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Parallel Transport over Path Spaces

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We develop a differential geometric framework for parallel transport over path spaces and a corresponding discrete theory, an integrated version of the continuum theory, using a category-theoretic framework.

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1. Introduction

A considerable body of literature has grown up around the notion of 'surface holonomy', or parallel transport on surfaces, motivated by the need to have a gauge theory of interaction between charged string-like objects. Approaches include direct geometric exploration of the space of paths of a manifold (Cattaneo et al. [5], for instance), and a very different, category-theory flavored development (Baez and Schreiber [2], for instance). In the present work we develop both a path-space ge-

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ometric theory as well as a category theoretic approach to surface holonomy, and describe some of the relationships between the two.

As is well known [1] from a group-theoretic argument and also from the fact that there is no canonical ordering of points on a surface, attempts to construct a group-valued parallel transport operator for surfaces leads to inconsistencies unless the group is abelian (or an abelian representation is used). So in our setting, there are *two* interconnected gauge groups G and H. We work with a fixed principal G-bundle $\pi : P \to M$ and connection $\overline{A}$; then, viewing the space of $\overline{A}$-horizontal paths itself as a bundle over the path space of M, we study a particular type of connection on this path-space bundle which is specified by means of a second connection A and a field B whose values are in the Lie algebra LH of H. We derive explicit formulas describing parallel-transport with respect to this connection. As far as we are aware, this is the first time an explicit description for the parallel transport operator has been obtained for a surface swept out by a path whose endpoints are not pinned. We obtain, in Theorem 2.1, conditions for the parallel-transport of a given point in path-space to be independent of the parametrization of that point, viewed as a path. We also discuss H-valued connections on the path space of M, constructed from the field B. In section 3 we show how the geometrical data, including the field B, lead to two categories. We prove several results for these categories and discuss how these categories may be viewed as 'integrated' versions of the differential geometric theory developed in section 2

In working with spaces of paths one is confronted with the problem of specifying a differential structure on such spaces. It appears best to proceed within a simpler formalism. Essentially, one continues to use terms such as 'tangent space' and 'differential form', except that in each case the specific notion is defined directly (for example, a tangent vector to a space of paths at a particular path γ is a vector field along γ) rather than by appeal to a general theory. Indeed, there is a good variety of choices for general frameworks in this philosophy (see, for instance, Stacey [16] and Viro [17]). For this reason we shall make no attempt to build a manifold structure on any space of paths.

Background and Motivation

Let us briefly discuss the physical background and motivation for this study. Traditional gauge fields govern interaction between point particles. Such a gauge field is, mathematically, a connection A on a bundle over spacetime, with the structure group of the bundle being the relevant internal symmetry group of the particle species. The amplitude of the interaction, along some path γ connecting the point particles, is often obtained from the particle wave functions ψ coupled together using quantities involving the path-ordered exponential integral $\mathcal{P}\exp(-\int_\gamma \overline{A})$, which is the same as the parallel-transport along the path γ by the connection $\overline{A}$. If we now change our point of view concerning particles, and assume that they are extended string-like entities, then each particle should be viewed not as a point entity but rather a path (segment) in spacetime. Thus, instead of the two particles located

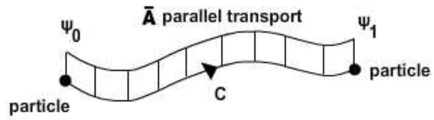

Fig. 1. Point particles interacting via a gauge field

at two points, we now have two paths γ_1 and γ_2; in place of a path connecting the two point particles we now have a parametrized path of paths, in other words a surface Γ, connecting γ_1 with γ_2. The interaction amplitudes would, one may expect, involve both the gauge field A, as expressed through the parallel transports along γ_1 and γ_2, and an interaction between these two parallel transport fields. This higher order, or higher dimensional interaction, could be described by means of a gauge field at the higher level: it would be a gauge field over the space of paths in spacetime.

Comparison with other works

The approach to higher gauge theory developed and explored by Baez [1], Baez and Schreiber [2,3], and Lahiri [13], and others cited in these papers, involves an abstract category theoretic framework of 2-connections and 2-bundles, which are higher-dimensional analogs of bundles and connections. There is also the framework of gerbes (Chatterjee [6], Breen and Messing [4], Murray [14]).

We develop both a differential geometric framework and category-theoretic structures. We prove in Theorem 2.1 that a requirement of parametrization invariance imposes a constraint on a quantity called the 'fake curvature' which has been observed in a related but more abstract context by Baez and Schreiber [2, Theorem 23]. Our differential geometric approach is close to the works of Cattaneo et al. [5], Pfeiffer [15], and Girelli and Pfeiffer [11]. However, we develop, in addition to the differential geometric aspects, the integrated version in terms of categories of diagrams, an aspect not addressed in [5]; also, it should be noted that our connection form is different from the one used in [5]. To link up with the integrated theory it is essential to explore the effect of the LH-valued field B. To this end we determine a 'bi-holonomy' associated to a path of paths (Theorem 2.2) in terms of the field B; this aspect of the theory is not studied in [5] or other works.

Our approach has the following special features:

- we develop the theory with two connections A and $\overline{A}$ as well as a 2-form B (with the connection $\overline{A}$ used for parallel-transport along any given string-like object, and the forms A and B used to construct parallel-transports between different strings);
- we determine, in Theorem 2.2, the 'bi-holonomy' associated to a path of paths using the B-field;
- we allow 'quadrilaterals' rather than simply bigons in the category theoretic

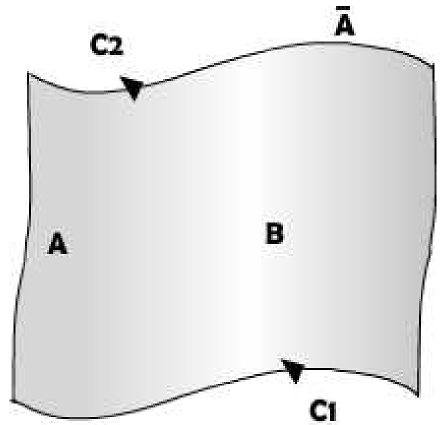

Fig. 2. Gauge fields along paths c_1 and c_2 interacting across a surface

formulation, corresponding to having strings with endpoints free to move rather than fixed-endpoint strings.

Our category theoretic considerations are related to notions about double categories introduced by Ehresmann [9,10] and explored further by Kelly and Street [12].

2. Connections on Path-space Bundles

In this section we will construct connections and parallel-transport for a pair of intertwined structures: path-space bundles with structure groups G and H, which are Lie groups intertwined as explained below in (2.1). For the physical motivation, it should be kept in mind that G denotes the gauge group for the gauge field along each path, or string, while H governs, along with G, the interaction between the gauge fields along different paths.

An important distinction between existing differential geometric approaches (such as Cattaneo et al. [5]) and the 'integrated theory' encoded in the category-theoretic framework is that the latter necessarily involves two gauge groups: a group G for parallel transport along paths, and another group H for parallel transport between paths (in path space). We shall develop the differential geometric framework using a pair of groups (G, H) so as to be consistent with the 'integrated' theory. Along with the groups G and H, we use a fixed smooth homomorphism $\tau : H \to G$ and a smooth map

$$G \times H \to H : (g, h) \mapsto \alpha(g)h$$

such that each $\alpha(g)$ is an automorphism of H, such that the identities

$$\begin{aligned} \tau\big(\alpha(g)h\big) &= g\tau(h)g^{-1} \\ \alpha\big(\tau(h)\big)h' &= hh'h^{-1} \end{aligned} \tag{2.1}$$

hold for all $g \in G$ and $h, h' \in H$. The derivatives $\tau'(e)$ and $\alpha'(e)$ will be denoted simply as $\tau : LH \to LG$ and $\alpha : LG \to LH$. (This structure is called a *Lie 2-group* in [1,2]).

To summarize very rapidly, anticipating some of the notions explained below, we work with a principal G-bundle $\pi : P \to M$ over a manifold M, equipped with connections A and $\overline{A}$, and an α-equivariant vertical 2-form B on P with values in the Lie algebra LH. We then consider the space $\mathcal{P}_{\overline{A}}P$ of $\overline{A}$-horizontal paths in P, which forms a principal G-bundle over the path-space $\mathcal{P}M$ in M. Then there is an associated vector bundle E over $\mathcal{P}M$ with fiber LH; using the 2-form B and the connection form $\overline{A}$ we construct, for any section σ of the bundle $P \to M$, an LH-valued 1-form θ^{σ} on $\mathcal{P}M$. This being a connection over the path-space in M with structure group H, parallel-transport by this connection associates elements of H to *parametrized surfaces* in M. Most of our work is devoted to studying a second connection form $\omega_{(A,B)}$, which is a connection on the bundle $\mathcal{P}_{\overline{A}}P$ which we construct using a second connection A on P. Parallel-transport by $\omega_{(A,B)}$ is related to parallel-transport by the LH-valued connection form θ^{σ}.

Principal bundle and the connection $\overline{A}$

Consider a principal G-bundle

$$\pi : P \to M$$

with the right-action of the Lie group G on P denoted

$$P \times G \to P : (p, g) \mapsto pg = R_g p.$$

Let $\overline{A}$ be a connection on this bundle. The space $\mathcal{P}_{\overline{A}}P$ of $\overline{A}$-horizontal paths in P may be viewed as a principal G-bundle over $\mathcal{P}M$, the space of smooth paths in M.

We will use the notation $pK \in T_pP$, for any point $p \in P$ and Lie-algebra element $K \in LG$, defined by

$$pK = \frac{d}{dt}\Big|_{t=0} p \cdot \exp(tK).$$

It will be convenient to keep in mind that we always use t to denote the parameter for a path on the base manifold M or in the bundle space P; we use the letter s to parametrize a path in path-space.

