

CDO TRANCHE SENSITIVITIES IN THE GAUSSIAN COPULA MODEL

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ABSTRACT. We derive explicit formulas for CDO tranche sensitivity to parameter variations, and prove results concerning the qualitative behavior of such tranche sensitivities, for a homogeneous portfolio governed by the one-factor Gaussian copula. Similar results are also derived for a Poisson-mixture model.

1. Introduction

Collateralized Debt Obligations (CDOs) are of central importance in the credit derivatives market. The size and health of the CDO market has enormous implications for the broader financial system, as is underlined by the central role of CDOs in the financial crisis of 2008. Rigorous investigation of the principles of pricing and risk management of CDOs is therefore of great importance. In this paper we present a mathematically rigorous development of some key aspects of the Gaussian copula model, a market standard model for homogeneous CDOs, with some additional results applying to a broader setting. A brief summary of the essential notions and terminology we use is presented at the end of this introduction.

The Gaussian copula model for managing CDO tranches became popular following the work of Li [18]. It is a widely-used foundational model which displays qualitative characteristics observed in practice and through simulations in other models. Our objective in this paper is to give mathematical proofs for several such features.

Copula models in general, and the Gaussian copula model in particular, have serious drawbacks from both theoretical and practical viewpoints. Just to note one basic feature, the one-factor Gaussian copula model has a single parameter, the correlation, and when matched to market data the correlation displays a clear ‘skew’ across different tranches of the same portfolio instead of being constant (see, for instance, [3, Figure 1]); this is analogous to ‘smiles’ for the Black-Scholes option pricing model. However, ease of implementation and flexibility have made these models essential in practice. Deeper stochastic models, involving large numbers of parameters, are hard to implement in practice and risk-management is extremely difficult. Thus, copula models, and especially the one-factor Gaussian model, are often chosen for practical reasons. Among copula models, the Gaussian copula model is fundamental; it is a starting point for more phenomenological

2000 *Mathematics Subject Classification.* AMS Primary 60G35.

Key words and phrases. Credit Derivatives, Gaussian Copula, CDS, CDO, Correlation.

* ANS acknowledges research supported from US NSF grant DMS-0601141.

variations on the model such as ‘factor-loading’ models (for instance [3]). It plays an important role both in practice and in obtaining a qualitative understanding of the default behavior of a portfolio of credits. It could perhaps be compared with the simple harmonic oscillator in quantum theory, which, while certainly not an accurate description of many physical phenomena, is, nonetheless crucial in obtaining a broader understanding of quantum phenomena.

There is a large and growing body of literature, both from industry and academic research, on various aspects of CDOs. The Gaussian copula model, growing out of the Credit Metrics/KMV model (see [4, page 83] for a description), became an industry standard, especially after the work of Li [18], but it is an idealized first approximation to real default correlations. Much of the literature (a very small sampling of which is included in the bibliography) focuses on developing, simulating, and evaluating alternative models which better address such issues as the correlation skew and term-structure of defaults. Our objective is more mathematical, in that we focus on rigorous proofs of some of the essential qualitative features of the single-factor Gaussian copula model for tranche pricing and risk management. The work of Cousin and Laurent [9] proves several results comparing the behavior of expectations of convex functions of portfolio losses under change of copula parameters, by using stochastic comparison methods developed in the actuarial mathematics literature. We refer to the bibliography in [9], and especially the book of Müller and Stoyan [21]. Since the writing of the first version of the present paper, literature on CDO tranche behavior has grown. We mention here only the work of Jarrow and van Deventer [15] and Ağça and Islam [1], both of which describe situations where the equity tranche is short correlation, which is contrary to the case we describe.

We now present a brief overview of the results we prove in this paper. We shall work in the context of homogeneous portfolios governed by a single-factor Gaussian copula default behavior and with zero recovery. Within this model we shall derive *closed-form expressions for tranche sensitivities*, and give mathematical proofs for the following features:

- (i) equity tranches are long correlation and senior tranches are short correlation;
- (ii) equity tranche deltas decrease (increase) when the index spread increases (decreases);
- (iii) tranche deltas, for index spread movements, form a probability measure on losses;
- (iv) the equity tranche is convex with respect to spread movements;
- (v) the normalized loss in a size N portfolio converges almost surely to a random variable, of known distribution, as $N \rightarrow \infty$.

Some of these results provide mathematical confirmation for observations which are known from simulations in industry practice (for instance, [13] and [16]). The qualitative feature (i) has also been proved by Cousin and Laurent [9] by using stochastic dominance techniques; we obtain, in addition to the qualitative feature, an exact formula for the tranche sensitivity to correlation.

We also obtain explicit formulas for tranche deltas and convexity for this model. In practice these quantities are computed numerically. However, closed-form expressions are always of some interest, even if in idealized models.

We then apply our methods to a different model, where losses are modeled with a Poisson distribution with parameter governed by a Gaussian variable. In this model, we can allow a degree of inhomogeneity in that different names can have different spreads. We prove results similar to that for the Gaussian copula model.

Our proofs rely on the specific form of the distributions (Gaussian, binomial, Poisson), and yield results, such as the behavior of tranche deltas, which are true across the full range of tranche cutoffs. This is unlikely to be true for other models. We leave to a future work the task of developing rigorous mathematical methods for examining (i)-(v), and related questions, for a broader class of models.

