PRICING FUNCTIONALS AND PRICING MEASURES

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ABSTRACT. We demonstrate how pricing functionals give rise to pricing measures, using a time-independent framework. For infinite market state spaces, the Gel'fand spectral theory is used to obtain the pricing measure. Pricing functionals with additional market information are shown, within this model, to be given by conditional expectations.

1. Introduction

The purpose of this paper is to present a method of constructing pricing measures that uses the Gel'fand theory of commutative C^*-algebras. This machinery has been used in quantum and statistical physics, but appears to be novel in the present economic context. Mathematically, the main results, especially Theorems 5.1 and 6.2, are in sections 5 and 6.

In addition to the specific mathematical results we also wish to emphasize that the more geometric approach of economic equilibrium theory as in Debreu [4] provides a conceptually clearer and more general framework for mathematical modeling of financial markets than the approach that has now become standard (and is summarized, for instance, in Karatzas and Shreve [12, 13]), which is heavily influenced by stock and bond market instruments. To view the fundamental pricing measure as a 'martingale measure', though certainly correct, places the time coordinate in a special role in the basic framework that is not necessary. In our presentation, the martingale feature is a special case of a more general property of prices: *the equilibrium market price of an asset under partial information about the market is the corresponding conditional expectation of the price.* A special case of this is obtained by taking 'partial information' to mean all information available till a given time t, and this yields the usual martingale property of prices. In brief, our framework is time-independent. (This point of view is developed more fully in [15].) While this point of view is not new (indeed it is more classical, based on economic equilibrium theory), it departs from the current orthodoxy. To make an analogy with quantum physics, if the standard stochastic framework for financial markets is compared with the Schrödinger picture wave function in coordinate space for a system of particles in space, then the point of view we are stressing corresponds to the general Hilbert space foundations of quantum theory. Example

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2.1 in section 2, and comments relating to it at several places later, explain how the ‘standard’ framework is a special case of ours.

Thus, this paper is concerned with finding a more general approach to the so-called *first fundamental theorem of asset pricing*, which essentially says that absence of arbitrage opportunities is equivalent to existence of equivalent martingale measures. The martingale ansatz was developed in Harrison and Kreps [10] and Harrison and Pliska [11], spawned an extensive literature, and has found a very general formulation in Delbaen and Schachermayer [5, 6].

A consequence of the focus on the first fundamental theorem is that we are mostly concerned with the absence of arbitrage and not so much with completeness of the market. Broadly speaking, completeness of a market is connected with uniqueness of the pricing measure. Completeness can also be understood in very simple terms as the possibility of replication of any payoff function by a suitable combination of a set of “basic” financial instruments. Market completeness is treated in what is often called the *second fundamental theorem of asset pricing*, the statement that a market is complete if and only if the equivalent martingale measure is unique [10, 11]. Battig and Jarrow [3] propose a decoupling of the issue of completeness from the martingale approach. A non-technical account of the issues addressed in the second fundamental theorem is provided in Flood [8].

The fundamental relationship between price and probability has been known since the earliest formulations of probability theory in terms of gambling returns. Briefly put, if I_B is an instrument or asset which yields one unit of time-t cash (or any chosen numeraire) in case event B occurs and yields nothing if B doesn't occur then

$$\text{price of asset } I_B = \text{ probability } Q(B) \text{ of event } B \tag{1.1}$$

where probability is assessed by the trader who is pricing the asset. For a general asset whose worth is $f(\omega)$ in market scenario ω, the corresponding equation is

$$\text{price of asset described by } f = \text{ the expected value } \int f\,dQ \text{ of } f \tag{1.2}$$

In ideal market equilibrium all traders agree on a common price for each traded instrument and this gives rise to a common assessment of probability, which is the *market equilibrium measure*. This measure describes the *market's view* of probabilities of events. Needless to say, no individual trader may truly agree with this ideal measure which is the result of consensus emerging out of trading rather than “real-life” probabilities (whatever that may be). Being based on the price at which a risky asset would be exchanged for another, this measure is the “risk-neutral measure” for the market.

There is one problem with the above analysis relating prices to probabilities: instruments like I_B will certainly not actually exist in the real market (though of course such “digital option” instruments would exist for certain types of events B). Thus the pricing measure Q will have to be *imputed* from prices of the instruments that are actually being traded in a market. This raises the theoretical question as to whether such a measure exists at all:

given prices of traded instruments, is there a probability measure Q on the market scenarios such that the prices result from Q as expected values as in (1.2) ? (1.3)

Our work is devoted to this question, and here is a summary of what we do:

- in section 2 we state the formal model we are considering, essentially that of a market with a set of traded instruments with a *consistent* set of prices (consistency includes a no-arbitrage condition);
- in section 3 we show that the answer to question (1.3) is *yes* if the market state-space is finite; this is of course a standard fact (see for instance Duffie [7, Chapter 1]), but there may be some value in the way we formulate the proof;
- in section 4 we give an example where the market state-space is infinite where the answer to (1.3) is *no*;
- in section 5, which is our main focus, we show how the market state-space, even if infinite, may be extended mathematically so that the answer to (1.3) is *yes*.;
- in section 6, we show how price functionals in the presence of market information are described by conditional expectations and how, as a very special case, this leads to the martingale property of market equilibrium prices. The results here are well-known but we take an approach that conforms to our model.

Expanding a state-space so that a probability measure can live on it is a common procedure in stochastic analysis (for example in the construction of Wiener measure) and in areas of physics. The method we have used in this paper is based on the Gel'fand theory of commutative C^*–algebras, a technique which has also been used in constructing measures in quantum field theories and statistical mechanics/thermodynamics. Sometimes projective systems of measure spaces are used to obtain limiting measures as an alternative to the Gel'fand transform method.