The tangent space to $\mathcal{P}_{\overline{A}}P$

The points of the space $\mathcal{P}_{\overline{A}}P$ are $\overline{A}$-horizontal paths in P. Although we call $\mathcal{P}_{\overline{A}}P$ a 'space' we do not discuss any topology or manifold structure on it. However, it is useful to introduce certain differential geometric notions such as tangent spaces on $\mathcal{P}_{\overline{A}}P$. It is intuitively clear that a tangent vector at a 'point' $\tilde{\gamma} \in \mathcal{P}_{\overline{A}}P$ ought to be a vector field on the path $\tilde{\gamma}$. We formalize this idea here (as has been done elsewhere as well, such as in Cattaneo et al. [5]).

If $\mathcal{P}X$ is a space of paths on a manifold X, we denote by ev_t the evaluation map

$$\mathrm{ev}_t : \mathcal{P}X \to X : \gamma \mapsto \mathrm{ev}_t(\gamma) = \gamma(t). \tag{2.2}$$

Our first step is to understand the tangent spaces to the bundle $\mathcal{P}_{\overline{A}}P$. The following result is preparation for the definition (see also [5, Theorem 2.1]).

Proposition 2.1. *Let $\overline{A}$ be a connection on a principal G-bundle $\pi : P \to M$, and*

$$\tilde{\Gamma} : [0,1] \times [0,1] \to P : (t,s) \mapsto \tilde{\Gamma}(t,s) = \tilde{\Gamma}_s(t)$$

a smooth map, and

$$\tilde{v}_s(t) = \partial_s \tilde{\Gamma}(t,s).$$

Then the following are equivalent:

(i) Each transverse path

$$\tilde{\Gamma}_s : [0,1] \to P : t \mapsto \tilde{\Gamma}(t,s)$$

is $\overline{A}$-horizontal.

(ii) The initial path $\tilde{\Gamma}_0$ is $\overline{A}$-horizontal, and the 'tangency condition'

$$\frac{\partial \overline{A}(\tilde{v}_s(t))}{\partial t} = F^{\overline{A}}\left(\partial_t \tilde{\Gamma}(t,s), \tilde{v}_s(t)\right) \tag{2.3}$$

holds, and thus also

$$\overline{A}\big(\tilde{v}_s(T)\big) - \overline{A}\big(\tilde{v}_s(0)\big) = \int_0^T F^{\overline{A}}\left(\partial_t \tilde{\Gamma}(t,s), \tilde{v}_s(t)\right)\, dt, \tag{2.4}$$

for every $T, s \in [0,1]$.

Equation (2.3), and variations on it, is sometimes referred to as the Duhamel formula and sometimes a 'non-abelian Stokes formula.' We can write it more compactly by using the notion of a Chen integral. With suitable regularity assumptions, a 2-form Θ on a space X yields a 1-form, denoted $\int \Theta$, on the space $\mathcal{P}X$ of smooth paths in X; if c is such a path, a 'tangent vector' $v \in T_c(\mathcal{P}X)$ is a vector field $t \mapsto v(t)$ along c, and the evaluation of the 1-form $\int \Theta$ on v is defined to be

$$\left(\int \Theta\right)_c v = \left(\int_c \Theta\right)(v) = \int_0^1 \Theta\big(c'(t), v(t)\big)\, dt. \tag{2.5}$$

The 1-form $\int \Theta$, or its localization to the tangent space $T_c(\mathcal{P}X)$, is called the Chen integral of Θ. Returning to our context, we then have

$$\mathrm{ev}_T^* \overline{A} - \mathrm{ev}_0^* \overline{A} = \int_0^T F^{\overline{A}}, \tag{2.6}$$

where the integral on the right is a Chen integral; here it is, by definition, the 1-form on $\mathcal{P}_{\overline{A}}P$ whose value on a vector $\tilde{v}_s \in T_{\tilde{\Gamma}_s}\mathcal{P}_{\overline{A}}P$ is given by the right side of (2.3). The pullback $\mathrm{ev}_t^* \overline{A}$ has the obvious meaning.

Proof. From the definition of the curvature form $F^{\overline{A}}$, we have

$$F^{\overline{A}}(\partial_t \tilde{\Gamma}, \partial_s \tilde{\Gamma}) = \partial_t\big(\overline{A}(\partial_s \tilde{\Gamma})\big) - \partial_s\big(\overline{A}(\partial_t \tilde{\Gamma})\big) - \overline{A}\big(\underbrace{[\partial_t \tilde{\Gamma}, \partial_s \tilde{\Gamma}]}_{0}\big) + \left[\overline{A}(\partial_t \tilde{\Gamma}), \overline{A}(\partial_s \tilde{\Gamma})\right].$$

So

$$\begin{aligned}\partial_t\big(\overline{A}(\partial_s\tilde{\Gamma})\big) - F^{\overline{A}}(\partial_t\tilde{\Gamma},\partial_s\tilde{\Gamma}) &= \partial_s\big(\overline{A}(\partial_t\tilde{\Gamma})\big) - \big[\overline{A}(\partial_t\tilde{\Gamma}),\overline{A}(\partial_s\tilde{\Gamma})\big] \\ &= 0 \qquad \text{if } \overline{A}(\partial_t\tilde{\Gamma}) = 0,\end{aligned} \tag{2.7}$$

thus proving (2.3) if (i) holds. The equation (2.4) then follows by integration.

Next suppose (ii) holds. Then, from the first line in (2.7), we have

$$\partial_s\big(\overline{A}(\partial_t\tilde{\Gamma})\big) - \big[\overline{A}(\partial_t\tilde{\Gamma}),\overline{A}(\partial_s\tilde{\Gamma})\big] = 0. \tag{2.8}$$

Now let $s \mapsto h(s) \in G$ describe parallel-transport along $s \mapsto \tilde{\Gamma}(s,t)$; then

$$h'(s)h(s)^{-1} = -\overline{A}\big(\partial_s\tilde{\Gamma}(s,t)\big), \quad \text{and } h(0) = e.$$

Then

$$\begin{aligned}&\partial_s\left(h(s)^{-1}\overline{A}\big(\partial_t\tilde{\Gamma}(t,s)\big)h(s)\right) \\ &\qquad = \mathrm{Ad}\big(h(s)^{-1}\big)\left[\partial_s\big(\overline{A}(\partial_t\tilde{\Gamma})\big) - \big[\overline{A}(\partial_t\tilde{\Gamma}),\overline{A}(\partial_s\tilde{\Gamma})\big]\right]\end{aligned} \tag{2.9}$$

and the right side here is 0, as seen in (2.8). Therefore,

$$h(s)^{-1}\overline{A}\big(\partial_t\tilde{\Gamma}(t,s)\big)h(s)$$

is independent of s, and hence is equal to its value at $s = 0$. Thus, if $\overline{A}$ vanishes on $\partial_t\tilde{\Gamma}(t,0)$ then it also vanishes in $\partial_t\tilde{\Gamma}(t,s)$ for all $s \in [0,1]$. In conclusion, if the initial path $\tilde{\Gamma}_0$ is $\overline{A}$-horizontal, and the tangency condition (2.3) holds, then each transverse path $\tilde{\Gamma}_s$ is $\overline{A}$-horizontal. □

In view of the preceding result, it is natural to define the tangent spaces to $\mathcal{P}_{\overline{A}}P$ as follows:

Definition 2.1. The tangent space to $\mathcal{P}_{\overline{A}}P$ at $\tilde{\gamma}$ is the linear space of all vector fields $t \mapsto \tilde{v}(t) \in T_{\tilde{\gamma}(t)}P$ along $\tilde{\gamma}$ for which

$$\boxed{\frac{\partial\overline{A}(\tilde{v}(t))}{\partial t} - F^{\overline{A}}\left(\tilde{\gamma}'(t),\tilde{v}(t)\right) = 0} \tag{2.10}$$

holds for all $t \in [0,1]$.

The *vertical subspace* in $T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P$ consists of all vectors $\tilde{v}(\cdot)$ for which $\tilde{v}(t)$ is vertical in $T_{\tilde{\gamma}(t)}P$ for every $t \in [0,1]$.

Let us note one consequence:

Lemma 2.1. *Suppose $\gamma : [0,1] \to M$ is a smooth path, and $\tilde{\gamma}$ an $\overline{A}$-horizontal lift. Let $v : [0,1] \to TM$ be a vector field along γ, and $\tilde{v}(0)$ any vector in $T_{\tilde{\gamma}(0)}P$ with $\pi_*\tilde{v}(0) = v(0)$. Then there is a unique vector field $\tilde{v} \in T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P$ whose projection down to M is the vector field v, and whose initial value is $\tilde{v}(0)$.*

Proof. The first-order differential equation (2.10) determines the vertical part of $\tilde{v}(t)$, from the initial value. Thus $\tilde{v}(t)$ is this vertical part plus the $\overline{A}$-horizontal lift of $v(t)$ to $T_{\tilde{\gamma}(t)}P$. □

Connections induced from B

All through our work, B will denote a vertical α-equivariant 2-form on P with values in LH. In more detail, this means that B is an LH-valued 2-form on P which is vertical in the sense that

$$B(u,v) = 0 \quad \text{if } u \text{ or } v \text{ is vertical,}$$

and α-equivariant in the sense that

$$R_g^* B = \alpha(g^{-1})B \quad \text{for all } g \in G$$

wherein $R_g : P \to P : p \mapsto pg$ is the right action of G on the principal bundle space P, and

$$\alpha(g^{-1})B = d\alpha(g^{-1})|_e B,$$

recalling that $\alpha(g^{-1})$ is an automorphism $H \to H$.