Before proceeding to the technical results let us briefly specify some of the terminology and notions. For more details on modeling and implementation see, for instance, Bluhm et al. [4], Lando [17] and Schönbucher [23].

We take a CDO to be a portfolio of N defaultable assets, called *names* or *credits* (for example, they may be credit default swaps); a default event for a name results in a loss of 1 unit in the portfolio. The default probability of a name, within a given time horizon, is reflected in its *spread*, which is, very roughly, the premium needed to insure unit notional of the credit against default. In our model, all names in the CDO are assumed to have the same spread. Most of our results will be proven for a *homogeneous* portfolio, with each name having the same default probability.

Losses are divided into *tranches*; for example, an *investor* in the 0 – 3% *tranche* will have to pay out losses up to 3% of the full portfolio notional in the portfolio and receives, in exchange, a *spread* payment periodically, until the 3% loss level or a specified time horizon is reached.

A tranche running from 0 loss to a particular *detachment* cut-off loss level is called an *equity tranche*, and its complement, the tranche upward of a given loss level will be called a *senior tranche*.

The difficulty in modeling tranche losses arises from *correlation* between the default behavior of different names.

A position, long or short, on a tranche is often hedged by an opposite position on the entire portfolio (which we refer to as the *index*); the notional in the index that hedges the tranche investment, against movement in the spreads, is the *delta* for the tranche (we discuss this with greater precision in section 3).

Often a tranche deal is executed simultaneously with the opposite index hedge position; the sensitivity of the combined hedged portfolio to changes in the index spread is measured by the *gamma* or convexity of the tranche.

When a credit defaults some amount of the notional is recoverable. This recovery factor is usually modeled as a constant or as a factor independent of the other random variables involved. For this reason we set the recovery factor to 0.

2. Sensitivity to Correlation in the Gaussian Model

In this section we derive, for the one-factor Gaussian copula model, exact formulas for the sensitivity of expected tranche losses to variation in correlation.

As consequence, we show that *equity tranches are long correlation and senior tranches are short correlation*. This means, for instance, that an investor in an equity tranche has lower expected loss payments when default correlation rises.

Here, by equity tranche, we mean any tranche of the form $[0, x]$, with attachment point 0; by a senior tranche we mean here any tranche $[x, 1]$ that runs from a given attachment point all the way to the maximum portfolio loss.

If correlation rises, the probability of very few defaults increases (as well as that for many defaults), and this ought to decrease the expected loss for a low-attachment equity tranche. However, it is not clear intuitively whether this ought to work for all equity tranches. Theorem 2.1 below establishes the result rigorously for the single-factor Gaussian copula model (2.1).

A CDO tranche deal involves periodic payments, of loss and spreads. Our discussion applies to the accumulated loss over *one* such period (typically 3 months). The loss payment for the full life of the CDO is a discounting-weighted sum of the single-period losses, and therefore may be deduced readily from the single-period loss analysis.

Before describing the specific model let us note that there is a large variety of models possible for studying credit defaults (see Lando [17], for instance). A more fundamental approach is to use a proxy variable $x_i(t)$ to denote the credit-state at time t of the i -th constituent of a portfolio containing N names, and then specify a stochastic differential equation for the \mathbb{R}^N -valued stochastic process

$$t \mapsto x(t) = (x_1(t), \dots, x_N(t)).$$

In this approach, name i defaults at time τ_i if this is the first time when $x_i(t)$ hits some threshold value x_i^* :

$$\tau_i = \inf\{t > 0 : x_i(t) \leq x_i^*\}$$

One would like to study the distribution of these hitting times. In the present paper we follow the more phenomenological approach of actually specifying default probabilities and their joint behavior. Specifying joint behavior of correlated variables whose marginals are known is the philosophy of copula theory. Our focus will be the one-factor Gaussian copula model popularized by Li [18]. It is best formulated directly in terms of correlated jointly-Gaussian variables than in terms of general copula theory.

Consider a homogeneous portfolio with N names, each with unit notional, with default behavior of the i -th name governed by the factor

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon_i, \tag{2.1}$$

where $Z, \epsilon_1, \dots, \epsilon_N$ are independent standard Gaussian variables. Name i defaults, within the given fixed time horizon, if X_i falls below a threshold c . Briefly,

$$X_i = \Phi(F_i^{-1}(\tau_i)),$$

where τ_i is the random default time for name i , and F_i is the distribution function of τ_i , assumed to be continuous and strictly increasing.

The model (2.1) encodes the assumption that the portfolio has a common correlation

$$\text{Corr}(X_i, X_j) = \mathbb{E}[X_i X_j] = \rho \geq 0, \text{ for all } i \neq j. \tag{2.2}$$

We use the notation

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^x \phi(s) ds, \quad (2.3)$$

for the standard Gaussian density and distribution.

The probability that exactly j names default is

$$p_j = \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} \phi(x) dx \quad (2.4)$$

where

$$p = \mathbb{P}[X_i \leq c | Z = x] = \mathbb{P}\left[\epsilon_i \leq \frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right] = \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \quad (2.5)$$

Consider now the tranche running from 0 loss to a loss of k names. Let l_k^e be the loss in this tranche (with superscript e signifying it is an equity tranche) in the given time horizon.