In [2], Balbás et al. have used a projective system of measure spaces to obtain a measure corresponding to a no-arbitrage system of prices. As in our method, they also need to expand the market state space in order to obtain the measure. The specific details of the framework are different, and we work in a more abstract setting, but the underlying issues are closely related.

2. The Model

The concepts we are formalizing are:

(i) a *market* which can exist in a certain set of *states*,
(ii) *assets* or *instruments* which have prices, in units of any chosen numeraire, in each market state,
(iii) a trader who, without exact information about the market state, associates *prices* to assets.

In general different traders would associate different prices to the same asset. Trading takes place when the bid and ask prices agree and so in ideal market equilibrium all traders would be using the same price mechanism.

We denote by

$$\Omega$$

the set of all market states. It is important to understand that an element of Ω may be thought of as describing one entire time evolution path for the market. However, our framework is flexible enough to also permit the more restrictive interpretation of an element of Ω being a market state at one particular time. (Such a dichotomy occurs also in quantum theory, where one has the Heisenberg picture versus the Schrödinger picture for states of a system.)

Again, let us stress that we do not need to single out the time coordinate in setting up the framework.

Example 2.1. For applications to stock and bond market instruments, one could take Ω to be the continuous path space $C_0([0,T],\mathbb{R}^N)$ (paths beginning at $0 \in \mathbb{R}^N$) to model a market with N underlying facors, all evolving over a time period $[0,T]$. An element $\omega \in \Omega$ is then a path $t \mapsto \omega(t) \in \mathbb{R}^N$. The state-dependent discount factor for time t is $e^{-\int_0^t r(\omega;u)\,du}$, where $r(\omega;u)$ is the risk-free interest rate at time u for market state ω. We will view assets as having a time-stamp; thus, a particular stock at a specific time $t \in [0,1]$ has worth $X(\omega;t)$ at time t in state ω in time-t money; converted to time-0 money, this asset's worth is described by the function $\omega \mapsto e^{-\int_0^t r(\omega;u)\,du} X(\omega;t)$.

Each traded asset has a specific worth (in units of a chosen numeraire) in each market state ω. Two assets which always have exactly the same worth in every scenario will be viewed as being effectively the same asset. Thus an asset can be modeled as a mapping

$$f : \Omega \to \mathbb{R} : \omega \mapsto f(\omega),$$

with $f(\omega)$ denoting the value of the asset f in market state ω.

As with the state space, there are two possible formulations here. If an element of Ω represents a market state through all time (as in Example 1.1 above), then the same physical asset/instrument will be described by different functions at different times. On the other hand, if an element of Ω represents a possible market state at any particular time, then it is the state which evolves in time through Ω while the asset functions remain the same.

We will work only with a class of assets for which the prices are given by *bounded* functions f. This is a technical assumption with no larger significance, as the price functional extends uniquely to unbounded functions, given some simple continuity assumption.

Different assets may be combined to form portfolios and we also assume that each asset can be scaled in any way. Thus the set of assets forms a vector space V. The sum of $f, g \in V$ is simply the pointwise sum $f+g$, the function $\Omega \to \mathbb{R}$ whose value at any state ω is $f(\omega)+g(\omega)$. If $f \in V$ and k is any real number then kf is the function on Ω whose value at any ω is $kf(\omega)$.

Lastly, we have the trader. The trader is not assumed to know which market state ω actually prevails but, based perhaps on his/her understanding or estimate

Table 1. Pricing Framework.

Mathematical object	Interpretation
A set Ω	elements of Ω correspond to market states
A vector space V of bounded functions $\Omega \to \mathbb{R}$	a function $f : \Omega \to \mathbb{R}$ corresponds to an instrument, with $f(\omega)$ being the price of f in market state ω
A linear functional $L : V \to \mathbb{R}$	$L(f)$ is the trader's *a priori* price for the instrument f
$\min f \leq Lf \leq \max f$ for all $f \in V$	this is the *no-arbitrage condition*

of which market state ω will be realized, associates to each asset f a price Lf the trader is willing to pay or receive in exchange for the asset. Clearly, Lf should be linear in f, i.e.

$$L(f + g) = L(f) + L(g) \text{ and } L(kf) = kL(f) \text{ for any constant scaling } k.$$

Thus the trader's role is summarized by his/her pricing procedure which is given by a linear functional

$$L : V \to \mathbb{R}$$

which we shall call the *pricing functional.*

Linearity of the pricing functional can be viewed as a form of no-arbitrage (or as lack of friction [11]).

Now we formalize the notion of no-arbitrage. An arbitrage opportunity would allow a market participant to purchase/short some asset h which would yield a sure profit in some scenario, with no risk of loss in any scenario, based on the available price Lh. Thus an asset h provides an arbitrage for the trader if the price Lh is strictly less than all possible values $h(\omega)$ or is strictly greater than all possible values $h(\omega)$. Thus non-existence of arbitrage means that L has the property that

$$\min f \leq Lf \leq \max f \text{ for all } f \in V \tag{2.1}$$

A stricter form would require that $\min f < Lf < \max f$ unless f is constant. There are, of course, other ways and contexts to formalize no-arbitrage. In some sense, even the linearity condition on L is a no-arbitrage condition.

Our model is summarized in Table 1. Note that the condition $\min f \leq Lf \leq \max f$ for all $f \in V$ is equivalent to $Lf \leq \max f$ for all $f \in V$ (just switch f to $-f$ in the latter condition to obtain $Lf \geq \min f$).

3. From Pricing Functional to Pricing Measure for Finite Ω

The results in this section are standard and well-known, but we include them for ease of readability and reference in the following sections. Proofs may be obtained by consulting, for example, Rudin's standard text [14]; complete proofs of the specific results are in Sengupta [15].