Consider an $\overline{A}$-horizontal $\tilde{\gamma} \in \mathcal{P}_{\overline{A}}P$, and a smooth vector field X along $\gamma = \pi \circ \tilde{\gamma}$; take any lift $\tilde{X}_{\tilde{\gamma}}$ of X along $\tilde{\gamma}$, and set

$$\theta_{\tilde{\gamma}}(X) \stackrel{\text{def}}{=} \left(\int_{\tilde{\gamma}} B\right)(\tilde{X}_{\tilde{\gamma}}) = \int_0^1 B\big(\tilde{\gamma}'(u), \tilde{X}_{\tilde{\gamma}}(u)\big)\, du. \tag{2.11}$$

This is independent of the choice of $\tilde{X}_{\tilde{\gamma}}$ (as any two choices differ by a vertical vector on which B vanishes) and specifies a linear form $\theta_{\tilde{\gamma}}$ on $T_\gamma(\mathcal{P}M)$ with values in LH. If we choose a different horizontal lift of γ, a path $\tilde{\gamma}g$, with $g \in G$, then

$$\theta_{\tilde{\gamma}g}(X) = \alpha(g^{-1})\theta_{\tilde{\gamma}}(X). \tag{2.12}$$

Thus, one may view $\tilde{\theta}$ to be a 1-form on $\mathcal{P}M$ with values in the vector bundle $E \to \mathcal{P}M$ associated to $\mathcal{P}_{\overline{A}}P \to \mathcal{P}M$ by the action α of G on LH.

Now fix a section $\sigma : M \to P$, and for any path $\gamma \in \mathcal{P}M$ let $\tilde{\sigma}(\gamma) \in \mathcal{P}_{\overline{A}}P$ be the $\overline{A}$-horizontal lift with initial point $\sigma\big(\gamma(0)\big)$. Thus, $\tilde{\sigma} : \mathcal{P}M \to \mathcal{P}_{\overline{A}}P$ is a section of the bundle $\mathcal{P}_{\overline{A}}P \to \mathcal{P}M$. Then we have the 1-form θ^σ on $\mathcal{P}M$ with values in LH given as follows: for any $X \in T_\gamma(\mathcal{P}M)$,

$$(\theta^\sigma)(X) = \theta_{\tilde{\sigma}(\gamma)}(X). \tag{2.13}$$

We shall view θ^σ as a connection form for the trivial H-bundle over $\mathcal{P}M$. Of course, it depends on the section σ of $\mathcal{P}_{\overline{A}}P \to \mathcal{P}M$, but in a 'controlled' manner, i.e., the behavior of θ^σ under change of σ is obtained using (2.12).

Constructing the connection $\omega_{(A,B)}$

Our next objective is to construct connection forms on $\mathcal{P}_{\overline{A}}P$. To this end, fix a connection A on P, in addition to the connection $\overline{A}$ and the α-equivariant vertical LH-valued 2-form B on P.

The evaluation map at any time $t \in [0,1]$, given by

$$\mathrm{ev}_t : \mathcal{P}_{\overline{A}}P \to P : \tilde{\gamma} \mapsto \tilde{\gamma}(t),$$

commutes with the projections $\mathcal{P}_{\overline{A}}P \to \mathcal{P}M$ and $P \to M$, and the evaluation map $\mathcal{P}M \to M$. We can pull back any connection A on the bundle P to a connection $\mathrm{ev}_t^* A$ on $\mathcal{P}_{\overline{A}}P$.

Given a 2-form B as discussed above, consider the LH-valued 1-form Z on $\mathcal{P}_{\overline{A}}P$ specified as follows. Its value on a vector $\tilde{v} \in T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P$ is defined to be

$$Z(\tilde{v}) = \int_0^1 B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right)\, dt. \tag{2.14}$$

Thus

$$Z = \int_0^1 B, \tag{2.15}$$

where on the right we have the Chen integral (discussed earlier in (2.5)) of the 2-form B on P, lifting it to an LH-valued 1-form on the space of ($\overline{A}$-horizontal) smooth paths $[0,1] \to P$. The Chen integral here is, by definition, the 1-form on $\mathcal{P}_{\overline{A}}P$ given by

$$\tilde{v} \in T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P \mapsto \int_0^1 B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right)\, dt.$$

Note that Z and the form θ are closely related:

$$Z(\tilde{v}) = \theta_{\tilde{\gamma}}(\pi_* \tilde{v}). \tag{2.16}$$

Now define the 1-form $\omega_{(A,B)}$ by

$$\boxed{\omega_{(A,B)} = \mathrm{ev}_1^* A + \tau(Z)} \tag{2.17}$$

Recall that $\tau : H \to G$ is a homomorphism, and, for any $X \in LH$, we are writing $\tau(X)$ to mean $\tau'(e)X$; here $\tau'(e) : LH \to LG$ is the derivative of τ at the identity. The utility of bringing in τ becomes clear only when connecting these developments to the category theoretic formulation of section 3. A similar construction, but using only one algebra LG, is described by Cattaneo et al. [5]. However, as we pointed out earlier, a parallel transport operator for a surface cannot be constructed using a single group unless the group is abelian. To allow non-abelian groups, we need to have two groups intertwined in the structure described in (2.1), and thus we need τ.

Note that $\omega_{(A,B)}$ is simply the connection $\mathrm{ev}_1^* A$ on the bundle $\mathcal{P}_{\overline{A}}P$, shifted by the 1-form $\tau(Z)$. In the finite-dimensional setting it is a standard fact that such a shift, by an equivariant form which vanishes on verticals, produces another connection; however, given that our setting is, technically, not identical to the finite-dimensional one, we shall prove this below in Proposition 2.2.

Thus,

$$\omega_{(A,B)}(\tilde{v}) = A\big(\tilde{v}(1)\big) + \int_0^1 \tau B\big(\tilde{\gamma}'(t), \tilde{v}(t)\big)\, dt. \tag{2.18}$$

We can rewrite this as

$$\omega_{(A,B)} = \mathrm{ev}_0^* A + \left[\mathrm{ev}_1^*(A - \overline{A}) - \mathrm{ev}_0^*(A - \overline{A})\right] + \int_0^1 \left(F^{\overline{A}} + \tau B\right). \tag{2.19}$$

To obtain this we have simply used the relation (2.4). The advantage in (2.19) is that it separates off the end point terms and expresses $\omega_{(A,B)}$ as a perturbation of the simple connection $\mathrm{ev}_0^* A$ by a vector in the tangent space $T_{\mathrm{ev}_0^* A}\mathcal{A}$, where $\mathcal{A}$ is the space of connections on the bundle $\mathcal{P}_{\overline{A}}P$. Here note that the 'tangent vectors' to the affine space $\mathcal{A}$ at a connection ω are the 1-forms $\omega_1 - \omega$, with ω_1 running over $\mathcal{A}$. A difference such as $\omega_1 - \omega$ is precisely an equivariant LG-valued 1-form which vanishes on vertical vectors.

Recall that the group G acts on P on the right

$$P \times G \to P : (p, g) \mapsto R_g p = pg$$

and this induces a natural right action of G on $\mathcal{P}_{\overline{A}}P$:

$$\mathcal{P}_{\overline{A}}P \times G \to \mathcal{P}_{\overline{A}}P : (\tilde{\gamma}, g) \mapsto R_g\tilde{\gamma} = \tilde{\gamma}g$$

Then for any vector X in the Lie algebra LG, we have a vertical vector

$$\tilde{X}(\tilde{\gamma}) \in T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P$$

given by

$$\tilde{X}(\tilde{\gamma})(t) = \frac{d}{du}\Big|_{u=0} \tilde{\gamma}(t)\exp(uX)$$

Proposition 2.2. *The form $\omega_{(A,B)}$ is a connection form on the principal G-bundle $\mathcal{P}_{\overline{A}}P \to \mathcal{P}M$. More precisely,*

$$\omega_{(A,B)}\big((R_g)_* v\big) = \mathrm{Ad}(g^{-1})\omega_{(A,B)}(v)$$

for every $g \in G$, $\tilde{v} \in T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big)$ and

$$\omega_{(A,B)}(\tilde{X}) = X$$

for every $X \in LG$.

Proof. It will suffice to show that for every $g \in G$,

$$Z\big((R_g)_* v\big) = \mathrm{Ad}(g^{-1})Z(v)$$

and every vector v tangent to $\mathcal{P}_{\overline{A}}P$, and

$$Z(\tilde{X}) = 0$$

for every $X \in LG$.

From (2.15) and the fact that B vanishes on verticals it is clear that $Z(\tilde{X})$ is 0. The equivariance under the G-action follows also from (2.15), on using the G-equivariance of the connection form A and of the 2-form B, and the fact that the right action of G carries $\overline{A}$-horizontal paths into $\overline{A}$-horizontal paths. ◻

Parallel transport by $\omega_{(A,B)}$

Let us examine how a path is parallel-transported by $\omega_{(A,B)}$. At the infinitesimal level, all we need is to be able to lift a given vector field $v : [0,1] \to TM$, along $\gamma \in \mathcal{P}M$, to a vector field $\tilde{v}$ along $\tilde{\gamma}$ such that:

(i) $\tilde{v}$ is a vector in $T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big)$, which means that it satisfies the equation (2.10):

$$\frac{\partial \overline{A}(\tilde{v}(t))}{\partial t} = F^{\overline{A}}\left(\tilde{\gamma}'(t), \tilde{v}(t)\right); \tag{2.20}$$

(ii) $\tilde{v}$ is $\omega_{(A,B)}$-horizontal, i.e. satisfies the equation

$$A\big(\tilde{v}(1)\big) + \int_0^1 \tau B\big(\tilde{\gamma}'(t), \tilde{v}(t)\big)\, dt = 0. \tag{2.21}$$

The following result gives a constructive description of $\tilde{v}$.

Proposition 2.3. *Assume that A, $\overline{A}$, B, and $\omega_{(A,B)}$ are as specified before. Let $\tilde{\gamma} \in \mathcal{P}_{\overline{A}}P$, and $\gamma = \pi \circ \tilde{\gamma} \in \mathcal{P}M$ its projection to a path on M, and consider any $v \in T_\gamma \mathcal{P}M$. Then the $\omega_{(A,B)}$-horizontal lift $\tilde{v} \in T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P$ is given by*

$$\tilde{v}(t) = \tilde{v}^{\mathrm{h}}_{\overline{A}}(t) + \tilde{v}^{\mathrm{v}}(t),$$

where $\tilde{v}^{\mathrm{h}}_{\overline{A}}(t) \in T_{\tilde{\gamma}(t)}P$ is the $\overline{A}$-horizontal lift of $v(t) \in T_{\gamma(t)}M$, and

$$\tilde{v}^{\mathrm{v}}(t) = \tilde{\gamma}(t)\left[\overline{A}\big(\tilde{v}(1)\big) - \int_t^1 F^{\overline{A}}\big(\tilde{\gamma}'(u), \tilde{v}^{h}_{\overline{A}}(u)\big)\, du\right] \tag{2.22}$$

wherein

$$\tilde{v}(1) = \tilde{v}^{h}_{A}(1) + \tilde{\gamma}(1)X, \tag{2.23}$$

with $\tilde{v}^{h}_{A}(1)$ being the A-horizontal lift of $v(1)$ in $T_{\tilde{\gamma}(1)}P$, and

$$X = -\int_0^1 \tau B\big(\tilde{\gamma}'(t), \tilde{v}^{h}_{\overline{A}}(t)\big)\, dt. \tag{2.24}$$

Note that X in (2.24) is $A\big(\tilde{v}(1)\big)$.

