A SUPPORT THEOREM FOR A GAUSSIAN RADON TRANSFORM IN INFINITE DIMENSIONS

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Abstract. We prove that in infinite dimensions, if a bounded continuous function has zero Gaussian integral over all hyperplanes outside a closed bounded convex set then the function is zero outside this set. This is an infinite-dimensional form of the well-known Helgason support theorem for Radon transforms in finite dimensions.

1. Introduction

The Radon transform [7] associates to a function f on the finite-dimensional space $\mathbb{R}^n$ the function Rf on the set of all hyperplanes in $\mathbb{R}^n$ whose value on any hyperplane P is the integral of f over P:

$$Rf(P) = \int_P f(x)\,dx, \tag{1.1}$$

the integration here being with respect to Lebesgue measure on P. This transform does not generalize directly to infinite dimensions because there is no useful notion of Lebesgue measure in infinite dimensions. However, there is a well-developed theory of Gaussian measures in infinite dimensions and so it is natural to extend the Radon transform to infinite dimensions using Gaussian measure:

$$Gf(P) = \int f\,d\mu_P, \tag{1.2}$$

where μ_P is Gaussian measure on any hyperplane P in a Hilbert space H_0. This transform was developed in [6] but we shall present a self-contained account below in subsection 2.1 and the Appendix.

A central feature of the classical Radon transform R is the Helgason support theorem (Helgason [5]): if f is a rapidly decreasing continuous function and $Rf(P)$ is 0 on every hyperplane P lying outside a compact convex set K, then

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f is 0 off K. In this paper we prove an infinite-dimensional version of this support theorem.

There are some substantial technical obstructions to proving the Radon transform support theorem in infinite dimensions. A support theorem, even in the finite dimensional case, works for a class of suitably regular functions, such as continuous functions of rapid decrease. In the infinite-dimensional setting it is first of all necessary to choose a framework for the Gaussian measures with respect to which the transform is defined. We use here the framework of nuclear spaces and duals. An alternative standard framework, that of Abstract Wiener Spaces, will require a separate development of the analytic aspects of our argument. Even within the setting of nuclear spaces, a choice has to be made about the class of functions to which the theorem would apply. In this paper we work with strongly (or even just sequentially) continuous, bounded functions; since the background measure is a Gaussian, boundedness implies rapid decrease in finite dimensions. (In Becnel [2] the result is proved for a class of functions called Hida test functions; these are not bounded but have smoothness and growth properties.) The topology and measurable structure on the infinite-dimensional space on which the underlying Gaussian measure is defined come into play. There are treacherous topological features of infinite dimensional spaces, such as absence of local compactness, non-metrizability (for the space we work with), and a distinction between continuity and sequential continuity. In addition to these analytic issues we also need, not surprisingly, some geometric results which are meaningful only in infinite dimensions.

The elegance and coherence of the infinite dimensional analytic and geometric ideas in the proof of the support theorem are satisfying, but leaves open one question about the Gauss-Radon transform: why should one study it? The finite-dimensional Radon transform is of central significance in tomography (albeit here only the three-dimensional theory matters), and there is a vast body of results of purely mathematical interest, including one of the great early results which specifies the range of the space of Schwartz functions. The authors freely acknowldege that the theory in infinite dimensions is in infancy, and at this stage can only express their hope that larger developments, techniques, and possible applications lie in the future. The motivation for our investigation of the infinite-dimensional theory arose in a stochastic context. Consider a random functional F, of suitable regularity, of a Brownian motion $t \mapsto B_t$; one may wish to recover information about F from the conditional expectation values $\mathbb{E}[F \mid \int_0^\infty f(t)\, dB_t = c]$, with f running over a suitable collection of functions and c over real numbers. Such a problem is essentially a problem concerning the Gauss-Radon transform in the setting of Gaussian measure over the Hilbert space $L^2([0,\infty))$. We shall not pursue this or other applications in the present paper where we develop the theory and central result in an abstract setting.

2. The Gauss-Radon Transform

The Gauss-Radon transform Gf of a function f for a real separable Hilbert space H_0 associates to each hyperplane P in H_0 the integral of f with respect to the Gaussian measure μ_P for the hyperplane P. In this section we will spell out the details of this, including a precise specification of the measure μ_P and the space on which f is defined.