The set of all functions $\Omega \to \mathbb{R}$ is denoted $\mathbb{R}^\Omega$. This is a real vector space by pointwise operations. Indeed if Ω is finite with N elements then $\mathbb{R}^\Omega$ is essentially $\mathbb{R}^N$. This a linear space V of real–valued functions on Ω is a subspace of $\Omega \to \mathbb{R}$.

Our goal is here is to show that if there are finitely many market states then the pricing functional L arises from a measure Q on the subsets of Ω. The argument is in two steps: (i) first we show how to extend L from V to a linear functional on all of $\mathbb{R}^\Omega$ and (ii) note that every linear functional on $\mathbb{R}^\Omega$ arises by integration with respect to a measure.

The first step is contained in:

Theorem 3.1. *Let V be a subspace of $\Omega \to \mathbb{R}$ consisting of bounded functions, where Ω is a non-empty set, and $L : V \to \mathbb{R}$ a linear functional satisfying the condition $Lf \leq \max f$ for all $f \in V$. Then there is a linear functional $L' : \mathbb{R}^\Omega \to \mathbb{R}$ which coincides with L on the subspace V and satisfies* $\min f \leq L'f \leq \max f$ *for all $f \in \mathbb{R}^\Omega$.*

Compare to Theorem 1 of Harrison and Kreps [10]. This is a consequence of the following slightly more general result (We omit the proof of this standard result for brevity.):

Theorem 3.2. *Let Z be a real vector space on which there is a function*

$$p : Z \to \mathbb{R}$$

such that

$$p(a + b) \leq p(a) + p(b)$$

holds for every $a, b \in Z$, and $p(ta) = tp(a)$ for every $a \in V$ and $t \geq 0$. Let V be a subspace of Z, and $L : V \to \mathbb{R}$ a linear functional satisfying the condition $Lf \leq p(f)$ for all $f \in V$. Then there is a linear functional $L' : Z \to \mathbb{R}$ which coincides with L on the subspace V and, moreover, satisfies $L'f \leq p(f)$ for all $f \in Z$.

To obtain Theorem 3.1 simply take $Z = \mathbb{R}^\Omega$ and $p(f) = \sup_{\omega\in\Omega} f(\omega)$.

The result above is essentially the Hahn-Banach theorem. The conditions imposed on the function $p : V \to \mathbb{R}$ imply that p is convex, and the result above produces from the given functional L a closed half-space $\{f \in V : L'(f) \leq 1\}$ containing the convex set $\{f \in V : p(f) \leq 1\}$.

In the application to the case where V is a space of functions and $p(h) = \max h$, the proof of Theorem 3.2 shows that L satisfies the min-max constraints:

$$\max_{h\in V} \min_{\omega\in\Omega} [Lh - \{h(\omega) - g(\omega)\}] \leq \min_{f\in V} \max_{\omega\in\Omega} [\{f(\omega) + g(\omega)\} - Lf] \tag{3.1}$$

In a 2-period model, this result follows from the Separating Hyperplane Theorem (e.g., Duffie [7], p. 4).

Next, we have the result which provides the pricing measure (we quote the result from Sengupta [15, Theorem 6.3.1] for ease of reference):

Theorem 3.3. *Let V be a subspace of $\Omega \to \mathbb{R}$, where Ω is a non-empty finite set, and $L : V \to \mathbb{R}$ a linear functional satisfying the condition $Lf \leq \max f$ for all $f \in V$. Then there is a probability measure Q on the set of all subsets of Ω such that*

$$Lf = \int f\,dQ \tag{3.2}$$

holds for all $f \in V$.

Proof. By Theorem 3.2, L extends to a linear functional $L' : \mathbb{R}^\Omega \to \mathbb{R}$ satisfying $L'f \leq \max f$ for all $f \in \mathbb{R}^\Omega$. From this, applied to f and then to $-f$, we have

$$\min f \leq L'f \leq \max f \text{ for all } f \in \mathbb{R}^\Omega. \tag{3.3}$$

In particular,

$$L'f \geq 0 \text{ whenever } f \geq 0.$$

Taking f to be the constant function 1, we have

$$L'1 = 1$$

The vector space $\mathbb{R}^\Omega$ has a finite basis consisting of the vectors δ_ω, with ω running over Ω, with δ_ω being the element of $\mathbb{R}^\Omega$ which has value 1 on ω and value 0 at all other points of Ω. Any linear functional on $\mathbb{R}^\Omega$ is uniquely specified by its values on the basis elements δ_ω. Define

$$q_\omega = L'(\delta_\omega)$$

The properties of L' noted above imply that $q_\omega \geq 0$ and

$$\sum_{\omega\in\Omega} q_\omega = \sum_{\omega\in\Omega} L'(\delta_\omega) = L'\left(\sum_{\omega\in\Omega} \delta_\omega\right). = L'1 = 1$$

So the numbers q_ω specify a probability measure Q on the subsets of Ω:

$$Q(E) = \textstyle\sum_{\omega\in E} q_\omega \text{ for all } E \subset \Omega \tag{3.4}$$

Then it is clear that

$$\int f\,dQ = L'f$$

for all $f \in \mathbb{R}^\Omega$, both sides being linear functionals of f which agree on the basis elements δ_ω. In particular, specializing to elements $f \in V$ we have the representation (3.2). □

Compare Theorem 3.3 to Theorem 2 in Harrison and Kreps [10], where the no-arbitrage condition enters through a martingale assumption on the price process, which introduces the emphasis on the time dimension. A generalization to a very general class of martingales is achieved in Delbaen and Schachermayer [5, 6]. Condition (3.3) generalizes in the sense that it is not necessary to specify whether a typical element $\omega \in \Omega$ is a trajectory or a state at one particular time.