Note also that since $\tilde{v}$ is tangent to $\mathcal{P}_{\overline{A}}P$, the vector $\tilde{v}^{\mathrm{v}}(t)$ is also given by

$$\tilde{v}^{\mathrm{v}}(t) = \tilde{\gamma}(t)\left[\overline{A}\big(\tilde{v}(0)\big) + \int_0^t F^{\overline{A}}\big(\tilde{\gamma}'(u), \tilde{v}^{h}_{\overline{A}}(u)\big)\, du\right] \tag{2.25}$$

Proof. The $\omega_{(A,B)}$ horizontal lift $\tilde{v}$ of v in $T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big)$ is the vector field $\tilde{v}$ along $\tilde{\gamma}$ which projects by π_* to v and satisfies the condition (2.21):

$$A\big(\tilde{v}(1)\big) + \int_0^1 \tau B\big(\tilde{\gamma}'(t), \tilde{v}(t)\big)\, dt = 0. \tag{2.26}$$

Now for each $t \in [0,1]$, we can split the vector $\tilde{v}(t)$ into an $\overline{A}$-horizontal part and a vertical part $\tilde{v}^{v}(t)$ which is essentially the element $\overline{A}\big(\tilde{v}^{v}(t)\big) \in LG$ viewed as a vector in the vertical subspace in $T_{\tilde{\gamma}(t)}P$:

$$\tilde{v}(t) = \tilde{v}^{h}_{\overline{A}}(t) + \tilde{v}^{v}(t)$$

and the vertical part here is given by

$$\tilde{v}^v(t) = \tilde{\gamma}(t)\overline{A}\big(\tilde{v}(t)\big).$$

Since the vector field $\tilde{v}$ is actually a vector in $T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big)$, we have, from (2.20), the relation

$$\overline{A}\big(\tilde{v}(t)\big) = \overline{A}\big(\tilde{v}(1)\big) - \int_t^1 F^{\overline{A}}\big(\tilde{\gamma}'(u), \tilde{v}^h_{\overline{A}}(u)\big)\, du.$$

We need now only verify the expression (2.23) for $\tilde{v}(1)$. To this end, we first split this into A-horizontal and a corresponding vertical part:

$$\tilde{v}(1) = \tilde{v}^h_A(1) + \tilde{\gamma}(1)A\big(\tilde{v}(1)\big)$$

The vector $A\big(\tilde{v}(1)\big)$ is obtained from (2.26), and thus proves (2.23). □

There is an observation to be made from Proposition 2.3. The equation (2.24) has, on the right side, the integral over the entire curve $\tilde{\gamma}$. Thus, if we were to consider parallel-transport of only, say, the 'left half' of $\tilde{\gamma}$, we would, in general, end up with a *different* path of paths!

Reparametrization Invariance

If a path is reparametrized, then, technically, it is a different point in path space. Does parallel-transport along a path of paths depend on the specific parametrization of the paths? We shall obtain conditions to ensure that there is no such dependence. Moreover, in this case, we shall also show that parallel transport by $\omega_{(A,B)}$ along a path of paths depends essentially on the surface swept out by this path of paths, rather than the specific parametrization of this surface.

For the following result, recall that we are working with Lie groups G, H, smooth homomorphism $\tau : H \to G$, smooth map $\alpha : G \times H \to H : (g, h) \mapsto \alpha(g)h$, where each $\alpha(g)$ is an automorphism of H, and the maps τ and α satisfy (2.1). Let $\pi : P \to M$ be a principal G-bundle, with connections A and $\overline{A}$, and B an LH-valued α-equivariant 2-form on P vanishing on vertical vectors. As before, on the space $\mathcal{P}_{\overline{A}}P$ of $\overline{A}$-horizontal paths, viewed as a principal G-bundle over the space $\mathcal{P}M$ of smooth paths in M, there is the connection form $\omega_{(A,B)}$ given by

$$\omega_{(A,B)} = \mathrm{ev}_1^* A + \int_0^1 \tau B.$$

By a 'smooth path' $s \mapsto \Gamma_s$ in $\mathcal{P}M$ we mean a smooth map

$$[0,1]^2 \to M : (t, s) \mapsto \Gamma(t, s) = \Gamma_s(t),$$

viewed as a path of paths $\Gamma_s \in \mathcal{P}M$.

With this notation and framework, we have:

Theorem 2.1. *Let*

$$\Phi : [0,1]^2 \to [0,1]^2 : (t, s) \mapsto (\Phi_s(t), \Phi^t(s))$$

be a smooth diffeomorphism which fixes each vertex of $[0,1]^2$. *Assume that* *(i)* *either*

$$F^{\overline{A}} + \tau(B) = 0 \tag{2.27}$$

and Φ *carries each* s*-fixed section* $[0,1] \times \{s\}$ *into an* s*-fixed section* $[0,1] \times \{\Phi^0(s)\}$*;*

(ii) *or*

$$\left[\mathrm{ev}_1^*(A - \overline{A}) - \mathrm{ev}_0^*(A - \overline{A})\right] + \int_0^1 \left(F^{\overline{A}} + \tau B\right) = 0, \tag{2.28}$$

Φ *maps each boundary edge of* $[0,1]^2$ *into itself, and* $\Phi^0(s) = \Phi^1(s)$ *for all* $s \in [0,1]$*.*

Then the $\omega_{(A,B)}$*-parallel-translate of the point* $\tilde{\Gamma}_0 \circ \Phi_0$ *along the path* $s \mapsto (\Gamma \circ \Phi)_s$*, is* $\tilde{\Gamma}_1 \circ \Phi_1$*, where* $\tilde{\Gamma}_1$ *is the* $\omega_{(A,B)}$*-parallel-translate of* $\tilde{\Gamma}_0$ *along* $s \mapsto \Gamma_s$*.*

As a special case, if the path $s \mapsto \Gamma_s$ *is constant and* Φ_0 *the identity map on* $[0,1]$*, so that* Γ_1 *is simply a reparametrization of* Γ_0*, then, under conditions (i) or (ii) above, the* $\omega_{(A,B)}$*-parallel-translate of the point* $\tilde{\Gamma}_0$ *along the path* $s \mapsto (\Gamma \circ \Phi)_s$*, is* $\tilde{\Gamma}_0 \circ \Phi_1$*, i.e., the appropriate reparametrizaton of the original path* $\tilde{\Gamma}_0$*.*

Note that the path $(\tilde{\Gamma} \circ \Phi)_0$ projects down to $(\Gamma \circ \Phi)_0$, which, by the boundary behavior of Φ, is actually that path $\Gamma_0 \circ \Phi_0$, in other words Γ_0 reparametrized. Similarly, $(\tilde{\Gamma} \circ \Phi)_1$ is an $\overline{A}$-horizontal lift of the path Γ_1, reparametrized by Φ_1.

If $A = \overline{A}$ then conditions (2.28) and (2.27) are the same, and so in this case the weaker condition on Φ in (ii) suffices.

Proof. Suppose (2.27) holds. Then the connection $\omega_{(A,B)}$ has the form

$$\mathrm{ev}_0^* A + \left[\mathrm{ev}_1^*(A - \overline{A}) - \mathrm{ev}_0^*(A - \overline{A})\right].$$

The crucial point is that this depends only on the end points, i.e., if $\tilde{\gamma} \in \mathcal{P}_{\overline{A}}P$ and $\tilde{V} \in T_{\tilde{\gamma}}\mathcal{P}_{\overline{A}}P$ then $\omega_{(A,B)}(\tilde{V})$ depends only on $\tilde{V}(0)$ and $\tilde{V}(1)$. If the conditions on Φ in (i) hold then reparametrization has the effect of replacing each $\tilde{\Gamma}_s$ with $\tilde{\Gamma}_{\Phi^0(s)} \circ \Phi_s$, which is in $\mathcal{P}_{\overline{A}}P$, and the vector field $t \mapsto \partial_s(\tilde{\Gamma}_{\Phi^0(s)} \circ \Phi_s(t))$ is an $\omega_{(A,B)}$-horizontal vector, because its end point values are those of $t \mapsto \partial_s(\tilde{\Gamma}_{\Phi^0(s)}(t))$, since $\Phi_s(t)$ equals t if t is 0 or 1.

Now suppose (2.28) holds. Then $\omega_{(A,B)}$ becomes simply $\mathrm{ev}_0^* A$. In this case $\omega_{(A,B)}(\tilde{V})$ depends on $\tilde{V}$ only through the initial value $\tilde{V}(0)$. Thus, the $\omega_{(A,B)}$-parallel-transport of $\tilde{\gamma} \in \mathcal{P}_{\overline{A}}P$, along a path $s \mapsto \Gamma_s \in \mathcal{P}M$, is obtained by A-parallel-transporting the initial point $\tilde{\gamma}(0)$ along the path $s \mapsto \Gamma^0(s)$, and shooting off $\overline{A}$-horizontal paths lying above the paths Γ_s. (Since the paths Γ_s do not necessarily have the second component fixed, their horizontal lifts need not be of the form $\tilde{\Gamma}_s \circ \Phi_s$, except at $s = 0$ and $s = 1$, when the composition $\tilde{\Gamma}_{\Phi_s} \circ \Phi_s$ is guaranteed to be meaningful.) From this it is clear that parallel translating $\tilde{\Gamma}_0 \circ \Phi_0$, by $\omega_{(A,B)}$ along the path $s \mapsto \Gamma_s$, results, at $s = 1$, in the path $\tilde{\Gamma}_1 \circ \Phi_1$. □

The curvature of $\omega_{(A,B)}$

We can compute the curvature of the connection $\omega_{(A,B)}$. This is, by definition,

$$\Omega_{(A,B)} = d\omega_{(A,B)} + \frac{1}{2}[\omega_{(A,B)} \wedge \omega_{(A,B)}],$$

where the exterior differential d is understood in a natural sense that will become clearer in the proof below. More technically, we are using here notions of calculus on smooth spaces; see, for instance, Stacey [16] for a survey, and Viro [17] for another approach.