In our model, the recovery rate is set equal to 0. Thus, default of a name results in loss of one unit to the portfolio. So the tranche loss is

$$l_k^e = \min\{\nu, k\} = 1_{[\nu=1]} + 21_{[\nu=2]} + \cdots + (k-1)1_{[\nu=k-1]} + k1_{[\nu \geq k]} \quad (2.6)$$

where ν is the number of defaults. Here, and always, we denote by 1_S the indicator function of an event S ; thus 1_S has value 1 if S occurs, and 0 if S does not occur. The loss for the senior tranche running from loss level $k+1$ all the way to N is:

$$l_k^s = \nu - \min\{\nu, k\} = 1_{[\nu=k+1]} + 21_{[\nu=k+2]} + \cdots + (N-k)1_{[\nu=N]} \quad (2.7)$$

Our main result for the sensitivity of expected tranche losses to the correlation parameter ρ is presented in the following theorem.

Theorem 2.1. *Assume that $Z, \epsilon_1, \dots, \epsilon_N$ are independent standard Gaussian variables, with $N > 1$, and let*

$$X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i \quad \text{for } i \in \{1, \dots, N\},$$

where $\rho \in (0, 1)$. Let $c \in \mathbb{R}$. Let ν be the random variable which counts the number of X_j with value $< c$:

$$\nu = \#\{j \in \{1, \dots, N\} : X_j < c\}$$

and, for $k \in \{1, \dots, N\}$,

$$l_k^e = \min\{\nu, k\} \quad \text{and} \quad l_k^s = \nu - \min\{\nu, k\}. \quad (2.8)$$

Then,

$$\frac{d\mathbb{E}[l_k^s]}{d\rho} = -\frac{d\mathbb{E}[l_k^e]}{d\rho} = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} \int_0^\infty J_k(\sqrt{\rho}w) \frac{w}{2\sqrt{\rho(1-\rho)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \quad (2.9)$$

where

$$J_k(u) = k(N-k) \binom{N}{k} \int_{\Phi(c\sqrt{1-\rho}-u)}^{\Phi(c\sqrt{1-\rho}+u)} t^k (1-t)^{N-k-1} dt \quad (2.10)$$

for all $u \in \mathbb{R}$. In particular,

$$\frac{d\mathbb{E}[l_k^e]}{d\rho} < 0, \quad \text{and} \quad \frac{d\mathbb{E}[l_k^s]}{d\rho} > 0,$$

for $1 \leq k < N$.

The rest of this section is devoted to proving this result.

Proof. Since

$$l_k^e + l_k^s = \nu,$$

we have

$$\mathbb{E}[l_k^e] + \mathbb{E}[l_k^s] = \mathbb{E}[\nu].$$

Observe that

$$\begin{aligned} \mathbb{E}[\nu] &= \mathbb{E}\left[\sum_{j=1}^N 1_{[X_j < c]}\right] \\ &= N\mathbb{P}[X_1 < c] = N\Phi(c). \end{aligned}$$

This is independent of ρ . Hence

$$\frac{d\mathbb{E}[l_k^s]}{d\rho} = -\frac{d\mathbb{E}[l_k^e]}{d\rho}.$$

From (2.6), the expected equity tranche loss is

$$L_k^e \stackrel{\text{def}}{=} \mathbb{E}[l_k^e] = p_1 + 2p_2 + \cdots + (k-1)p_{k-1} + k[1 - p_0 - \cdots - p_{k-1}], \quad (2.11)$$

which can be rewritten as

$$L_k^e = k - \sum_{j=0}^k (k-j)p_j. \quad (2.12)$$

From this, and the expression (2.4) for p_j , we have

$$\begin{aligned} \frac{dL_k^e}{d\rho} &= -\sum_{j=0}^k (k-j) \binom{N}{j} \int_{\mathbb{R}} [jp^{j-1}(1-p)^{N-j} \\ &\quad - (N-j)p^j(1-p)^{N-j-1}] \frac{\partial p}{\partial \rho} \phi(x) dx \\ &= \int_{\mathbb{R}} I_k(p) \frac{\partial p}{\partial \rho} \phi(x) dx, \end{aligned} \quad (2.13)$$

where p , dependent on x , is as in (2.5), and I_k is the function specified by

$$I_k(s) \stackrel{\text{def}}{=} -\sum_{j=0}^k \binom{N}{j} (k-j) [js^{j-1}(1-s)^{N-j} - (N-j)s^j(1-s)^{N-j-1}] \quad (2.14)$$

for all $s \in [0, 1]$. (Note that the integrand in the expression for $dL_k^e/d\rho$ contains an exponentially decreasing term in x^2 , which ensures that $d/d\rho$ and $\int_{\mathbb{R}} \dots dx$ can be interchanged.)