2.1. Gaussian Measure for Affine Subspaces. Throughout this paper H_0 is a separable real Hilbert space. The inner-product on H_0 will be denoted $\langle \cdot, \cdot \rangle_0$ or simply $\langle \cdot, \cdot \rangle$, and the corresponding inner product by $\| \cdot \|_0$.

An affine subspace of H_0 is a translate of a closed subspace, i.e. a subset of the form $u + F$, for some closed subspace F of H_0 and some vector $u \in H_0$.

As we prove in the Appendix, *there is a measurable space* $(\Omega, \mathcal{F})$ *and a linear map* $x \mapsto \hat{x}$ *taking vectors* x *in some dense subspace of* H_0 *to measurable functions on* Ω*, such that for any closed subspace* $F \subset H_0$*, and any* $u \in F$*, there is a probability measure* $\mu_{u+F^\perp}$ *on* $(\Omega, \mathcal{F})$ *satisfying*

$$\int e^{i\hat{x}}\, d\mu_{u+F^\perp} = e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_{F^\perp}\|_0^2}, \tag{2.1}$$

where $x_{F^\perp}$ *denotes the orthogonal projection of* x *onto* $F^\perp$.

The mapping $x \mapsto \hat{x}$ then extends, for each u and F, to a continuous linear map

$$H_0 \to L^2(\mu_{u+F^\perp}) : x \mapsto \hat{x}$$

and then (2.1) holds for all $x \in H_0$.

A construction of $(\Omega, \mathcal{F})$ is carried out in the Appendix, with Ω being the infinite product $\mathbb{R}^{\{1,2,3,\ldots\}}$ and $\mathcal{F}$ the σ-algebra of subsets of Ω generated by all the coordinate projections $\mathbb{R}^{\{1,2,3,\ldots\}} \to \mathbb{R}$.

To carry out meaningful analysis with respect to measures in infinite dimensions, the underlying space should be realized as a topological vector space with suitable analytic structure. One convenient, standard choice for Ω is as the dual space $\mathcal{H}'$ of a topological vector space $\mathcal{H}$ which sits inside H_0 as a dense subspace; moreover, it is assumed that $\mathcal{H}$ is a *nuclear space* (as spelled out in Proposition 2.1 below). The dual space $\mathcal{H}'$, consists of all continuous linear maps

$$x' : \mathcal{H} \to \mathbb{R} : x \mapsto \langle x', x \rangle.$$

The *weak topology* on $\mathcal{H}'$ is the smallest one for which the evaluation map

$$\hat{x} : \mathcal{H}' \to \mathbb{R} : x' \mapsto \langle x', x \rangle$$

is continuous for every $x \in \mathcal{H}$. The *strong* topology on $\mathcal{H}'$ is the smallest one for which all translates of

$$N(D;\epsilon) = \{x' \in \mathcal{H}' \,:\, \sup_{x\in D} |\langle x', x\rangle| < \epsilon\}$$

are open, for all $\epsilon > 0$ and all bounded $D \subset \mathcal{H}$ (a subset of $\mathcal{H}$ is bounded if it lies inside some multiple of any given open set). The *cylindrical σ-algebra* for $\mathcal{H}'$ is the smallest σ-algebra generated by the maps $\hat{x}$ with x running over $\mathcal{H}$.

We isolate here the minimal features of this structure which we shall need (a construction is given in the Appendix):

Proposition 2.1. *Let H_0 be a separable real Hilbert space. Then there is a dense subspace $\mathcal{H} \subset H_0$ such that:*

(i) *there is a sequence of inner-products $\langle\cdot,\cdot\rangle_p$, with $p \in \{1,2,3,...\}$, on $\mathcal{H}$ such that the corresponding norms $\|\cdot\|_p$ satisfy*

$$\|\cdot\|_0 \le \|\cdot\|_1 \le \|\cdot\|_2 \le ..., \tag{2.2}$$

and, with H_p denoting the subspace of H_0 given by the completion of $\mathcal{H}$ with respect to the norm $\|\cdot\|_p$,

$$\mathcal{H} = \bigcap_{p\in\{1,2,3,...\}} H_p \subset \cdots \subset H_2 \subset H_1 \subset H_0, \tag{2.3}$$

and, for every $p \in \{1,2,3,...\}$ and any orthonormal basis $v_1, v_2, ...$ for H_p,

$$\sum_{n=1}^{\infty} \|v_n\|_{p-1}^2 < \infty \tag{2.4}$$

(that is, the inclusion map $H_p \to H_{p-1}$ is Hilbert-Schmidt);