First we describe some notation about Chen integrals in the present context. If B is a 2-form on P, with values in a Lie algebra, then its Chen integral $\int_0^1 B$, restricted to $\mathcal{P}_{\overline{A}}P$, is a 1-form on $\mathcal{P}_{\overline{A}}P$ given on the vector $\tilde{V} \in T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big)$ by

$$\left(\int_0^1 B\right)(\tilde{V}) = \int_0^1 B\big(\tilde{\gamma}'(t), \tilde{V}(t)\big)\, dt.$$

If C is also a 2-form on P with values in the same Lie algebra, we have a product 2-form on the path space $\mathcal{P}_{\overline{A}}P$ given on $\tilde{X}, \tilde{Y} \in T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big)$ by

$$\begin{aligned}
&\left(\int_0^1\right)^2 [B \wedge C](\tilde{X}, \tilde{Y}) \\
&\quad = \int_{0 \le u < v \le 1} \Big[B\big(\tilde{\gamma}'(u), \tilde{X}(u)\big), C\big(\tilde{\gamma}'(v), \tilde{Y}(v)\big)\Big]\, du\, dv \\
&\qquad - \int_{0 \le u < v \le 1} \Big[C\big(\tilde{\gamma}'(u), \tilde{X}(u)\big), B\big(\tilde{\gamma}'(v), \tilde{Y}(v)\big)\Big]\, du\, dv \\
&\quad = \int_0^1 \int_0^1 \Big[B\big(\tilde{\gamma}'(u), \tilde{X}(u)\big), C\big(\tilde{\gamma}'(v), \tilde{Y}(v)\big)\Big]\, du\, dv.
\end{aligned} \tag{2.29}$$

Proposition 2.4. *The curvature of $\omega_{(A,B)}$ is*

$$\begin{aligned}
\Omega^{\omega_{(A,B)}} = \mathrm{ev}_1^* F^A + d\left(\int_0^1 \tau B\right) \\
+ \left[\mathrm{ev}_1^* A \wedge \int_0^1 \tau B\right] + \left(\int_0^1\right)^2 [\tau B \wedge \tau B],
\end{aligned} \tag{2.30}$$

where the integrals are Chen integrals.

Proof. From

$$\omega_{(A,B)} = \mathrm{ev}_1^* A + \int_0^1 \tau B,$$

we have

$$\begin{aligned}
\Omega^{\omega_{(A,B)}} &= d\omega_{(A,B)} + \frac{1}{2}[\omega_{(A,B)} \wedge \omega_{(A,B)}] \\
&= \mathrm{ev}_1^* dA + d\int_0^1 \tau B + W,
\end{aligned} \tag{2.31}$$

where

$$
\begin{aligned}
W(\tilde{X},\tilde{Y}) &= [\omega_{(A,B)}(\tilde{X}),\omega_{(A,B)}(\tilde{Y})] \\
&= [\mathrm{ev}_1^*A(\tilde{X}),\mathrm{ev}_1^*A(\tilde{Y})] \\
&\quad + \left[\mathrm{ev}_1^*A(\tilde{X}),\int_0^1 \tau B\big(\tilde{\gamma}'(t),\tilde{Y}(t)\big)\,dt\right] \\
&\quad + \left[\int_0^1 \tau B\big(\tilde{\gamma}'(t),\tilde{X}(t)\big)\,dt,\mathrm{ev}_1^*A(\tilde{Y})\right] \\
&\quad + \int_0^1\int_0^1 \Big[\tau B\big(\tilde{\gamma}'(u),\tilde{X}(u)\big),\tau B\big(\tilde{\gamma}'(v),\tilde{Y}(v)\big)\Big]\,du\,dv \\
&= [\mathrm{ev}_1^*A,\mathrm{ev}_1^*A](\tilde{X},\tilde{Y}) + \left[\mathrm{ev}_1^*A\wedge\int_0^1 \tau B\right](\tilde{X},\tilde{Y}) \\
&\quad + \left(\int_0^1\right)^2 [\tau B\wedge\tau B](\tilde{X},\tilde{Y}).
\end{aligned}
\tag{2.32}
$$

□

In the case $A=\overline{A}$, and without τ, the expression for the curvature can be expressed in terms of the 'fake curvature' $F^{\overline{A}}+B$. For a result of this type, for a related connection form, see Cattaneo et al. [5, Theorem 2.6] have calculated a similar formula for curvature of a related connection form.

A more detailed exploration of the fake curvature would be of interest.

Parallel-transport of horizontal paths

As before, A and $\overline{A}$ are connections on a principal G-bundle $\pi: P\to M$, and B is an LH-valued α-equivariant 2-form on P vanishing on vertical vectors. Also $\mathcal{P}X$ is the space of smooth paths $[0,1]\to X$ in a space X, and $\mathcal{P}_{\overline{A}}P$ is the space of smooth $\overline{A}$-horizontal paths in P.

Our objective now is to express parallel-transport along paths in $\mathcal{P}M$ in terms of a smooth local section of the bundle $P\to M$:

$$\sigma: U\to P$$

where U is an open set in M. We will focus only on paths lying entirely inside U.

The section σ determines a section $\tilde{\sigma}$ for the bundle $\mathcal{P}_{\overline{A}}P\to\mathcal{P}M$: if $\gamma\in\mathcal{P}M$ then $\tilde{\sigma}(\gamma)$ is the unique $\overline{A}$-horizontal path in P, with initial point $\sigma\big(\gamma(0)\big)$, which projects down to γ. Thus,

$$\tilde{\sigma}(\gamma)(t)=\sigma(\gamma(t))\overline{a}(t), \tag{2.33}$$

for all $t\in[0,1]$, where $\overline{a}(t)\in G$ satisfies the differential equation

$$\overline{a}(t)^{-1}\overline{a}'(t) = -\mathrm{Ad}\big(\overline{a}(t)^{-1}\big)\overline{A}\,((\sigma\circ\gamma)'(t)) \tag{2.34}$$

for $t\in[0,1]$, and the initial value $\overline{a}(0)$ is e.

Recall that a tangent vector $V\in T_\gamma\big(\mathcal{P}M\big)$ is a smooth vector field along the path γ. Let us denote $\tilde{\sigma}(\gamma)$ by $\tilde{\gamma}$:

$$\tilde{\gamma}\stackrel{\text{def}}{=}\tilde{\sigma}(\gamma).$$

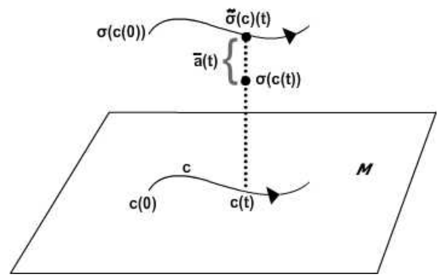

Fig. 3. The section $\tilde{\sigma}$ applied to a path c

Note, for later use, that

$$\tilde{\gamma}'(t) = \sigma_*\big(\gamma'(t)\big)\overline{a}(t) + \underbrace{\tilde{\gamma}(t)\overline{a}(t)^{-1}\overline{a}'(t)}_{\text{vertical}}. \tag{2.35}$$

Now define the vector

$$\tilde{V} = \tilde{\sigma}_*(V) \in T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big) \tag{2.36}$$

to be the vector $\tilde{V}$ in $T_{\tilde{\gamma}}\big(\mathcal{P}_{\overline{A}}P\big)$ whose initial value $\tilde{V}(0)$ is

$$\tilde{V}(0) = \sigma_*\big(V(0)\big).$$

The existence and uniqueness of $\tilde{V}$ was proved in Lemma 2.1.

Note that $\tilde{V}(t) \in T_{\tilde{\gamma}(t)}P$ and $(\sigma_*V)(t) \in T_{\sigma(\gamma(t))}P$, are generally different vectors. However, $(\sigma_*V)(t)\overline{a}(t)$ and $\tilde{V}(t)$ are both in $T_{\tilde{\gamma}(t)}P$ and differ by a vertical vector because they have the same projection $V(t)$ under π_*:

$$\tilde{V}(t) = (\sigma_*V)(t)\overline{a}(t) + \text{vertical vector}. \tag{2.37}$$

Our objective now is to determine the LG-valued 1-form

$$\omega_{(\overline{A},A,B)} = \tilde{\sigma}^*\omega_{(A,B)} \tag{2.38}$$

on $\mathcal{P}M$, defined on any vector $V \in T_\gamma(\mathcal{P}M)$ by

$$\omega_{(\overline{A},A,B)}(V) = \omega_{(A,B)}\big(\tilde{\sigma}_*V\big). \tag{2.39}$$

We can now work out an explicit expression for this 1-form.