We can now compute the derivative $\partial p/\partial \rho$ from (2.5):

$$\begin{aligned} \frac{\partial p}{\partial \rho} &= \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right) \frac{\sqrt{1 - \rho} \left\{-\frac{1}{2\sqrt{\rho}}x\right\} - (c - \sqrt{\rho}x) \left\{-\frac{1}{2\sqrt{1 - \rho}}\right\}}{1 - \rho} \\ &= -\frac{(1 - \rho)x - \sqrt{\rho}(c - \sqrt{\rho}x)}{2\sqrt{\rho}(1 - \rho)^{3/2}} \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right) \\ &= -\frac{x - c\sqrt{\rho}}{2\sqrt{\rho}(1 - \rho)^{3/2}} \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right). \end{aligned}$$

So

$$\begin{aligned} \frac{dL_k^e}{d\rho} &= -\int_{\mathbb{R}} I_k(p) \frac{(x - c\sqrt{\rho})}{2\sqrt{\rho}(1 - \rho)^{3/2}} \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right) \phi(x) dx \\ &\stackrel{\underbrace{\hspace{1.5cm}}_{y=x-c\sqrt{\rho}}}{=} -\int_{\mathbb{R}} I_k(p) \frac{y}{2(1 - \rho)^{3/2}\sqrt{\rho}} \phi\left(\frac{c(1 - \rho) - \sqrt{\rho}y}{\sqrt{1 - \rho}}\right) \phi(y + c\sqrt{\rho}) dy \\ &= -\int_{\mathbb{R}} I_k(p) \frac{y}{2(1 - \rho)^{3/2}\sqrt{\rho}} \frac{1}{2\pi} e^{-\frac{y^2}{2(1 - \rho)} - \frac{c^2}{2}} dy. \end{aligned}$$

Looking back at (2.5), let us write

$$p(y) = p = \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right) = \Phi\left(\frac{c(1 - \rho) - \sqrt{\rho}y}{\sqrt{1 - \rho}}\right) \quad (2.15)$$

Note that this is clearly monotonically decreasing in y .

Returning again to the derivative $dL_k^e/d\rho$, we have:

$$\frac{dL_k^e}{d\rho} = -\int_0^\infty [I_k(p(y)) - I_k(p(-y))] \frac{y}{2(1 - \rho)^{3/2}\sqrt{\rho}} \frac{1}{2\pi} e^{-\frac{y^2}{2(1 - \rho)} - \frac{c^2}{2}} dy \quad (2.16)$$

Substituting w for $y/\sqrt{1 - \rho}$ and following some algebraic manipulation, which we outsource to Lemma 2.2 below, yields the expression (2.9). Also from Lemma 2.2, the function $I_k(\cdot)$ is monotonically decreasing. Now, as noted above, for $y > 0$, we have $p(y) < p(-y)$. Hence,

$$I_k(p(y)) - I_k(p(-y)) > 0 \quad \text{for any } y > 0.$$

This implies, from (2.16), that

$$dL_k^e/d\rho < 0,$$

which is the result we had set out to prove. \square

We have used the following observation about $I_k(p)$:

Lemma 2.2. *Let*

$$I_k(p) = -\sum_{j=0}^k \binom{N}{j} (k - j) [jp^{j-1}(1 - p)^{N-j} - (N - j)p^j(1 - p)^{N-j-1}]$$

where N and k are positive integers, with $k \leq N$, and $p \in [0, 1]$. Then

$$\begin{aligned} I_k(p) &= N \sum_{j=1}^k \binom{N-1}{j-1} p^{j-1} (1-p)^{N-j} \\ &= k(N-k) \binom{N}{k} \int_p^1 t^{k-1} (1-t)^{N-k-1} dt \end{aligned} \quad (2.17)$$

In particular, $I_k(p)$ is monotonically decreasing with p , if $1 \leq k < N$.

Proof. First let us rework the expression for $I_k(p)$:

$$\begin{aligned} I_k(p) &\stackrel{\text{def}}{=} - \sum_{j=0}^k \binom{N}{j} (k-j) [j p^{j-1} (1-p)^{N-j} - (N-j) p^j (1-p)^{N-j-1}] \\ &= - \sum_{j=0}^{k-1} \left[\binom{N}{j+1} (k-j-1)(j+1) \right. \\ &\quad \left. - \binom{N}{j} (k-j)(N-j) \right] p^j (1-p)^{N-j-1} \\ &= \sum_{j=0}^{k-1} \binom{N}{j} (N-j) p^j (1-p)^{N-j-1} \\ &= N \sum_{j=1}^k \binom{N-1}{j-1} p^{j-1} (1-p)^{N-j} \\ &= N(1-p)^{N-1} + \sum_{j=1}^{k-1} \binom{N}{j} (N-j) p^j (1-p)^{N-j-1}. \end{aligned}$$

Taking the derivative, we obtain

$$\begin{aligned} I'_k(p) &= \sum_{j=1}^{k-1} \binom{N}{j} (N-j) j p^{j-1} (1-p)^{N-j-1} \\ &\quad - \sum_{j=0}^{k-1} \binom{N}{j} (N-j) p^j (N-j-1) (1-p)^{N-j-2} \\ &= \sum_{j=0}^{k-2} \underbrace{\left\{ \binom{N}{j+1} (N-j-1)(j+1) - \binom{N}{j} (N-j)(N-j-1) \right\}}_0 p^j (1-p)^{N-j-2} \\ &\quad - \binom{N}{k-1} (N-k+1)(N-k) p^{k-1} (1-p)^{N-k-1} \end{aligned}$$

Rewriting the last term, we have

$$I'_k(p) = -(N-k)k \binom{N}{k} p^{k-1} (1-p)^{N-k-1}$$

Integrating, and using the value N for $I_k(0)$, we obtain (2.17). \square

The work of Cousin and Laurent [9], specifically their Proposition 3.9, which applies to a broad class of copulas, provides an alternative proof of the monotonicity part of our Theorem 2.1, by using the concept of supermodular order (for this see, for instance, [21]). However, we have obtained an explicit expression for the derivative (2.9).