4. A Counterexample for Infinite Ω

Our purpose in this section is to demonstrate that the hypothesis of finiteness of Ω cannot be dropped from Theorem 3.3. (See also Back and Pliska [1] for a similar phenomenon.)

Let Ω be any countably infinite set and let V be the set of all real-valued functions on Ω which are constant outside finite sets, i.e. V consists of all functions $f : \Omega \to \mathbb{R}$ such that there is a finite set S_f outside which f is constant. Clearly V is a linear space under the pointwise operations. Let $L : V \to \mathbb{R}$ be the functional which associates to each $f \in V$ the constant value which f takes outside some finite set. Again it is readily seen that L is a linear functional. Our point now is:

Proposition 4.1. *There is no measure Q on any σ-algebra of subsets relative to which each $f \in V$ is measurable and for which $Lf = \int f\, dQ$ holds for every $f \in V$.*

Proof. Lets first check that the σ–algebra $\sigma(V)$ with respect to which all functions in V are measurable is in fact the set of all subsets of Ω. For any point $p \in \Omega$ the indicator function $1_{\{p\}}$ is in V. So the one-point set $\{p\}$, being the set on which $1_{\{p\}}$ takes the value 1, must be in the $\sigma(V)$. Since Ω is countable it follows that every subset of Ω is in $\sigma(V)$.

Next suppose there is a measure Q on $\sigma(V)$ such that $Lf = \int f\, dQ$ holds for every $f \in V$. Then, since Ω is the countable union of all the one-point sets $\{p\}$, we have

$$Q(\Omega) = \sum_{p \in \Omega} Q(\{p\}) \tag{4.1}$$

But

$$Q(\{p\}) = \int 1_{\{p\}}\, dQ = L(1_{\{p\}}) = 0$$

from the definition of L, since the function $1_{\{p\}}$ is, by definition, 0 at all points outside $\{p\}$. On the other hand,

$$Q(\Omega) = \int 1\, dQ = L(1) = 1,$$

the last equality again being a direct consequence of the definition of L. The two preceding equations contradict the relation (4.1). The contradiction shows that no Q having the desired properties can exist. $\square$

5. Pricing Measure on the Gel'fand Spectrum

In this section we shall show that in case Ω is infinite, it is possible to extend Ω in such a way that a probability measure Q with the desired properties exists for the extended space.

Standard notions and theorems about Gel'fand theory which we shall use below are available in Rudin [14, Chapters 10, 11].

As usual, we start with a non-empty set Ω, a linear space V of real-valued *bounded* functions on Ω, and a linear functional $L : V \to \mathbb{R}$ which satisfies the no-arbitrage condition $Lf \le \max f$ for all $f \in V$.

Our objective in this section is to prove the following result:

Theorem 5.1. *Let Ω be a non-empty set, V a linear space of bounded real-valued functions on Ω, and L a linear functional on V such that $Lf \leq \max f$ for all $f \in V$. Then there is*

(a) *a set $\hat{\Omega}$,*
(b) *a linear space $\hat{V}$ of functions on $\hat{\Omega}$,*
(c) *a probability measure Q on a σ–algebra of subsets of $\hat{\Omega}$ with respect to which all the functions in $\hat{V}$ are measurable,*
(d) *a linear isomorphism $V \to \hat{V} : f \mapsto \hat{f}$*
(e) *and a map $i : \Omega \to \hat{\Omega}$,*

such that:

(i) *for every $f \in V$, the value Lf is the average value of $\hat{f}$ with respect to the probability measure Q:*
$$L(f) = \int_{\hat{\Omega}} \hat{f}\, dQ \text{ for every } f \in V, \tag{5.1}$$
(ii) *the range of the function $\hat{f}$ is the closure of the range of f, for each function $f \in V$,*
(iii) *the measure Q is defined on the completed σ–algebra generated by the functions $\hat{f}$ with f running over V*
(iv) *the map $i : \Omega \to \hat{\Omega}$ identifies any two states in Ω which have the property that all elements $f \in V$ take the same value in the two states, i.e. $i(\omega) = i(\omega')$ if $f(\omega) = f(\omega')$ for all $f \in V$.*

Moreover, there is a topology on $\hat{\Omega}$ which makes it a compact Hausdorff space, the function $\hat{f}$ is continuous on $\hat{\Omega}$ for each $f \in V$, and $i(\Omega)$ is a dense subset of $\hat{\Omega}$.

Suppose, in the preceding framework, that there is a non-negative function $h \in V$ such that $Lh = 0$ but $h \neq 0$. Then we have $\hat{h} \geq 0$ on $\hat{\Omega}$ and $\int \hat{h}\, dQ = 0$. This implies that
$$Q[\{h > 0\}] = 0$$
Thus the measure Q assigns zero probability to such events, i.e. to events allowing a profit from instruments which have zero cost. Suppose we start with an initially given measure P (for example, a model probability measure for real-life uncertainties) such that assets such as I_B for events with $P(B) = 0$ have zero price. Then events which have zero P–probability continue to have zero Q–probability. With some more structure, one would have Q *equivalent* as a measure to the given measure P.

Before proceeding to the proof of Theorem 5.1 we shall discuss its meaning.

5.1. Interpretation of Theorem 5.1. As before, Ω consists of the set of all possible states of a market.

The elements of V are functions on Ω, and for such a function f the value $f(\omega)$ is the worth of the asset when market state ω is realized. The linear functional $L : V \to \mathbb{R}$ associates to each $f \in V$ the value Lf, which is what a particular trader would pay for the instrument f without knowledge of which scenario/state will actually be realized. In market equilibrium all traders use the same functional L. As we have seen before, when there are infinitely many market scenarios the price Lf may not arise as the expectation value of f (for all $f \in V$ simultaneously) relative to some probability measure Q on the set of market scenarios. Theorem 5.1 solves this problem by providing a probability measure, but at a cost: this measure lives on a possibly *larger* space $\hat{\Omega}$. (The space $\hat{\Omega}$ is not *necessarily* larger than Ω; indeed, for the setting of Example 2.1, the market equilibrium pricing measure is often taken to be Wiener measure on $\Omega = C_0([0,T];\mathbb{R}^N)$, and asset worths specified through martingale processes, as explained in section 6 below.)