(ii) *$\mathcal{H}$ is a locally convex topological vector space when equipped with the topology in which open sets are unions of translates of open balls of the form $\{x \in \mathcal{H} : \|x\|_p < \epsilon\}$ with $p \in \{0,1,2,3,...\}$ and $\epsilon > 0$;*

(iii) *for every every closed subspace $F \subset H_0$ and every $u \in F$ there is a probability measure $\mu_{u+F^\perp}$ on the cylindrical σ-algebra of $\mathcal{H}'$ such that*

$$\int_{\mathcal{H}'} e^{i\hat{x}}\, d\mu_{u+F^\perp} = e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_{F^\perp}\|_0^2} \tag{2.5}$$

for all $x \in \mathcal{H}$, where $x_{F^\perp}$ is the orthogonal projection of x onto $F^\perp$.

The conditions in (i) are summarized by saying that $\mathcal{H}$ is a *nuclear space.*

Among numerous extraordinarily convenient features of nuclear spaces is the fact that the notions of weak convergence and strong convergence in the dual space $\mathcal{H}'$ coincide (see Gel'fand et al. [3, 4][Section 6.4 in Volume 2 and Section 3.4 in Volume 4]). Moreover, the Borel σ-algebra generated by the weak topology and that by the strong topology coincide (see, for instance, Becnel [1]). There is, potentially, a third topology of interest on $\mathcal{H}'$, called the *inductive limit* topology, which is the largest locally convex topological vector space structure on $\mathcal{H}'$ for which the naturally induced injections $H_p' \to \mathcal{H}'$ are continuous. This happens to agree with the strong topology [1].

The characteristic function of $\mu_{u+F^\perp}$ provided by (2.5) implies that, with respect to the probability measure $\mu_{u+F^\perp}$, the random variable $\hat{x}$ has Gaussian distribution with mean $\langle x, u\rangle_0$ and variance $\|x_{F^\perp}\|_0^2$. Hence,

$$\|\hat{x}\|^2_{L^2(\mu_{u+F^\perp})} = |\langle u, x\rangle_0|^2 + \|x_{F^\perp}\|_0^2 \leq (\|u\|_0^2 + 1)\|x\|_0^2. \tag{2.6}$$

Thus, $x \mapsto \hat{x}$ is continuous as a map $\mathcal{H} \to L^2(\mathcal{H}', \mu_{u+F^\perp})$, and so extends to a continuous linear map

$$H_0 \to L^2(\mu_{u+F^\perp}) : x \mapsto \hat{x},$$

with $\hat{x}$ satisfying (2.5), i.e. $\hat{x}$ is a Gaussian variable with mean $\langle x, u\rangle_0$ and variance $\|x_{F^\perp}\|_0^2$, and hence also the bound (2.6). Note, however, that for x in H_0 outside $\mathcal{H}$, the definition of $\hat{x}$ depends on $\mu_{u+F^\perp}$ and hence on the affine subspace $u + F^\perp$.

The measure $\mu_{u+F^\perp}$ is indeed concentrated 'on' the affine subspace $u+F^\perp$, but only in the following special sense:

Proposition 2.2. *If F is a closed subspace of the real separable Hilbert space H_0 and $u \in F$ then for any $x \in F$ the random variable $\hat{x}$ is equal to the constant $\langle x, u\rangle_0$ almost surely with respect to the measure $\mu_{u+F^\perp}$.*

Proof. For any $u, x \in F$, and $t \in \mathbb{R}$ we have

$$\int e^{it\hat{x}}\, d\mu_{u+F^\perp} = e^{it\langle x,u\rangle_0 - \frac{t^2}{2}\|x_{F^\perp}\|_0^2} = e^{it\langle x,u\rangle_0}, \tag{2.7}$$

which, as a function of t, is the characteristic function of the random variable whose value is the constant $\langle x, u\rangle_0$. Hence $\hat{x}$ is equal to the constant $\langle x, u\rangle_0$ almost everywhere with respect to $\mu_{u+F^\perp}$. □