Proposition 2.5. *With notation as above, and* $V \in T_\gamma\big(\mathcal{P}M\big)$,

$$\omega_{(\overline{A},A,B)}(V) = \mathrm{Ad}\big(\overline{a}(1)^{-1}\big)A_\sigma\left(V(1)\right) + \int_0^1 \mathrm{Ad}\big(\overline{a}(t)^{-1}\big)\tau B_\sigma\big(\gamma'(t), V(t)\big)\,dt, \tag{2.40}$$

where C_σ *denotes the pullback* σ^*C *on* M *of a form* C *on* P*, and* $\overline{a} : [0,1] \to G$ *describes parallel-transport along* γ*, i.e., satisfies*

$$\overline{a}(t)^{-1}\overline{a}'(t) = -\mathrm{Ad}\big(\overline{a}(t)^{-1}\big)\overline{A}_\sigma\big(\gamma'(t)\big)$$

with initial condition $\overline{a}(0) = e$*. The formula for* $\omega_{(\overline{A},A,B)}(V)$ *can also be expressed as*

$$\begin{aligned}\omega_{(\overline{A},A,B)}&(V)\\ &= A_\sigma\left(V(0)\right) + \left[\mathrm{Ad}\big(\overline{a}(1)^{-1}\big)(A_\sigma - \overline{A}_\sigma)\left(V(1)\right) - (A_\sigma - \overline{A}_\sigma)\left(V(0)\right)\right]\\ &\qquad + \int_0^1 \mathrm{Ad}\big(\overline{a}(t)^{-1}\big)\big(F_\sigma^{\overline{A}} + \tau B_\sigma\big)\big(\gamma'(t), V(t)\big)\,dt.\end{aligned} \tag{2.41}$$

Note that in (2.41), the terms involving $\overline{A}_\sigma$ and $F_\sigma^{\overline{A}}$ cancel each other out.

Proof. From the definition of $\omega_{(A,B)}$ in (2.17) and (2.14), we see that we need only focus on the B term. To this end we have, from (2.35) and (2.37):

$$\begin{aligned}B\big(\tilde\gamma'(t), \tilde V(t)\big) &= B\big(\sigma_*\big(\gamma'(t)\big)\overline{a}(t) + \text{vertical}, (\sigma_* V)(t)\overline{a}(t) + \text{vertical}\big)\\ &= B\big(\sigma_*\big(\gamma'(t)\big)\overline{a}(t), (\sigma_* V)(t)\overline{a}(t)\big)\\ &= \alpha\big(\overline{a}(t)^{-1}\big)B_\sigma\big(\gamma'(t), V(t)\big).\end{aligned} \tag{2.42}$$

Now recall the relation (2.1)

$$\tau\big(\alpha(g)h\big) = g\tau(h)g^{-1}, \text{ for all } g \in G \text{ and } h \in H,$$

which implies

$$\tau\big(\alpha(g)K\big) = \mathrm{Ad}(g)\tau(K) \text{ for all } g \in G \text{ and } K \in LH.$$

As usual, we are denoting the derivatives of τ and α by τ and α again. Applying this to (2.42) we have

$$\tau B\big(\tilde\gamma'(t), \tilde V(t)\big) = \mathrm{Ad}\big(\overline{a}(t)^{-1}\big)\tau B_\sigma\big(\gamma'(t), V(t)\big),$$

and this yields the result. □

Suppose

$$\tilde\Gamma : [0,1]^2 \to P : (t,s) \mapsto \tilde\Gamma(t,s) = \tilde\Gamma_s(t) = \tilde\Gamma^t(s)$$

is smooth, with each $\tilde\Gamma_s$ being $\overline{A}$-horizontal, and the path $s \mapsto \tilde\Gamma(0,s)$ being A-horizontal. Let $\Gamma = \pi \circ \tilde\Gamma$. We will need to use the *bi-holonomy* $g(t,s)$ which is specified as follows: parallel translate $\tilde\Gamma(0,0)$ along $\Gamma_0|[0,t]$ by $\overline{A}$, then up the path $\Gamma^t|[0,s]$ by A, back along Γ_s-reversed by $\overline{A}$ and then down $\Gamma^0|[0,s]$ by A; then the resulting point is

$$\tilde\Gamma(0,0)g(t,s). \tag{2.43}$$

The path

$$s \mapsto \tilde\Gamma_s$$

describes parallel transport of the initial path $\tilde\Gamma_0$ using the connection $\mathrm{ev}_0^* A$. In what follows we will compare this with the path

$$s \mapsto \hat\Gamma_s$$

which is the parallel transport of $\hat{\Gamma}_0 = \tilde{\Gamma}_0$ using the connection ev_1^*A. The following result describes the 'difference' between these two connections.

Proposition 2.6. *Suppose*

$$\tilde{\Gamma} : [0,1]^2 \to P : (t,s) \mapsto \tilde{\Gamma}(t,s) = \tilde{\Gamma}_s(t) = \tilde{\Gamma}^t(s)$$

is smooth, with each $\tilde{\Gamma}_s$ *being* $\overline{A}$*-horizontal, and the path* $s \mapsto \tilde{\Gamma}(0,s)$ *being* A*-horizontal. Then the parallel translate of* $\tilde{\Gamma}_0$ *by the connection* ev_1^*A *along the path* $[0,s] \to \mathcal{P}M : u \mapsto \Gamma_u$*, where* $\Gamma = \pi \circ \tilde{\Gamma}$*, results in* $\tilde{\Gamma}_s g(1,s)$*, with* $g(1,s)$ *being the 'bi-holonomy' specified as in (2.43).*

Proof. Let $\hat{\Gamma}_s$ be the parallel translate of $\tilde{\Gamma}_0$ by ev_1^*A along the path $[0,s] \to \mathcal{P}M : u \mapsto \Gamma_u$. Then the right end point $\hat{\Gamma}_s(1)$ traces out an A-horizontal path, starting at $\tilde{\Gamma}_0(1)$. Thus, $\hat{\Gamma}_s(1)$ is the result of parallel transporting $\tilde{\Gamma}(0,0)$ by $\overline{A}$ along Γ_0 then up the path $\Gamma^1|[0,s]$ by A. If we then parallel transport $\hat{\Gamma}_s(1)$ back by $\overline{A}$ along $\Gamma_s|[0,1]$-reversed then we obtain the initial point $\hat{\Gamma}_s(0)$. This point is of the form $\tilde{\Gamma}_s(0)b$, for some $b \in G$, and so

$$\hat{\Gamma}_s = \tilde{\Gamma}_s b.$$

Then, parallel-transporting $\hat{\Gamma}_s(0)$ back down $\Gamma^0|[0,s]$-reversed, by A, produces the point $\tilde{\Gamma}(0,0)b$. This shows that b is the bi-holonomy $g(1,s)$. □

Now we can turn to determining the parallel-transport process by the connection $\omega_{(A,B)}$. With $\tilde{\Gamma}$ as above, let now $\check{\Gamma}_s$ be the $\omega_{(A,B)}$-parallel-translate of $\tilde{\Gamma}_0$ along $[0,s] \to \mathcal{P}M : u \mapsto \Gamma_u$. Since $\check{\Gamma}_s$ and $\tilde{\Gamma}_s$ are both $\overline{A}$-horizontal and project by π_* down to Γ_s, we have

$$\check{\Gamma}_s = \hat{\Gamma}_s b_s,$$

for some $b_s \in G$. Since $\omega_{(A,B)} = \mathrm{ev}_1^*A + \tau(Z)$ applied to the s-derivative of $\check{\Gamma}_s$ is 0, and ev_1^*A applied to the s-derivative of $\hat{\Gamma}_s$ is 0, we have

$$b_s^{-1}\partial_s b_s + \mathrm{Ad}(b_s^{-1})\tau Z(\partial_s \hat{\Gamma}_s) = 0 \tag{2.44}$$

Thus, $s \mapsto b_s$ describes parallel transport by θ^σ where the section σ satisfies $\sigma \circ \Gamma = \hat{\Gamma}$.

Since $\hat{\Gamma}_s = \tilde{\Gamma}_s g(1,s)$, we then have

$$\begin{aligned} \frac{db_s}{ds} b_s^{-1} &= -\mathrm{Ad}\big(g(1,s)^{-1}\big)\tau Z(\partial_s \tilde{\Gamma}_s) \\ &= -\mathrm{Ad}\big(g(1,s)^{-1}\big) \int_0^1 \tau B\big(\partial_t \tilde{\Gamma}(t,s), \partial_s \tilde{\Gamma}(t,s)\big)\, dt \end{aligned} \tag{2.45}$$

To summarize:

Theorem 2.2. *Suppose*

$$\tilde{\Gamma} : [0,1]^2 \to P : (t,s) \mapsto \tilde{\Gamma}(t,s) = \tilde{\Gamma}_s(t) = \tilde{\Gamma}^t(s)$$

is smooth, with each $\tilde{\Gamma}_s$ being $\overline{A}$-horizontal, and the path $s \mapsto \tilde{\Gamma}(0,s)$ being A-horizontal. Then the parallel translate of $\tilde{\Gamma}_0$ by the connection $\omega_{(A,B)}$ along the path $[0,s] \to \mathcal{P}M : u \mapsto \Gamma_u$, where $\Gamma = \pi \circ \tilde{\Gamma}$, results in

$$\tilde{\Gamma}_s g(1,s)\tau\big(h_0(s)\big), \tag{2.46}$$

with $g(1,s)$ being the 'bi-holonomy' specified as in (2.43), and $s \mapsto h_0(s) \in H$ solving the differential equation

$$\frac{dh_0(s)}{ds} h_0(s)^{-1} = -\alpha\big(g(1,s)^{-1}\big) \int_0^1 B\big(\partial_t\tilde{\Gamma}(t,s), \partial_s\tilde{\Gamma}(t,s)\big)\, dt \tag{2.47}$$

with initial condition $h_0(0)$ being the identity in H.

Let σ be a smooth section of the bundle $P \to M$ in a neighborhood of $\Gamma([0,1]^2)$.

Let $a_t(s) \in G$ specify parallel transport by A up the path $[0,s] \to M : v \mapsto \Gamma(t,v)$, i.e., the A-parallel-translate of $\sigma\Gamma(t,0)$ up the path $[0,s] \to M : v \mapsto \Gamma(t,v)$ results in $\sigma(\Gamma(t,s))a_t(s)$.