3. Tranche Deltas

We shall calculate the delta of a tranche with respect to a uniform credit spread movement across the index. Recall that in our homogeneous portfolio, this spread is a function of the default threshold c , and so we shall be concerned with sensitivity of the tranche loss and index loss to variation in c .

As before, L_k^e denotes the expected loss in the $[0, k]$ tranche, and now let L_N be the expected loss for the entire portfolio:

$$L_N = \mathbb{E}[l_N], \quad (3.1)$$

where

$$l_N = \sum_{j=1}^N 1_{[X_j \leq c]}. \quad (3.2)$$

By the *delta*, which we denote $\Delta_{k,\text{spread}}$, of the $[0, k]$ tranche we shall mean the factor such that the portfolio

long $[0, k]$ tranche + short $\Delta_{k,\text{spread}}$ times the full index

is stationary to first order, against variation of the default threshold c . Thus,

$$\Delta_{k,\text{spread}} = \frac{\frac{\partial L_k^e}{\partial c}}{\frac{\partial L_N}{\partial c}} \quad (3.3)$$

It is important for the practitioner to note that what we are calling ‘delta’ is often the displayed ‘tranche leverage’ scaled by the tranche width.

For the Gaussian copula, the expected loss in the index is

$$L_N = E(l_N) = N\Phi(c), \quad (3.4)$$

and so

$$\frac{\partial L_N}{\partial c} = N\phi(c) = \frac{N}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}. \quad (3.5)$$

Recall that the expected loss L_k^e in the equity $0 - k$ tranche is

$$L_k^e = k - \sum_{j=0}^k (k-j)p_j, \quad (3.6)$$

where

$$p_j = \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} \phi(x) dx, \quad (3.7)$$

and

$$p = \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \quad (3.8)$$

Theorem 3.1. *The delta of the $[0, k]$ -tranche is*

$$\Delta_{k,\text{spread}} = \Delta_{\text{spread}}(\{0, 1, \dots, k\})$$

where Δ_{spread} is the measure on subsets of $\{0, 1, \dots, N\}$ given by

$$\Delta_{\text{spread}}(S) = \sum_{k \in S} p_{\Delta_s}(k), \quad (3.9)$$

for any $S \subset \{0, \dots, N\}$, where

$$p_{\Delta_s}(k) = \int_{\mathbb{R}} \binom{N-1}{k-1} p(y)^{k-1} (1-p(y))^{N-1-(k-1)} \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy \quad (3.10)$$

for $k \in \{1, \dots, N\}$, and $p_{\Delta_s}(0)$ is 0 by definition.

Proof. Let us first work with an equity tranche with losses $\leq k$. From (3.6) we have

$$\begin{aligned} \frac{\partial L_k^e}{\partial c} &= - \sum_{j=0}^k (k-j) \binom{N}{j} \int_{\mathbb{R}} [jp^{j-1}(1-p)^{N-j} \\ &\quad - (N-j)p^j(1-p)^{N-j-1}] \frac{\partial p}{\partial c} \phi(x) dx \\ &= \int_{\mathbb{R}} I_k(p) \frac{\partial p}{\partial c} \phi(x) dx, \end{aligned} \quad (3.11)$$

where, from Lemma 2.2,

$$I_k(p) = N \sum_{j=1}^k \binom{N-1}{j-1} p^{j-1} (1-p)^{N-j}. \quad (3.12)$$

Now

$$\frac{\partial p}{\partial c} = \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{(c-\sqrt{\rho}x)^2}{2(1-\rho)}},$$

and, by simple algebra,

$$\frac{\partial p}{\partial c} \phi(x) = \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{(x-c\sqrt{\rho})^2}{2(1-\rho)} - \frac{c^2}{2}}.$$

Setting $y = x - c\sqrt{\rho}$, we have

$$\begin{aligned} \frac{dL_k^e}{dc} &= \int_{\mathbb{R}} I_k(p) \frac{\partial p}{\partial c} \phi(x) dx \\ &= \int_{\mathbb{R}} I_k(p) \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{(x-c\sqrt{\rho})^2}{2(1-\rho)} - \frac{c^2}{2}} dx \\ &= \int_{\mathbb{R}} I_k(p(y)) \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy. \end{aligned}$$

Therefore

$$\begin{aligned} \Delta_{k,\text{spread}} &= \frac{\partial L_k^e}{\partial c} / \frac{\partial L_N}{\partial c} \\ &= \frac{1}{N} \int_{\mathbb{R}} I_k(p(y)) \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy. \end{aligned} \quad (3.13)$$

Substituting in the integrand (3.13) the expression for $I(p)$ from (3.12) shows that

$$\Delta_{k,\text{spread}} = \sum_{j \in \{0,1,\dots,k\}} p_{\Delta_s}(j),$$

where

$$p_{\Delta_s}(j) = \int_{\mathbb{R}} \binom{N-1}{j-1} p(y)^{j-1} (1-p(y))^{N-j} \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy,$$

understood to be 0 when j is 0. This simplifies to $p_{\Delta_s}(j)$ as given by (3.10). \square

This result confirms, for the Gaussian copula model, a possibly general phenomenon that the delta with respect to index spread movements is a probability measure on the loss levels.