2.2. The Gauss-Radon Transform.

Definition 2.1. If f is a bounded Borel measurable function on $\mathcal{H}'$ then its *Gauss-Radon transform* Gf is the function which associates to each hyperplane P in H_0 the value

$$Gf(P) = \int_{\mathcal{H}'} f\, d\mu_P \tag{2.8}$$

Our framework thus far has been infinite-dimensional. The Gauss-Radon transform of a function f on a finite-dimensional real Hilbert space H is the function Gf which associates to each hyperplane P in H the value

$$G_H f(P) = \int f\, d\mu_P,$$

where now μ_P is the usual standard Gaussian measure on the hyperplane P, specified by the characteristic function

$$\int_H e^{i\langle x,y\rangle_0}\, d\mu_P(y) = e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_\perp\|_0^2} \tag{2.9}$$

for all $x \in H$, where $u \in P$ is orthogonal to P, and $x_\perp$ is the component of x orthogonal to the hyperplane P. Note that here the measure μ_P is defined simply on the original finite-dimensional space H itself (more technically, it is the dual of H, identified with H by means of the inner-product $\langle \cdot, \cdot \rangle_0$ on H) and is concentrated on the hyperplane P.

3. Some geometric and limiting results

As before, we work with a separable real Hilbert space H_0, a dense subspace $\mathcal{H}$ which is a topological vector space, and the measures $\mu_{u+F^\perp}$ on the dual $\mathcal{H}'$. For any $p \in \{1, 2, ...\}$ we denote by H_p the subspace of H_0 obtained by completing $\mathcal{H}$ with respect to the norm $\|\cdot\|_p$. Let $\mathcal{H}'_p$ be the subspace of $\mathcal{H}'$ consisting of all linear functionals on $\mathcal{H}$ which are continuous with respect to $\|\cdot\|_p$. The space $\mathcal{H}$ has the structure described in Proposition 2.1 above. Thus we have, dual to the chain of inclusions (2.3), the chain of inclusions

$$H_0 = \mathcal{H}'_0 \subset \mathcal{H}'_1 \subset \mathcal{H}'_2 \subset \cdots \subset \bigcup_{p \geq 0} \mathcal{H}'_p = \mathcal{H}'. \tag{3.1}$$

Note that any element of $\mathcal{H}'_p$ extends to a unique element of the Hilbert-space dual H'_p, for any $p \in \{0, 1, 2, ...\}$, and, conversely, any element in H'_p restricts to a linear functional on $\mathcal{H}$ which is continuous with respect to $\|\cdot\|_p$, i.e. an element of $\mathcal{H}'_p$. Thus we can and will make the identification

$$\mathcal{H}'_p \simeq H'_p \tag{3.2}$$

for all $p \in \{0, 1, 2, ...\}$. Thus we may evaluate an element $x' \in \mathcal{H}'_p$ on any vector in $H_p \subset H_0$.

As may be readily verified, a *continuous* linear functional x' on $\mathcal{H}$ is continuous with respect to a particular norm $\|\cdot\|_p$ if and only if

$$\|x'\|^2_{-p} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} |\langle x', v_n \rangle|^2 < \infty \tag{3.3}$$

for some (and hence any) maximal $\langle \cdot, \cdot \rangle_p$-orthonormal sequence of vectors v_n in $\mathcal{H}$. The value of $\|x'\|_{-p} \geq 0$ is independent of the choice of the vectors v_n; if finite, it is simply the norm of x' viewed as an element of the dual Hilbert space H'_p. Note that

$$\|x'\|^2_{-p} = \left(\sum_{n=1}^{\infty} \hat{v}_n^2 \right)(x'), \tag{3.4}$$

where on the right we have the evaluation of the, possibly ∞-valued function $\sum_{n=1}^{\infty} \hat{v}_n^2$ on the element $x' \in \mathcal{H}'$.

We will use the closed ball of radius R in $\mathcal{H}'_p$:

$$D_{-p}(R) = \{x' \in \mathcal{H}' \,:\, \|x'\|_{-p} \leq R\}, \tag{3.5}$$

for $p \in \{1, 2, 3, ...\}$. As in the Banach-Alaoglu theorem, this ball is a compact subset of $\mathcal{H}'$ in the weak topology.

Note that, for $p \in \{0, 1, 2, 3, ...\}$, a continuous linear functional x' on $\mathcal{H}$ lies in $D_{-p}(R)$ if and only if

$$\sum_{n=1}^{\infty} |\langle x', v_n\rangle|^2 \leq R^2 \tag{3.6}$$

for some, and hence any, orthonormal basis $v_1, v_2, ...$ in H_p. The finiteness of the sum in the left here implies that x' is continuous with respect to the $\|\cdot\|_p$-norm and then (3.6) says that the dual norm $\|x'\|_{-p}$ is $\leq R$.