Part (d) of the theorem assures us that the instruments/assets f continue to exist on the new market-scenario space $\hat{\Omega}$, the new versions being the functions $\hat{f}$.

Part (i) tells us that the desired fundamental relationship:

price of an instrument=average of possible potential values in all scenarios

now holds, with the averaging done with respect to the measure Q.

Part (iii) assures us that the original collection of market events are expanded only minimially in the new setting, at most by including events which are viewed as having probability zero.

Part (iv) actually fixes a problem that may have been present in the original formulation. Surely two market states in which every possible asset has the same worth should be considered as essentially the same market state, and this is precisely the case in $\hat{\Omega}$.

Lastly, there is a topology on the new market state space $\hat{\Omega}$ and relative to this the original set of states is a dense subset, thereby showing again that the new state space is only a minimal expansion of the original one. However, it must be said that the topology on $\hat{\Omega}$ can be quite strange and need not be viewed as having any practical significance.

The measure Q need not be uniquely defined, an issue that connects to the problem of market completeness and that we take up again in the Conclusion. Compare Theorem 5.1 to Theorem 3 in Harrison and Kreps [10], which shows the existence of a measure for one special case of an infinite state space Ω, the case where the price process is a diffusion. The corresponding result in Delbaen and Schachermayer [5] is Theorem 1.1.

5.2. Proof of Theorem 5.1. The proof will use Gel'fand's spectral theory of commutative C^*–algebras. We have kept the material below reasonably self-contained but a detailed account, including a proof of the fundamental theorem of Gel'fand and Naimark, are available in Rudin [14].

Let B be the algebra of all complex-valued bounded functions on Ω. On B we have the sup-norm:

$$\|f\|_{\sup} = \sup_{\omega\in\Omega} |f(\omega)|$$

Then B is a complex, commutative, Banach $*$–algebra with unit 1. The involution $* : B \to B$ is simply the conjugation $f^* = \overline{f}$. Note that

$$\|ff^*\|_{\sup} = \|f\|^2_{\sup} \tag{5.2}$$

Let B_V be the closure of the complex subalgebra generated by all elements $f \in V$ along with the constant function 1. This is, of course, also a complex, commutative, Banach $*$–algebra with unit. The *Gel'fand spectrum* of B_V is the set

$\hat{\Omega}$ of all non-zero complex homomorphisms of the algebra B_V. In more detail, the elements of $\hat{\Omega}$ are all non-zero linear maps $\phi : B_V \to \mathbb{C}$ for which $\phi(fg) = \phi(f)\phi(g)$ holds for all $f, g \in B_V$.

Thus for each $f \in B_V$ there is a function $\hat{f}$ on $\hat{\Omega}$ given by

$$\hat{f}(\phi) = \phi(f) \tag{5.3}$$

This is called the *Gel'fand transform* of f.

The set $\hat{\Omega}$ is equipped with the smallest topology with respect to which all the functions $\hat{f}$, as f runs over B_V, are continuous. Let $C(\hat{\Omega})$ be the set of all complex-valued continuous functions on $\hat{\Omega}$. This is a complex commutative Banach $*$–algebra under the sup-norm, and equation (5.2) holds.

The fundamental theorem of Gel'fand and Naimark [14, Theorem 11.18] applied to this situation says that the Gel'fand transform

$$B_V \to C(\hat{\Omega}) : f \mapsto \hat{f}$$

is an isometric algebra isomorphism which preserves the conjugation operation.

Lemma 5.2. *The Gel'fand transform carries the set of real-valued functions in B_V onto the set real-valued functions in $C(\hat{\Omega})$. Moreover, it maps the set of non-negative elements in B_V onto the set of non-negative elements in $C(\hat{\Omega})$.*

Proof. An element $f \in B_V$ is real-valued if and only if $f = f^*$, and this translates by the Gel'fand transform to $\hat{f} = (\hat{f})^*$ which is the condition for $\hat{f}$ being real-valued.

Next suppose $f \in B_V$ is a real-valued non-negative function on Ω. By the Weierstrass theorem there is a sequence of polynomials $p_n(x)$ such that $p_n(x) \to \sqrt{x}$ uniformly on the compact interval $[0, \|f\|_{\text{sup}}]$. So the sequence of elements $p_n \circ f$, which belong to B_V, converge uniformly to $\sqrt{f}$. So $\sqrt{f}$ is an element of B_V. Thus, writing $h = \sqrt{f}$ we have $h \in B_V$, $h = h^*$ and $f = hh^*$. Applying the Gel'fand transform, which preserves the conjugation operation $*$, we see that $\hat{f} = gg^* = |g|^2$, where $g = \hat{h} \in C(\Delta)$. So $\hat{f} \geq 0$. The same argument can be applied to the inverse Gel'fand transform to obtain the converse result. □

The order preserving nature of the Gel'fand transform noted above implies that

$$\max \hat{f} = \max f \text{ for all real-valued } f \in B_V$$

because for any real number c, we have $f \leq c$ if and only if $\hat{f} \leq \hat{c}$ and, since the Gelfand transform preserves 1 we have $\hat{c} = c$.