On the other hand, $\overline{a}_s(t)$ will specify parallel transport by $\overline{A}$ along $[0,t] \to M : u \mapsto \Gamma(u,s)$. Thus,

$$\tilde{\Gamma}(t,s) = \sigma\big(\Gamma(t,s)\big)a_0(s)\overline{a}_s(t) \tag{2.48}$$

The bi-holonomy is given by

$$g(1,s) = a_0(s)^{-1}\overline{a}_s(1)^{-1}a_1(s)\overline{a}_0(1).$$

Let us look at parallel-transport along the path $s \mapsto \Gamma_s$, by the connection $\omega_{(A,B)}$, in terms of the trivialization σ. Let $\hat{\Gamma}_s \in \mathcal{P}_{\overline{A}}P$ be obtained by parallel transporting $\tilde{\Gamma}_0 = \tilde{\sigma}(\Gamma_0) \in \mathcal{P}_{\overline{A}}P$ along the path

$$[0,s] \to M : u \mapsto \Gamma^0(u) = \Gamma(0,u).$$

This transport is described through a map

$$[0,1] \to G : s \mapsto c(s),$$

specified through

$$\hat{\Gamma}_s = \tilde{\sigma}(\Gamma_s)c(s) = \tilde{\Gamma}_s a_0(s)^{-1}c(s). \tag{2.49}$$

Then $c(0) = e$ and

$$c(s)^{-1}c'(s) = -\mathrm{Ad}\big(c(s)^{-1}\big)\omega_{(\overline{A},A,B)}\big(V(s)\big), \tag{2.50}$$

where $V_s \in T_{\Gamma_s}\mathcal{P}M$ is the vector field along Γ_s given by

$$V_s(t) = V(s,t) = \partial_s\Gamma(t,s) \qquad \text{for all } t \in [0,1].$$

Equation (2.50), written out in more detail, is

$$\begin{aligned} c(s)^{-1}c'(s) = -\mathrm{Ad}\big(c(s)^{-1}\big)\Big[&\mathrm{Ad}\big(\overline{a}_s(1)^{-1}\big)A_\sigma\big(V_s(1)\big) \\ &+ \int_0^1 \mathrm{Ad}\big(\overline{a}_s(t)^{-1}\big)\tau B_\sigma\big(\Gamma_s'(t), V_s(t)\big)\, dt\Big], \end{aligned} \tag{2.51}$$

where $\overline{a}_s(t) \in G$ describes $\overline{A}_\sigma$-parallel-transport along $\Gamma_s|[0,t]$. By (2.46), $c(s)$ is given by

$$c(s) = a_0(s)g(1,s)\tau(h_0(s)),$$

where $s \mapsto h_0(s)$ solves

$$\frac{dh_0(s)}{ds}h_0(s)^{-1} = -\int_0^1 \alpha\big(\overline{a}_s(t)a_0(s)g(1,s)\big)^{-1}B_\sigma\big(\partial_t\Gamma(t,s), \partial_s\Gamma(t,s)\big)\,dt, \quad (2.52)$$

with initial condition $h_0(0)$ being the identity in H. The geometric meaning of $\overline{a}_s(t)a_0(s)$ is that it describes parallel-transport first by A_σ up from $(0,0)$ to $(0,s)$ and then to the right by $\overline{A}_\sigma$ from $(0,s)$ to (t,s).

3. Two categories from plaquettes

In this section we introduce two categories motivated by the differential geometric framework we have discussed in the preceding sections. We show that the geometric framework naturally connects with certain category theoretic structures introduced by Ehresmann [9,10] and developed further by Kelley and Street [12].

We work with the pair of Lie groups G and H, along with maps τ and α satisfying (2.1), and construct two categories. These categories will have the same set of objects, and also the same set of morphisms.

The set of objects is simply the group G:

$$\mathbf{Obj} = G.$$

The set of morphisms is

$$\mathbf{Mor} = G^4 \times H,$$

with a typical element denoted

$$(a, b, c, d; h).$$

It is convenient to visualize a morphism as a plaquette labeled with elements of G:

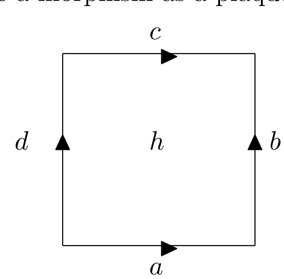

Fig. 4. Plaquette

To connect with the theory of the preceding sections, we should think of a and c as giving $\overline{A}$-parallel-transports, d and b as A-parallel-transports, and h should be thought of as corresponding to $h_0(1)$ of Theorem 2.2. However, this is only a rough guide; we shall return to this matter later in this section.

For the category **Vert**, the *source* (domain) and *target* (co-domain) of a morphism are:

$$s_{\mathbf{Vert}}(a, b, c, d; h) = a$$
$$t_{\mathbf{Vert}}(a, b, c, d; h) = c$$

For the category **Horz**

$$s_{\mathbf{Horz}}(a, b, c, d; h) = d$$
$$t_{\mathbf{Horz}}(a, b, c, d; h) = b$$

We define vertical composition, that is composition in **Vert**, using Figure 5. In this figure, the upper morphism is being applied first and then the lower.

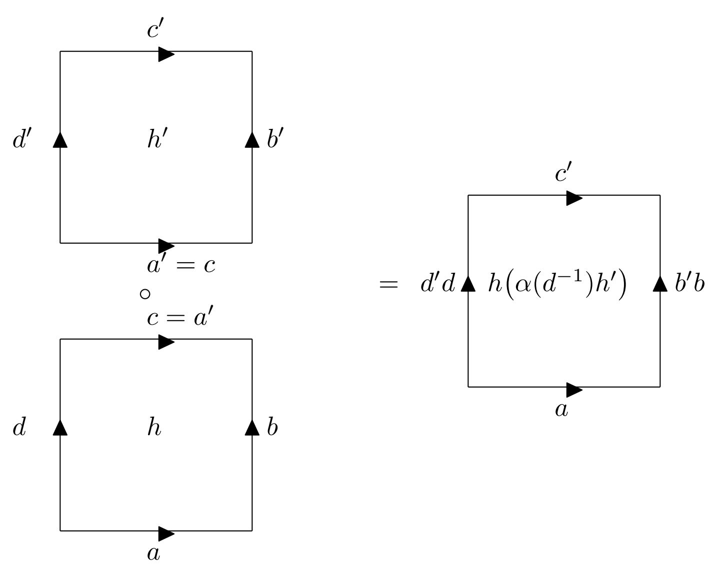

Fig. 5. Vertical Composition

Horizontal composition is specified through Figure 6. In this figure we have used the notation $\circ_{\rm opp}$ to stress that, as morphisms, it is the one to the left which is applied first and then the one to the right.

Our first observation is:

Proposition 3.1. *Both* ***Vert*** *and* ***Horz*** *are categories, under the specified composition laws. In both categories, all morphisms are invertible.*

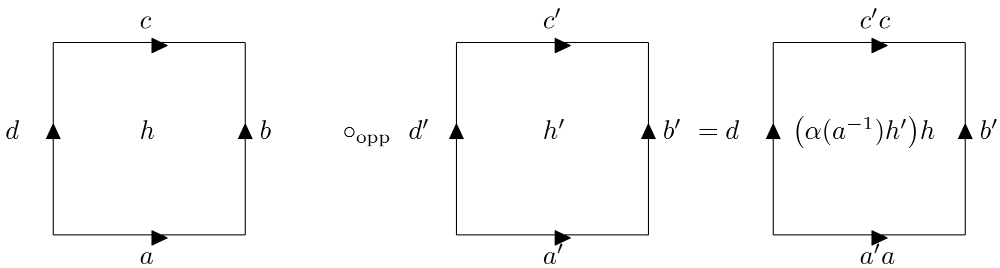

Fig. 6. Horizontal Composition (for $b = d'$).

Proof. It is straightforward to verify that the composition laws are associative. The identity map $a \to a$ in **Vert** is $(a, e, a, e; e)$, and in **Horz** it is $(e, a, e, a; e)$. These are displayed in in Figure 7. The inverse of the morphism $(a, b, c, d; h)$ in **Vert** is $(c, b^{-1}, a, d^{-1}; \alpha(d)h^{-1})$; the inverse in **Horz** is $(a^{-1}, d, c^{-1}, b; \alpha(a)h^{-1})$. $\square$

The two categories are isomorphic, but it is best not to identify them.

We use $\circ_H$ to denote horizontal composition, and $\circ_V$ to denote vertical composition.

We have seen earlier that if A, $\overline{A}$ and B are such that $\omega_{(A,B)}$ reduces to $\mathrm{ev}_0^* A$ (for example, if $A = \overline{A}$ and $F^{\overline{A}} + \tau(B)$ is 0) then all plaquettes $(a, b, c, d; h)$ arising from the connections A and $\omega_{(A,B)}$, satisfy

$$\tau(h) = a^{-1}b^{-1}cd.$$

Motivated by this observation, we could consider those morphisms $(a, b, c, d; h)$ which satisfy

$$\tau(h) = a^{-1}b^{-1}cd \tag{3.1}$$

However, we can look at a broader class of morphisms as well. Suppose

$$\underline{\mathrm{h}} \mapsto z(\underline{\mathrm{h}}) \in Z(G)$$

is a mapping of the morphisms in the category **Horz** or in **Vert** into the center $Z(G)$ of G, which carries composition of morphisms to products in $Z(G)$:

$$z(\underline{\mathrm{h}} \circ \underline{\mathrm{h}}') = z(\underline{\mathrm{h}})z(\underline{\mathrm{h}}').$$

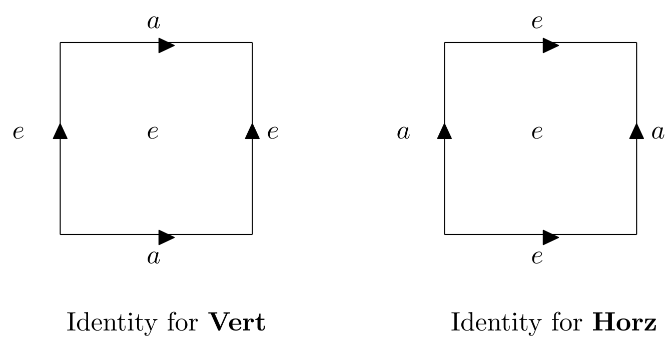

Fig. 7. Identity Maps.