Proposition 3.1. *Let F be a finite-dimensional subspace of the separable real Hilbert space H_0, and U a bounded subset of F. Then for any $p \in \{1, 2, 3, ...\}$, and any $\epsilon > 0$, there is an $R \in (0, \infty)$ such that*

$$\mu_{u+F^\perp}[D_{-p}(R)] > 1 - \epsilon, \tag{3.7}$$

for all $u \in U$. Consequently,

$$\mu_{u+F^\perp}(\mathcal{H}'_p) = 1.$$

Proof. Fix an orthonormal basis $v_1, v_2, ...$ of H_p lying in the dense subspace $\mathcal{H}$ (we can choose a maximal $\langle\cdot,\cdot\rangle_p$-orthonormal set in $\mathcal{H}$). The complement $D_{-p}(R)^c$ is given by

$$D_{-p}(R)^c = \left[\frac{1}{R^2}\sum_{n=1}^{\infty} \hat{v}_n^2 > 1\right] \subset \mathcal{H}'. \tag{3.8}$$

So, as in Chebyshev's inequality,

$$\begin{aligned}
\mu_{u+F^\perp}[D_{-p}(R)^c] &\leq \int_{\mathcal{H}'} \frac{1}{R^2}\sum_{n=1}^{\infty} \hat{v}_n^2 \, d\mu_{u+F^\perp} \\
&= \frac{1}{R^2}\sum_{n=1}^{\infty} \|\hat{v}_n\|^2_{L^2(\mu_{u+F^\perp})} \\
&\leq \frac{1}{R^2}\sum_{n=1}^{\infty} (\|u\|_0^2 + 1)\|v_n\|_0^2 \qquad \text{(by (2.6))} \\
&\leq \frac{1}{R^2}\left(\sum_{n=1}^{\infty} \|v_n\|^2_{p-1}\right)(\sup_{u\in U} \|u\|_0^2 + 1) \quad (\text{using } \|\cdot\|_0 \leq \|\cdot\|_{p-1})
\end{aligned} \tag{3.9}$$

This is $< \epsilon$ when R large enough, since U is bounded and $\sum_{n=1}^{\infty} \|v_n\|^2_{p-1}$ is finite for $p \geq 1$. □

Here is a very convenient alternative way to look at hyperplane integrals:

Proposition 3.2. *If F is a closed subspace of the real separable Hilbert space H_0 and $u \in F$ then*

$$\int f(x')\, d\mu_{u+F^\perp}(x') = \int f(x'+u)\, d\mu_{F^\perp}(x') \tag{3.10}$$

whenever f is a measurable function on $\mathcal{H}'$ and the equality here holds in the sense that if either side is defined so is the other and the integrals are then equal.

Observe that F sits inside H_0, which we are viewing as a subspace of $\mathcal{H}'$; thus $u \in F$ is also a linear functional on $\mathcal{H}$, mapping any $x \in \mathcal{H}$ to $\langle u, x\rangle_0$.

Proof. On taking $f = e^{i\hat{x}}$, with $x \in \mathcal{H}$, we have

$$\begin{aligned}\int \underbrace{e^{i\hat{x}(x'+u)}}_{f(x'+u)}\, d\mu_{F^\perp}(x') &= e^{i\langle x,u\rangle} \int e^{i\langle x',x\rangle}\, d\mu_{F^\perp}(x') \\ &= e^{i\langle x,u\rangle} e^{-\frac{1}{2}\|x_{F^\perp}\|_0^2} \\ &= \int \underbrace{e^{i\hat{x}}}_{f}\, d\mu_{u+F^\perp}\end{aligned} \tag{3.11}$$

The rest of the argument is fairly routine but we work through it for the sake of completeness and to be sure no infinite-dimensional issues are overlooked.