Recall that B_V is the sup-norm closed algebra of functions on Ω generated by all the functions in V and the constant function 1. Now the algebra B_V, viewed as a real vector space, contains V as a subspace. Let $\hat{V}$ be the image of V in $C(\hat{\Omega})$ under the Gel'fand transform; it is a real subspace of $C(\hat{\Omega})_{\text{real}}$, this being the algebra of real-valued continuous functions on $C(\hat{\Omega})$. The real-linear functional $L : V \to \mathbb{R}$ goes over to a real-linear functional:

$$\hat{L} : \hat{V} \to \mathbb{R} : f \mapsto Lf$$

The no-arbitrage condition $Lf \leq \max f$ valid for $f \in V$, implies that $\hat{L}h \leq \max h$ is valid for all $h \in \hat{V}$. The extension result given by Theorem 3.2 then provides a linear functional $\hat{L}'$ on $C(\hat{\Omega})_{\mathrm{real}}$ such that

$$\hat{L}'f \leq \max f \text{ for all } f \in C(\hat{\Omega})_{\mathrm{real}}$$

Switching f to $-f$ gives

$$\hat{L}'f \geq \min f$$

In particular, we have

$$\hat{L}'f \geq 0 \text{ for all non-negative } f \in C(\hat{\Omega}) \tag{5.4}$$

as well as

$$\hat{L}'1 = 1$$

The Riesz-Markov theorem then implies that there is a measure Q on the Borel σ–algebra of $\hat{\Omega}$ such that

$$\hat{L}'f = \int f\,dQ \text{ for every } f \in C(\hat{\Omega})_{\mathrm{real}}. \tag{5.5}$$

The definition of Q starts with setting

$$Q(K) = \inf_{\{f \in C(\hat{\Omega})_{\mathrm{real}}, f \geq 1_K\}} \hat{L}f \tag{5.6}$$

for every *compact* $K \subset \hat{\Omega}$ and then showing that this extends to a measure on the entire Borel σ–algebra.

Setting $f = 1$ in (5.5) shows that $Q(\hat{\Omega}) = 1$, i.e. that Q is a probability measure.

Recall that if a measure μ is given on a σ–algebra $\mathcal{F}$ of subsets of some set X then it is often useful to work with the larger σ–algebra $\mathcal{F}_\mu$ generated by the sets of $\mathcal{F}$ and all subsets of sets of μ–measure 0. The σ–algebra $\mathcal{F}_\mu$ is the *completion* of $\mathcal{F}$ by μ, and below we shall use the notation of subscripting by a measure to denote completion.

Proposition 5.3. *The Q–completed Borel σ–algebra of $\hat{\Omega}$ is generated by the functions $\hat{f}$ for f running over V along with the sets of Q–measure* 0.

Proof. Let $\mathcal{B}$ be the Borel σ–algebra of $\hat{\Omega}$, and $\mathcal{B}'$ the sub-algebra generated by the functions $\hat{f}$ with f running over V. Our goal is to show that

$$\mathcal{B}_Q = \mathcal{B}'_Q \tag{5.7}$$

By definition of B_V, each function in B_V is the uniform limit of a sequence of functions of the form $P(h_1, ..., h_m)$, with P being polynomial in m variables and $h_1, ..., h_m \in V$, and m varying over positive integers. Applying the Gelfand transform to this observation shows that, in particular, the σ–algebra of subsets of $\hat{\Omega}$ generated by the continuous functions *coincides* with the σ–algebra generated by just the functions $\hat{f}$ with f running over $\hat{V}$.

Thus, to prove that $\mathcal{B} \subset \mathcal{B}'_Q$, it will suffice to show that for any closed set $D \subset \hat{\Omega}$ there is a sequence of real-valued continuous functions f_n on $\hat{\Omega}$ such that $f_n(x) \to 1_D(x)$ for Q–almost every $x \in \hat{\Omega}$. [This follows from Lusin's theorem but we include a proof specialized to the present context.] Since $\hat{\Omega}$ is compact

Hausdorff, closed subsets of $\hat{\Omega}$ are compact. As noted in (5.6) the definition of Q requires that

$$Q(D) = \inf_{f \in C(\hat{\Omega})_{\text{real}}, 1_D \leq f} \hat{L} f$$

So there is a sequence of real-valued continuous functions f_n on $\hat{\Omega}$ with $f_n \geq 1_D$ such that

$$Q(D) = \lim_{n \to \infty} \hat{L}(f_n)$$

Replacing f_n by $\min\{f_1, ..., f_n\}$ we may assume that $f_1 \geq f_2 \geq ....$ So the pointwise limit $\lim_{n \to \infty} f_n(x)$ exists for every $x \in \hat{\Omega}$; denote this limit by $f(x)$. Then f is the pointwise limit of a sequence of continuous functions and $f \geq 1_D$. Moreover,

$$\int f \, dQ = \lim_{n \to \infty} \int f_n \, dQ$$

by dominated convergence, and so, since $\int f_n \, dQ = \hat{L} f_n$ it follows that

$$\int f \, dQ = Q(D)$$

Since $f \geq 1_D$ we conclude that f must actually be equal to 1_D almost everywhere with respect to Q.

The preceding arguments prove that for any closed subset D of $\hat{\Omega}$, the indicator function 1_D is Q–almost-everywhere the pointwise limit of a sequence of continuous functions and each such function is itself a uniform limit of a sequence of functions expressible as polynomials in the elements of $\hat{V}$. Consequently, 1_D is measurable with respect to the Q–completed σ–algebra $\mathcal{B}'_Q$ generated by all the functions in $\hat{V}$. So D itself is in the latter σ–algebra. Since the closed sets generate the Borel σ–algebra it follows that every Borel set is in $\mathcal{B}'_Q$, i.e. $\mathcal{B} \subset \mathcal{B}'_Q$. It follows that

$$\mathcal{B}_Q \subset \mathcal{B}'_Q$$

Conversely, since $\mathcal{B}'$ is generated by a family of continuous functions we have $\mathcal{B}' \subset \mathcal{B}$ and so also $\mathcal{B}'_Q \subset \mathcal{B}_Q$. This completes the proof of (5.7). □

Finally we include a proof for the claim that the range of $\hat{f}$ is the closure of the range of f:

Proposition 5.4. *For any $f \in V$ the range of the Gel'fand transform $\hat{f}$ is the closure of the range of f.*

Proof. The function $\hat{f}$ being continuous on the compact space $\hat{\Omega}$ has compact image.