As an immediate consequence of the theorem, we have:

Corollary 3.2. *The delta of an equity tranche decreases monotonically with increasing spread. The delta of a senior tranche increases with increasing spread.*

Proof. As $I_k(p)$ is monotonically decreasing with p and $p(y) = \Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)$ is monotonically increasing with c , the delta (3.13) decreases with increasing c . The senior tranche (loss level $> k$) loss is the index loss minus the equity tranche (loss level $\leq k$) loss, and so the delta for the senior tranche is 1 minus the equity delta. \square

4. Convexity of Equity Tranches

Consider the portfolio which is long one equity tranche $[0, k]$ and short h units of CDO index (i.e. the notional is h times the full index notional). The expected loss of the portfolio is the negative of

$$V_k(h) = hL_N - L_k^e.$$

We define the *convexity* of the $[0, k]$ tranche to be

$$\Gamma_k = \frac{\partial^2 V_k(h)}{\partial c^2} \Big|_{h=\Delta_{k,\text{spread}}}. \tag{4.1}$$

This gives the second-order increment in the mark-to-market for the investor in the equity tranche who has hedged the tranche investment with protection on the index.

It is important to note that the term ‘gamma’ is used in related but different forms. In particular, we have taken the derivative with respect to the threshold c ; it would be more meaningful to compute derivatives with respect to the index spread. However, the index spread is related in a specific monotonic way to c , and one derivative may be computed from the other without difficulty.

The main qualitative observation about Γ_k is that it is positive:

Theorem 4.1. *With notation as above, $\Gamma_k > 0$ for all $k \in \{1, \dots, N-1\}$. Thus, for small changes in c , the quantity $V_k(h)$ increases.*

Proof. Recall that

$$\frac{\partial L_k^e}{\partial c} = \int_{\mathbb{R}} I_k(p(y)) \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy \quad (4.2)$$

and

$$\frac{\partial L_N}{\partial c} = \frac{N}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}, \quad (4.3)$$

where

$$\begin{aligned} I_k(p) &= N - (N-k)k \binom{N}{k} \int_0^p t^{k-1} (1-t)^{N-k-1} dt \\ &= (N-k)k \binom{N}{k} \int_p^1 t^{k-1} (1-t)^{N-k-1} dt \end{aligned} \quad (4.4)$$

and

$$p(y) = \Phi \left(\frac{c(1-\rho) - \sqrt{\rho}y}{\sqrt{1-\rho}} \right). \quad (4.5)$$

As we have seen, the delta of the tranche is:

$$\Delta_{k,\text{spread}} = \frac{1}{N} \int_{\mathbb{R}} I_k(p(y)) \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy. \quad (4.6)$$

Taking the derivative of (4.2), and writing p for $p(y)$, we have

$$\frac{\partial^2 L_k^e}{\partial c^2} = \frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \left[\frac{\partial I_k(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} + (-c)I_k(p) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} \right] dy,$$

and

$$\frac{\partial I_k(p)}{\partial c} = -(N-k)k \binom{N}{k} p^{k-1} (1-p)^{N-k-1} \cdot \frac{\sqrt{1-\rho}}{\sqrt{2\pi}} e^{-\frac{(c(1-\rho) - \sqrt{\rho}y)^2}{2(1-\rho)}} \quad (4.7)$$

On the other hand,

$$\frac{\partial^2 L_N}{\partial c^2} = \frac{-cN}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}.$$

Therefore

$$\begin{aligned} \Gamma_k &= \Delta_{k,\text{spread}} \frac{\partial^2 L_N}{\partial c^2} - \frac{\partial^2 L_k^e}{\partial c^2} \\ &= \left(\frac{-cN}{\sqrt{2\pi}} \right) e^{-c^2/2} \frac{1}{N\sqrt{2\pi(1-\rho)}} \int_{\mathbb{R}} I_k(p) e^{-\frac{y^2}{2(1-\rho)}} dy \\ &\quad - \frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \left[\frac{\partial I_k(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} + (-c)I_k(p) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} \right] dy \\ &= -\frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \frac{\partial I_k(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy. \end{aligned}$$

From the expression (4.7), we see that, since $k \in \{1, \dots, N-1\}$,

$$\frac{\partial I_k(p)}{\partial c} < 0 \text{ for all } y \in \mathbb{R},$$

and so we conclude that Γ_k is positive for $k \in \{1, \dots, N-1\}$. \square

5. The Large- N Limit

As before, we work with the standard Gaussian copula for a portfolio of size N . Since a typical CDO portfolio, at least initially, has over a hundred names, it is of interest to examine the behavior of the portfolio in the limit as $N \rightarrow \infty$.

The large- N behavior has been studied through simulations for various copula models, for instance by Schönbucher [22]. Other related works include Andersen and Sidenius [3, Section 3.3], Cousin and Laurent [9], and Frey [12].

As before, default of name i occurs when a random variable X_i falls below a threshold level c . We assume, as model, that there are independent random variables $Z, \epsilon_1, \epsilon_2, \dots$, such that

$$X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i$$

where $\rho > 0$ is the correlation between any pair of names in the portfolio.