Consider a C^∞ function g on $\mathbb{R}^N$ having compact support. Then g is the Fourier transform of a rapidly decreasing smooth function and so, in particular, it is the Fourier transform of a complex Borel measure ν_g on $\mathbb{R}^N$:

$$g(t) = \int_{\mathbb{R}^N} e^{it\cdot w}\, d\nu_g(w)$$

Then for any $x_1, ..., x_N \in \mathcal{H}$, the function $g(\hat{x}_1, ..., \hat{x}_N)$ on $\mathcal{H}'$ can be expressed as

$$\begin{aligned}g(\hat{x}_1, ..., \hat{x}_N)(x') &= \int_{\mathbb{R}^N} e^{iw_1\langle x',x_1\rangle+\cdots+iw_N\langle x',x_N\rangle}\, d\nu_g(w_1, ..., w_N) \\ &= \int_{\mathbb{R}^N} e^{i\langle x', w_1x_1+\cdots+w_Nx_N\rangle}\, d\nu_g(w_1, ..., w_N).\end{aligned} \tag{3.12}$$

The function

$$\mathcal{H}' \times \mathbb{R}^N : (x', (w_1, ..., w_N)) \mapsto \sum_{j=1}^N w_j\langle x', x_j\rangle$$

is measurable with respect to the product of the Borel σ-algebras on $\mathcal{H}'$ and $\mathbb{R}^N$. So we can apply Fubini's theorem, along with (3.11), to conclude that the identity (3.10) holds when f is of the form $g(\hat{x}_1, ..., \hat{x}_N)$.

Now the indicator function 1_C of a compact cube C in $\mathbb{R}^N$ is the pointwise limit of a uniformly bounded sequence of C^∞ functions of compact support on $\mathbb{R}^N$, and so the result holds also for f of the form $1_C(\hat{x}_1, ..., \hat{x}_N)$, i.e. the indicator function of $(\hat{x}_1, ..., \hat{x}_N)^{-1}(C)$. Then, by the Dynkin π-λ theorem it holds for the indicator functions of all sets in the σ-algebra generated by the functions $\hat{x}$ with x running over $\mathcal{H}$, i.e. all Borel sets. Then, taking linear combinations and applying monotone convergence, the result holds for all non-negative measurable f on $\mathcal{H}'$. □

The space $\mathcal{H}'$ need not be metrizable in the weak or strong topologies and so, even though every continuous function is sequentially continuous, the converse need not be true for the weak or strong topologies. There is, however, one simplification: though the weak and strong topologies on $\mathcal{H}'$ are different, it is a remarkable fact that a sequence converges weakly if and only if it converges strongly. For this reason, there is no distinction between weak sequential continuity and strong sequential continuity of a function. See Gel'fand and Vilenkin [4, Section 3.4] for these facts.

We say that a function f on $\mathcal{H}'$ is *sequentially continuous* at a point x' if for any sequence $x'_n \in \mathcal{H}'$ converging to x', the values $f(x'_n)$ converge to $f(x')$. If f is sequentially continuous everywhere we say it is sequentially continuous.

Note that a strongly continuous function is automatically sequentially continuous. In the converse direction, we can only say that a sequentially continuous function on $\mathcal{H}'$ is continuous on each of the Hilbert spaces H_{-p}, because these Hillbert spaces are metrizable and so there is no distinction between continuity and sequential continuity.

Proposition 3.3. *Suppose that $f : \mathcal{H}' \to \mathbb{R}$ is a bounded, sequentially continuous function on $\mathcal{H}'$. Then for every finite-dimensional subspace F of $\mathcal{H}$, the function $u \mapsto \int f \, d\mu_{u+F^\perp}$ is continuous and bounded in $u \in F$.*

Proof. Let U be a bounded neighborhood of u in F. Let $\epsilon > 0$. Then by Proposition 3.1, there is an $R \in (0, \infty)$ such that

$$\mu_{y+F^\perp}[D_{-1}(R)] > 1 - \epsilon$$

for all $y \in U$, where $D_{-1}(R)$ is the closed R-ball in $H_{-1} \subset \mathcal{H}'$:

$$D_{-1}(R) = \{x' \in \mathcal{H}' : \|x'\|_{-1} \le R\}.$$

Then, for any $v \in U$, we have

$$\begin{aligned} \left| \int f \, d\mu_{u+F^\perp} - \int f \, d\mu_{v+F^\perp} \right| &\le \int_{D_{-1}(R)} |f(x'+u) - f(x'+v)| \, d\mu_{F^\perp}(x') \\ &\quad + 2\epsilon \|f\|_{\sup} \quad \text{(by Proposition 3.2)} \\ &\le \sup_{x' \in D_{-1}(R)} |f(x'+u) - f(x'+v)| + 2\epsilon \|f\|_{\sup}. \end{aligned} \tag{3.13}$$