Suppose k is a real number lying in the closure of the range of f. Then f takes values arbitrarily close to k or has the value k itself in the range; so $1/(f-k)$ is either not defined everywhere or is unbounded. In fact, k is in the closure of the range of f if and only if $f - k$ is not invertible in B_V. Passing over isomorphically to $C(\hat{\Omega})$, this is equivalent to $\hat{f} - k$ having no inverse in $C(\hat{\Omega})$, which in turn is equivalent to k being in the closure of the range of $\hat{f}$. Since the range of $\hat{f}$ is closed, we conclude that k is in the closure of the range of f if and only if it is in the range of $\hat{f}$. □

6. Pricing with Additional Information

We shall now consider pricing in the presence of information. In the preceding sections we have analyzed prices of instruments decided on an *a priori* basis. The task now is to analyze prices of instruments based on knowledge of the values of a certain given set of instruments. For example, the given instruments may be all market instruments at a particular time, and the task is to understand prices at a later time.

In the first few paragraphs below we study the situation to isolate a mathematical model and then we prove a result within this model and briefly indicate its ramifications.

As before, we have our set up consisting of the market state space Ω, the space V of functions on Ω corresponding to traded instruments, and the functional L on V for which Lf is the *a priori* price the trader would pay for instrument f. We now consider the price the trader would be pay if the values of a certain collection A of instruments were known. For simplicity of exposition, let us think of A as being a finite set $A = \{X_1, ..., X_n\}$. The price based on knowledge of the values of the functions X_i, would be described by a functional $f \mapsto L_A f$, where now $L_A f$ is determined by the values of $X_1, ..., X_n$, i.e.

$$\textit{the value of } L_A f \textit{ is a function of the values of } X_1, ..., X_n. \tag{6.1}$$

Let us now switch over to the setting of $\hat{\Omega}$, the functional $\hat{L}$ and the corresponding probability measure Q on the Borel σ–algebra $\mathcal{B}$ of $\hat{\Omega}$. It will be convenient to define

$$\hat{L}f = \int f\, dQ$$

for all Borel functions f on Q for which the integral exists (certainly this is in agreement with the case for continuous functions). It will also be convenient to work with the *real* Hilbert space $L^2_r(\hat{\Omega}, \mathcal{B}, Q)$ of Borel-measurable functions $g : \hat{\Omega} \to \mathbb{R}$ which are square-integrable, i.e. for which $\int g^2\, dQ < \infty$. The inner-product on $L^2_r(\hat{\Omega}, \mathcal{B}, Q)$ is given, as usual, by

$$\langle g, h\rangle = \int gh\, dQ$$

There is a convenient way to capture the notion of a function being "a function of" a given collection of functions. To this end, let A be a given collection of Borel functions on $\hat{\Omega}$ and $\mathcal{A}$ the σ–algebra generated by these functions. A function f on $\hat{\Omega}$ is a "function of" the given collection A if f is measurable with respect to $\mathcal{A}$. In view of this, statement (6.1) can be rewritten as

$$\hat{L}_A f \textit{ is an } \mathcal{A}\textit{–measurable function.} \tag{6.2}$$

Price consistency requires that $\hat{L}_A$ be *linear*. Actually, now something more should be true: the equation

$$\hat{L}_A(kf) = k\hat{L}_A f$$

TABLE 2. Pricing Framework with Information.

Mathematical object	Interpretation
A set of A of Borel functions on $\hat{\Omega}$	A corresponds to a given set of assets whose values will be known to the trader
The σ–algebra algebra $\mathcal{A}$ generated by the functions in A	Events in $\mathcal{A}$ are those determined by the given set A of asset prices
A linear operator L_A on $L^2_r(\hat{\Omega}, \mathcal{B}, Q)$	$\hat{L}_A f$ is the price for asset f based on knowledge of the values of the given assets
$\hat{L}_A(kf) = k\hat{L}_A f$ for all $f \in L^2_r(\hat{\Omega}, \mathcal{B}, Q)$ and all $k \in L^2_r(\hat{\Omega}, \mathcal{A}, Q)$	k being $\mathcal{A}$–measurable means k is effectively a constant, being determined by the given information
$\hat{L}\left(\hat{L}_A f\right) = \hat{L} f$ for all $f \in L^2_r(\hat{\Omega}, \mathcal{B}, Q)$	this is a compatibility condition between *a priori* pricing by $\hat{L}$ and pricing by $\hat{L}_A$ based on given information

should hold not only for all constants k, but all k which are functions of $X_1, ..., X_n$; thus (focusing on bounded k for technical convenience) we should require that

$$\hat{L}_A(kf) = k\hat{L}_A f \text{ for all } f \in L^2_r(\hat{\Omega}, \mathcal{B}, Q) \text{ and all } k \in L^\infty_r(\mathcal{B}, Q) \tag{6.3}$$

(here $L^\infty_r(\mathcal{B}, Q)$ is the space of essentially bounded Borel measurable functions). The reason for this is that such k are effectively constants or "known" quantities for the trader pricing with knowledge of the values of $X_1, ..., X_n$.

