F, f) considered equivalent to (F', f') if $F' = F\rho(\phi)$ and $f' = \rho(\phi^{-1})f$ for some morphism ϕ of $\mathbf{K}$. The target for $[F, f]$ is $[t(F), t(f)]$, and source is $[s(F), s(f)]$. There is a well-defined projection functor

$$\mathbf{P} \times_\rho \mathbf{V} \to \mathbf{P}$$

taking $[p, v]$ to $\pi_P(p)$ and $[F, f]$ to $\pi_P(F)$. We view this as the *associated vector bundle* for the given structures.

Given a connection on $\mathbf{P}$ we can construct parallel transport on $\mathbf{P} \times_\rho \mathbf{V}$ as follows. Consider a morphism $f : x \to y$ in $\mathbf{B}$ and $(p, v) \in \mathrm{Obj}(\mathbf{P}) \times \mathrm{Obj}(\mathbf{V})$ such that $\pi_P(p) = x$. Then we define the *parallel transport* of $[p, v]$ along f to be $[t(F), v]$, where F is the horizontal lift of f through p.

We apply the abstract constructions above to the specific principal categorical bundles we have studied before.

Let $\overline{\mathbf{A}_0}$ be the categorical connection from Example CC1, associated with a connection $\overline{A}$ on a principal G-bundle $\pi : P \to B$. A morphism of $\mathbf{P}_0 \times_\rho \mathbf{V}$ is an equivalence class of ordered pairs/triples

$$((\gamma; p, q); v, w),$$

where γ is a backtrack erased path on M from $\pi(p)$ to $\pi(q)$, and $v, w \in V$. We think of v as being located at the source of γ and w being located at the target of γ. The equivalence relation between such triples is given by

$$((\gamma; p, q); v, w) \sim \big((\gamma; pg, qg); g^{-1}(v, w)\big). \tag{11.1}$$

From a connection $\overline{A}$ on $\pi : P \to B$ we have a connection $\overline{\mathbf{A}}_0$ on $\pi : \mathbf{P}_0 \to \mathbf{B}$, and then the corresponding parallel-transport of $[p, v]$ along γ results in $[q, v]$, where q is the target of the $\overline{A}$-horizontal lift of $\overline{A}$ initiating at p. This is exactly in accordance with the traditional notion of parallel transport for associated bundles.

12 Concluding Remarks

In this paper we have developed a general theory of categorical connections on categorical bundles and explored several families of examples, including a new class of examples that we call 'decorated' bundles. We have explained how traditional connection forms along with 2-forms on a classical principal bundle give rise to categorical connections that describe parallel-transport of decorated paths over path spaces. Our work is related to that of Martins and Picken [20, 21] but goes beyond the study of holonomies to the study of parallel-transport and we provide new families of examples of these categorical geometric structures. The large families of examples we have constructed may be specialized by imposing specific consistency or flatness conditions and would in that way connect to the notions of categorical connections that are currently in use in the literature. More precisely, the categorical connections we have constructed in section 10 using 1-forms and 2-forms on a traditional bundle $\pi : P \to M$, involving a hierarchy of categorical groups, could be specialized by imposing suitable flatness constraints on these forms. Our approach also appears to open up the possibility of building a full hierarchy of decorated bundles and connections over spaces of parametrized submanifolds (viewed as paths in lower order path spaces) of M of all dimensions.

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