The portfolio loss, truncated at size N and scaled by N , is

$$L^{(N)} = \frac{1}{N} \sum_{i=1}^N 1_{[X_i \leq c]}$$

where we have explicitly indicated N on the left. We have then

Theorem 5.1. *The sequence $L^{(N)}$ converges with probability 1 to the random variable $\Phi\left(\frac{c-\sqrt{\rho}Z}{\sqrt{1-\rho}}\right)$:*

$$L^{(N)} \rightarrow l^{(\infty)} \stackrel{\text{def}}{=} \Phi\left(\frac{c-\sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \quad \text{almost surely.}$$

Moreover,

$$L^{(N)} \rightarrow l^{(\infty)}$$

in L^2 .

Proof. The variable $L^{(N)}$ is a function of the Gaussian variable $(Z, \epsilon_1, \dots, \epsilon_N)$. For each fixed value for Z , it is the average of N independent, identically distributed (bounded) variables. So, by the law of large numbers, for each fixed value z of Z ,

$$\lim_{N \rightarrow \infty} L^{(N)} = \mathbb{P}[X_1 \leq c \mid Z = z] = \mathbb{P}\left[\epsilon_1 \leq \frac{c - \sqrt{1-\rho}z}{\sqrt{\rho}}\right] = \Phi\left(\frac{c - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

almost surely in $(\epsilon_1, \dots, \epsilon_N)$. Therefore, by Fubini's theorem (which guarantees that a measurable set with all sections of full measure is itself of full measure),

$$\lim_{N \rightarrow \infty} L^{(N)} = \Phi\left(\frac{c - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right)$$

holds almost everywhere.

For L^2 convergence, denote $\mathbb{P}[X_i \leq c \mid Z = z]$ by $p(z)$, so that $l^\infty = p(Z)$. Then, by expressing the expected value as the expectation of the conditional expectation (on Z), we have:

$$\mathbb{E} \left[\left(\frac{1}{N} \sum_{i=1}^N 1_{[X_i \leq c]} - p(Z) \right)^2 \right] = \mathbb{E} \left[\frac{p(Z)(1-p(Z))}{N} \right] \leq \frac{1}{N} \rightarrow 0,$$

as $N \rightarrow \infty$. □

Assuming that Z is also standard Gaussian, the distribution of the limiting average loss $l^{(\infty)}$ is thus

$$\mathbb{P}[l^{(\infty)} \leq x] = \Phi \left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - c}{\sqrt{\rho}} \right) \quad (5.1)$$

This agrees with Schönbucher [22, Eq. (23)] and also the early work of Vasicek [26, page 3]. Similar questions and distributions appear in investigations motivated by physical contexts (see, for instance, Thurner and Hanel [25]).

Now let us consider an extension of this, permitting a *distribution of default thresholds*. To this end suppose that W_1, W_2, \dots are independent standard Gaussians, independent of the variables $(Z, \epsilon_1, \epsilon_2, \dots)$.

Suppose that the threshold C_i 's are all independent random variables with standard Gaussian distribution, and they are also independent of the Gaussian variables $Z, \epsilon_1, \epsilon_2, \dots$. Suppose $c \in \mathbb{R}$ and $\sigma > 0$ and assume that the default threshold for name i is

$$C_i = c + \sigma W_i.$$

The proportion of defaulting names, among the first N , is

$$L^{(N)} = \frac{1}{N} \sum_{i=1}^N 1_{[X_i \leq C_i]}.$$

By the law of large numbers, for each fixed value z of Z ,

$$\begin{aligned} \lim_{N \rightarrow \infty} L^{(N)} &= \mathbb{P}[X_1 \leq C_1 | Z = z] \\ &= \mathbb{P}[\sqrt{1-\rho}\epsilon_1 - \sigma W_1 \leq c - \sqrt{\rho}z] \\ &= \Phi \left(\frac{c - \sqrt{\rho}z}{\sqrt{1-\rho + \sigma^2}} \right). \end{aligned}$$

Thus,

$$L^{(N)} \rightarrow L^{(\infty)} \stackrel{\text{def}}{=} \Phi \left(\frac{c - \sqrt{\rho}Z}{\sqrt{1-\rho + \sigma^2}} \right) \quad \text{almost surely.} \quad (5.2)$$

The special case $\sigma = 0$ yields Theorem 5.1.

6. A Poisson-mixture Model

Variations of the Gaussian copula model include models which use the Poisson distribution (see, for instance, Burtschell [6]). In this section we show that some of the properties proved for the single-factor Gaussian copula model also hold for a ‘Poisson mixture’ model. A model of this general type is explored in the context of credit defaults in the book of Bluhm et al. [4].

We have again a portfolio of N names. Now, however, we assume that each credit can suffer multi-level ‘defaults’ (such as downgrades). Conditional on a

common global factor, modeled by a standard Gaussian Z , the defaults are independent, and the i -th name's loss distribution is Poisson with mean λ_i given by

$$\lambda_i = \lambda_i(x) = \mathbb{P}[X_i \leq c_i | Z = x] = \Phi\left(\frac{c_i - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right). \quad (6.1)$$

Here again, we have, for each i , a threshold $c_i \in \mathbb{R}$, and a variable

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon_i, \quad (6.2)$$

where $Z, \epsilon_1, \dots, \epsilon_N$ are independent standard Gaussians. Now, however, the threshold is used simply to determine λ_i .