Lastly, we need to relate $\hat{L}_A$ with the *a priori* pricing functional $\hat{L}$. The consistency condition we impose is

$$\hat{L}\left(\hat{L}_A f\right) = \hat{L} f \text{ for all } f \in L^2_r(\hat{\Omega}, \mathcal{B}, Q) \tag{6.4}$$

This may be understood conceptually, but in the end it is an additional assumption on the way pricing with information relates to pricing without information and we take (6.4) as an axiom.

We shall view $\hat{L}_A$ as a linear operator on $L^2_r(\hat{\Omega}, \mathcal{B}, Q)$ with range in the closed subspace $L^2_r(\hat{\Omega}, \mathcal{A}, Q)$.

Table 2 summarizes our model for pricing with information.

We then have the following geometrical description of the operator $\hat{L}_A$:

Proposition 6.1. *The operator $\hat{L}_A$is given by the orthogonal projection of the Hilbert space $L^2(\hat{\Omega}, \mathcal{B}, Q)$ onto the closed subspace $L^2(\hat{\Omega}, \mathcal{A}, Q)$.*

Proof. For any $f \in L^2(\hat{\Omega}, \mathcal{B}, Q)$ and any bounded g in $L^2(\hat{\Omega}, \mathcal{A}, Q)$ we have

$$\begin{aligned}
\langle f - \hat{L}_A f, g\rangle &= \int fg\,dQ - \int (\hat{L}_A f)g\,dQ \\
&= \hat{L}(fg) - \hat{L}\left(g\hat{L}_A f\right) \\
&= \hat{L}(fg) - \hat{L}\left(\hat{L}_A(gf)\right) \qquad \text{because } g \text{ is } \mathcal{A}\text{–measurable} \\
&= \hat{L}(fg) - \hat{L}(fg) \\
&= 0
\end{aligned}$$

Any $h \in L^2_r(\hat{\Omega}, \mathcal{A}, Q)$ is the L^2–limit of the bounded functions obtained by truncating h off above at N and below at $-N$, with $N \to \infty$. So it follows that

$$\langle f - \hat{L}_A f, h\rangle = 0$$

for every $h \in L^2_r(\hat{\Omega}, \mathcal{A}, Q)$. This says exactly that the element $\hat{L}_A f$ in $L^2_r(\hat{\Omega}, \mathcal{A}, Q)$ has the property that $f - \hat{L}_A f$ is orthogonal to the subspace $L^2_r(\hat{\Omega}, \mathcal{A}, Q)$. Thus, $\hat{L}_A f$ is the orthogonal projection of f onto $L^2_r(\hat{\Omega}, \mathcal{A}, Q)$. □

There is a standard interpretation of orthogonal projections in the above setting as conditional expectations. Using the relation

$$\int fg\,dQ = \int g\hat{L}_A f\,dQ$$

with $g = 1_E$ for any event $E \in \mathcal{A}$, we see that $\hat{L}_A f$ is the $\mathcal{A}$–measurable function for which

$$\int_E f\,dQ = \int_E \hat{L}_A f\,dQ$$

holds for all events E in $\mathcal{A}$. Thus we have the following probabilistic view of $\hat{L}_A$:

Theorem 6.2. *For any $f \in L^2_r(\hat{\Omega}, \mathcal{B}, Q)$, the price $\hat{L}_A f$ based on knowledge given by the σ–algebra $\mathcal{A}$ is the conditional expectation $E_Q(f|\mathcal{A})$:*

$$\hat{L}_A f = E_Q(f|\mathcal{A}) \tag{6.5}$$

The geometrical significance of the conditional pricing functional $\hat{L}_A$, or the probabilistic interpretation given above, has the following consequence:

Proposition 6.3. *If A and B are collections of instruments with $A \subset B$ then:*

$$\hat{L}_A\left(\hat{L}_B f\right) = \hat{L}_A(f) \tag{6.6}$$

Let us consider a special case. Take A to be all market instruments up till time s and B to be all market instruments up till a later time $t > s$. Then, denoting $\hat{L}_A f$ by f_s, and similarly for B, we have the well-known *martingale* condition for prices:

$$E_Q[f_t|\mathcal{F}_s] = f_s \tag{6.7}$$

where $\mathcal{F}_s$ is the collection of all market events up till time s. More specifically, in the setting of Example 2.1, this says that the discounted prices of traded instruments are martingales.

7. Conclusion and Open Questions

We show in this paper that the *first fundamental theorem of asset pricing*, namely the equivalence of the condition of no-arbitrage and existence of a pricing measure, can be achieved in the framework of the Gel'fand transform. This allows to de-emphasize the time dimension and the martingale property of the price process compared to the standard approach. We show in our framework how the martingale feature arises as a special case of the more general property that the equilibrium market price of an asset under partial information about the market is the corresponding conditional expectation of the price. Partial information can but need not be indexed by time.

The question of market completeness, sometimes referred to as the *second fundamental theorem of asset pricing* and connected to uniqueness of the pricing measure, is not addressed in this paper and left for future research. It is particularly challenging to disentangle this issue from the time dimension since in the standard framework, completeness is usually defined to mean that for every payoff function f of a "basic" instrument X at maturity T, there is a self-financing portfolio p such that $p(0)$ is the fair market price at time 0 and that $p(T) = f(X(T))$. Self-financing means that between 0 and T, p reaches the value $f(X(T))$ without injection or extraction of funds except for the price $p(0)$. Thus, the standard definition of completeness is essentially a time concept. The possibility and desirability to achieve a definition of completeness that does not rely on the martingale approach has been recognized before, however (Battig and Jarrow [3]). As noted earlier in the Introduction, completeness of a market is, broadly, connected with uniqueness of the pricing measure. However, exactly how this uniqueness is formulated may depend on the technical framework. We leave the task of exploring market completeness issues in the Gel'fand transform framework for future work.

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