Then we say that a morphism $\underline{\text{h}} = (a, b, c, d; h)$ is *quasi-flat* with respect to z if

$$\tau(h) = (a^{-1}b^{-1}cd)z(\underline{\text{h}}) \tag{3.2}$$

A larger class of morphisms could also be considered, by replacing $Z(G)$ by an abelian normal subgroup, but we shall not explore this here.

Proposition 3.2. *Composition of quasi-flat morphisms is quasi-flat. Thus, the quasi-flat morphisms form a subcategory in both* ***Horz*** *and* ***Vert****.*

Proof. Let $\underline{\text{h}} = (a, b, c, d; h)$ and $\underline{\text{h}}' = (a', b', c', d'; h')$ be quasi-flat morphisms in **Horz**, such that the horizontal composition $\underline{\text{h}}' \circ_H \underline{\text{h}}$ is defined, i.e., $b = d'$. Then

$$\underline{\text{h}}' \circ_H \underline{\text{h}} = (a'a, b', c'c, d; \{\alpha(a^{-1})h'\}h).$$

Applying τ to the last component in this, we have

$$\begin{aligned} a^{-1}\tau(h')a\tau(h) &= a^{-1}(a'^{-1}b'^{-1}c'd')a(a^{-1}b^{-1}cd)z(\underline{\text{h}})z(\underline{\text{h}}') \\ &= \big((a'a)^{-1}b'^{-1}(c'c)d\big)z(\underline{\text{h}}' \circ_H \underline{\text{h}}), \end{aligned} \tag{3.3}$$

which says that $\underline{\text{h}}' \circ_H \underline{\text{h}}$ is quasi-flat.

Now suppose $\underline{\text{h}} = (a, b, c, d; h)$ and $\underline{\text{h}}' = (a', b', c', d'; h')$ are quasi-flat morphisms in **Vert**, such that the vertical composition $\underline{\text{h}}' \circ_V \underline{\text{h}}$ is defined, i.e., $c = a'$. Then

$$\underline{\text{h}}' \circ_V \underline{\text{h}} = (a, b'b, c', d'd; h\{\alpha(d^{-1})h'\}).$$

Applying τ to the last component in this, we have

$$\begin{aligned}\tau(h)d^{-1}\tau(h')d &= (a^{-1}b^{-1}cd)d^{-1}(a'^{-1}b'^{-1}c'd')dz(\underline{\mathrm{h}})z(\underline{\mathrm{h}}') \\ &= \big(a'^{-1}(b'b)^{-1}c'd'd\big)z(\underline{\mathrm{h}}'\circ_V \underline{\mathrm{h}}),\end{aligned} \tag{3.4}$$

which says that $\underline{\mathrm{h}}'\circ_V \underline{\mathrm{h}}$ is quasi-flat. □

For a morphism $\underline{\mathrm{h}} = (a, b, c, d; h)$ we set

$$\tau(\underline{\mathrm{h}}) = \tau(h).$$

If $\underline{\mathrm{h}} = (a, b, c, d; h)$ and $\underline{\mathrm{h}}' = (a', b', c', d'; h')$ are morphisms then we say that they are τ-equivalent,

$$\underline{\mathrm{h}} =_\tau \underline{\mathrm{h}}'$$

if $a = a'$, $b = b'$., $c = c'$, $d = d'$, and $\tau(h) = \tau(h')$.

Proposition 3.3. *If $\underline{h}, \underline{h}', \underline{h}'', \underline{h}''$ are quasi-flat morphisms for which the compositions on both sides of (3.5) are meaningful, then*

$$(\underline{h}'''\circ_H \underline{h}'')\circ_V (\underline{h}'\circ_H \underline{h}) =_\tau (\underline{h}'''\circ_V \underline{h}')\circ_H (\underline{h}''\circ_V \underline{h}) \tag{3.5}$$

whenever all the compositions on both sides are meaningful.

Thus, the structures we are using here correspond to double categories as described by Kelly and Street [12, section 1.1]

Proof. This is a lengthy but straight forward verification. We refer to Figure 8. For a morphism $\underline{\mathrm{h}} = (a, b, c, d; h)$, let us write

$$\tau_\partial(\underline{\mathrm{h}}) = a^{-1}b^{-1}cd.$$

For the left side of (3.5), we have

$$\begin{aligned}(\underline{\mathrm{h}}'\circ_H \underline{\mathrm{h}}) &= (a'a, b', c'c, d; \{\alpha(a^{-1})h'\}h) \\ (\underline{\mathrm{h}}'''\circ_H \underline{\mathrm{h}}'') &= (c'c, b'', f'f, d'; \{\alpha(c^{-1})h'''\}h'') \\ \underline{\mathrm{h}}^* \stackrel{\text{def}}{=} (\underline{\mathrm{h}}'''\circ_H \underline{\mathrm{h}}'')\circ_V (\underline{\mathrm{h}}'\circ_H \underline{\mathrm{h}}) &= (a'a, b''b', f'f, d'd; h^*),\end{aligned} \tag{3.6}$$

where

$$h^* = \{\alpha(a^{-1})h'\}h\{\alpha(d^{-1}c^{-1})h'''\}\{\alpha(d^{-1})h''\} \tag{3.7}$$

Applying τ gives

$$\begin{aligned}\tau(h^*) &= a^{-1}\tau(h')z(\underline{\mathrm{h}}')a\cdot\tau(h)z(\underline{\mathrm{h}})d^{-1}c^{-1}\tau(h''')cd\cdot \\ &\qquad z(\underline{\mathrm{h}}''')\cdot d^{-1}\tau(h'')dz(\underline{\mathrm{h}}'') \\ &= (a'a)^{-1}(b''b')^{-1}(f'f)(d'd)z(\underline{\mathrm{h}}^*),\end{aligned} \tag{3.8}$$

where we have used the fact, from (2.1), that α is converted to a conjugation on applying τ, and the last line follows after algebraic simplification. Thus,

$$\tau(h^*) = \tau_\partial(\underline{\mathrm{h}}^*)z(\underline{\mathrm{h}}^*) \tag{3.9}$$

On the other hand, by an entirely similar computation, we obtain

$$\underline{\mathrm{h}}_* \stackrel{\mathrm{def}}{=} (\underline{\mathrm{h}}''' \circ_V \underline{\mathrm{h}}') \circ_H (\underline{\mathrm{h}}'' \circ_V \underline{\mathrm{h}}) = (a'a, b''b', f'f, d'd; h_*), \tag{3.10}$$

where

$$h_* = \{\alpha(a^{-1})h'\}\{\alpha(a^{-1}b^{-1})h'''\}h\{\alpha(d^{-1})h''\} \tag{3.11}$$

Applying τ to this yields, after using (2.1) and computation,

$$\tau(h_*) = \tau_\partial(\underline{\mathrm{h}}_*)z(\underline{\mathrm{h}}_*)$$

Since $\tau(h_*)$ is equal to $\tau(h^*)$, the result (3.5) follows. □

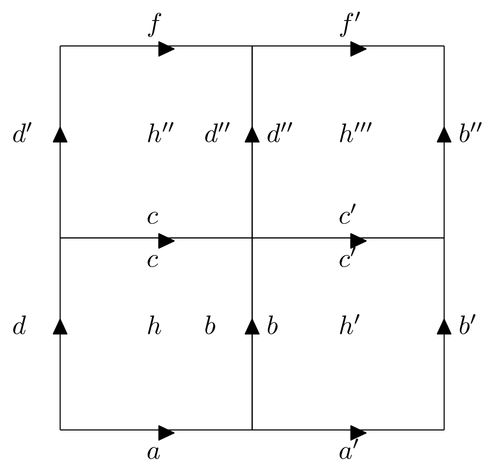

Fig. 8. Consistency of Horizontal and Vertical Compositions.

Ideally, a discrete model would be the exact 'integrated' version of the differential geometric connection $\omega_{(A,B)}$. However, it is not clear if such an ideal transcription is feasible for any such connection $\omega_{(A,B)}$ on the path-space bundle. To make contact with the differential picture we have developed in earlier sections, we should compare quasi-flat morphisms with parallel translation by $\omega_{(A,B)}$ in the case where B is such that $\omega_{(A,B)}$ reduces to ev_0^*A (for instance, if $A = \overline{A}$ and the fake curvature $F^{\overline{A}} + \tau(B)$ vanishes); more precisely, the h for quasi-flat morphisms (taking all $z(\mathrm{h})$ to be the identity) corresponds to the quantity $h_0(1)$ specified through the differential equation (2.47). It would be desirable to have a more thorough relationship between the discrete structures and the differential geometric constructions, even in the case when $z(\cdot)$ is not the identity. We hope to address this in future work.

4. Concluding Remarks

We have constructed in (2.17) a connection $\omega_{(A,B)}$ from a connection A on a principal G-bundle P over M, and a 2-form B taking values in the Lie algebra of a second structure group H. The connection $\omega_{(A,B)}$ lives on a bundle of $\overline{A}$-horizontal paths, where $\overline{A}$ is another connection on P which may be viewed as governing the gauge theoretic interaction along each curve. Associated to each path $s \mapsto \Gamma_s$ of paths, beginning with an initial path Γ_0 and ending in a final path Γ_1 in M, is a parallel transport process by the connection $\omega_{(A,B)}$. We have studied conditions (in Theorem 2.1) under which this transport is 'surface-determined', that is, depends more on the surface Γ swept out by the path of paths than on the specific parametrization, given by Γ, of this surface. We also described connections over the path space of M with values in the Lie algebra LH obtained from the $\overline{A}$ and B. We developed an 'integrated' version, or a discrete version, of this theory, which is most conveniently formulated in terms of categories of quadrilateral diagrams. These diagrams, or morphisms, arise from parallel transport by $\omega_{(A,B)}$ when B has a special form which makes the parallel transports surface-determined.

Our results and constructions extend a body of literature ranging from differential geometric investigations to category theoretic ones. We have developed both aspects, clarifying their relationship.

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