Note that X_i is standard Gaussian, and so

$$\mathbb{P}[X_i \leq c_i] = \Phi(c_i). \quad (6.3)$$

For small values of λ_i , the probability of multiple losses in an individual name is small, and so the preceding model can be taken as an approximate description for a synthetic CDO of the type we have considered before.

The probability that the portfolio loss ν has value $j \in \{0, 1, 2, \dots\}$ is then

$$p_j \stackrel{\text{def}}{=} \mathbb{P}[\nu = j] = \int_{\mathbb{R}} e^{-\lambda} \frac{\lambda^j}{j!} \phi(x) dx, \quad (6.4)$$

where

$$\lambda = \lambda(x) = \sum_{i=1}^N \lambda_i. \quad (6.5)$$

Then, following the same procedure as for the Gaussian copula, for the expected loss

$$L_k^e = \mathbb{E}[\min\{\nu, k\}],$$

of the $[0, k]$ -tranche, we have

$$\frac{\partial L_k^e}{\partial \rho} = \int_{\mathbb{R}} I_{P,k}(\lambda) \frac{\partial \lambda}{\partial \rho} \phi(x) dx \quad (6.6)$$

where

$$I_{P,k}(\lambda) = \sum_{j=0}^k (k-j) e^{-\lambda} \left[\frac{\lambda^j}{j!} - \frac{\lambda^{j-1}}{(j-1)!} \right], \quad (6.7)$$

with the second term in $[\dots]$ here is taken to be 0 when $j = 0$. The subscript P in $I_{P,k}$ is for Poisson.

Lemma 6.1. *For $I_{P,k}$ as in (6.7) we have*

$$I_{P,k}(\lambda) = e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} = 1 - \int_0^\lambda e^{-t} \frac{t^{k-1}}{(k-1)!} dt \quad (6.8)$$

In particular, $I_{P,k}(\lambda)$ is monotonically decreasing with λ , if $1 \leq k < N$.

Proof. Algebraic simplification gives

$$I_{P,k}(\lambda) = e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} \quad (6.9)$$

The derivative of this is

$$I'_{P,k}(\lambda) = -e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} + e^{-\lambda} \sum_{j=1}^{k-1} \frac{\lambda^{j-1}}{(j-1)!} = -e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}.$$

Integrating, and using the value 1 for $I_{P,k}(0)$ by (6.9), we obtain equation (6.8). \square

Now by using reasoning similar to that used for the binomial (Gaussian copula) case, we see that

$$\frac{dL_k^e}{d\rho} < 0, \quad \text{and} \quad \frac{dL_k^s}{d\rho} > 0. \quad (6.10)$$

For this model we can also study deltas for individual names. We take, as definition,

$$\Delta_{\text{spread}}^i([0, k]) = \frac{\partial L_k^e}{\partial c_i} / \frac{\partial \Phi(c_i)}{\partial c_i} \quad (6.11)$$

where the denominator is, roughly, the sensitivity of the expected loss for a unit notional credit-default swap on the name i to changes in the threshold c_i .

Using the same method as for the binomial (Gaussian copula) case, but now using Lemma 6.1, we obtain:

$$\Delta_{\text{spread}}^i([0, k]) = \sum_{j \in [0, k]} \int_{\mathbb{R}} e^{-\lambda_i(y)} \frac{\lambda_i(y)^{j-1}}{(j-1)!} \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy \quad (6.12)$$

where the summand on the right side is understood to be 0 when $j = 0$, and, also, on the right $\lambda_i(y)$ is

$$\lambda_i(y) = \sum_{k=1}^N \Phi \left(\frac{(c_k - c_i \rho) - \sqrt{\rho} y}{\sqrt{1-\rho}} \right). \quad (6.13)$$

Notice that $\lambda_i(y)$ is lower for a higher value of c_i (a riskier credit). Then, by Lemma 6.1, $I_{P,k}(\lambda_i(y))$ is higher, and so $\Delta_{\text{spread}}^i([0, k])$ is higher; thus,

riskier credits have higher deltas in equity tranches.

This confirms intuition (riskier credits default likely sooner, and therefore impact an equity tranche more) and results of simulations (see, for instance, [16, Chart 8]).

This model permits further evaluation of quantities such as iGammas (convexity effect of individual credits), but we leave such questions to future work.

7. Concluding Remarks

CDOs are credit derivative instruments of great importance in the global financial system. In this paper we have given mathematically rigorous proofs of a variety of key features of CDO tranche prices and tranche risk management characteristics which are often taken for granted in market practice. Our results are mainly, but not solely, in the context of the standard single-factor Gaussian copula

model, and this may be viewed as an initial step towards a thorough investigation of CDO copula models.

Acknowledgments. Research support for ANS from US NSF grant DMS 0601141 is gratefully acknowledged. ANS also thanks Morten Andersen for many enlightening discussions. Some of the results in this paper grew out of the dissertation of Meng [19]. We are grateful to A. Cousin for bringing to our attention their work [9]. We are also thankful to the anonymous referee for mentioning the reference item [15] and for several useful suggestions.

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