Now we claim uniform continuity: *there is a $\delta > 0$ such that if $v \in F$ with $\|v - u\|_0 < \delta$ then $\sup_{x' \in D_{-1}(R)} |f(x'+u) - f(x'+v)|$ is $< \epsilon$.*

Assume the contrary. Then there is a sequence of points $v_n \in F$ converging to u and a sequence of points $x'_n \in D_{-1}(R)$ such that $|f(x'_n+u)-f(x'_n+v_n)|$ is $\geq \epsilon$. Now, by the Banach-Alaoglu theorem and separability, $D_{-1}(R)$ is compact and metrizable in the weak topology of $\mathcal{H}'$ and so we may assume that $x'_n \to x'$ weakly for some $x' \in D_{-1}(R)$. Hence ($x'_n \to x'$ strongly, and) also $x'_n + v_n \to x' + u$. Since f is sequentially continuous, it follows that

$$f(x'_n+u)-f(x'_n+v_n) \to f(x+u)-f(x+u)=0,$$

contradicting the assumption made.

Thus, indeed, for all v in some neighborhood of u in F we have

$$\sup_{x'\in D_{-1}(R)} |f(x'+u)-f(x'+v)| < \epsilon.$$

Then we have for such v, from (3.13),

$$\left| \int f\, d\mu_{u+F^\perp} - \int f\, d\mu_{v+F^\perp} \right| \leq (1+2\|f\|_{\sup})\epsilon, \tag{3.14}$$

which shows that $\int f\, d\mu_{u+F^\perp}$ depends continuously on $u \in F$. □

Next we have a key limiting result:

Proposition 3.4. *Suppose that f is a bounded Borel function on $\mathcal{H}'$, sequentially continuous at $u \in H_0 \subset \mathcal{H}'$. Then:*

$$\lim_{u\in F\to H_0} \int f\, d\mu_{u+F^\perp} = f(u),$$

in the sense that if $F_1 \subset F_2 \subset \cdots$ is a sequence of finite-dimensional subspaces, with $u \in F_1$, such that $\cup_{n\geq 1} F_n$ is dense in H_0, then

$$\lim_{n\to\infty} \int f\, d\mu_{u+F_n^\perp} = f(u) \tag{3.15}$$

Proof. Observe first that since f is sequentially continuous at u on the Hilbert space $H_{-1} \subset \mathcal{H}'$ it is continuous on H_{-1} at u. Let $\epsilon > 0$. Then there is a closed ball $D_{-1}(R)$ in the Hilbert space H_{-1}, centered at 0 and having radius some $R \in (0,\infty)$, such that:

$$\sup_{x'\in D_{-1}(R)} |f(u+x')-f(u)| < \epsilon. \tag{3.16}$$

Let $v_1, v_2, ..$ be an orthonormal basis of H_1 lying in $\mathcal{H}$. For any finite-dimensional subspace F in H_0, we have

$$\begin{aligned} \mu_{F^\perp}\big(D_{-1}(R)^c\big) &= \mu_{F^\perp}\left[\sum_{m=1}^{\infty} \hat{v}_m^2 > R^2\right] \\ &\leq \frac{1}{R^2}\sum_{m=1}^{\infty}\int \hat{v}_m^2\, d\mu_{F^\perp} \\ &= \frac{1}{R^2}\sum_{m=1}^{\infty} \|(v_m)_{F^\perp}\|_0^2 \qquad \text{(by (2.6))} \end{aligned} \tag{3.17}$$

Note that the series above is dominated termwise by the convergent series

$$\sum_{m=1}^{\infty} \|v_m\|_0^2 < \infty.$$

Now, to apply this to the subspaces F_n, observe first that the projection $x_{F_n^\perp} = x - x_{F_n}$ converges to 0 (choose an orthonormal basis of H_0 comprised of bases in the F_n). Hence, by dominated convergence,

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} \|(v_m)_{F_n^\perp}\|_0^2 = 0.$$

Consequently,

$$\lim_{n\to\infty} \mu_{F_n^\perp}\big(D_{-1}(R)^c\big) = 0. \tag{3.18}$$

Hence,

$$\begin{aligned} \left|\int f\, d\mu_{u+F_n^\perp} - f(u)\right| &\leq \int |f(u+x') - f(u)|\, d\mu_{F_n^\perp}(x') \\ &\leq \sup_{x'\in D_{-1}(R)} |f(u+x') - f(u)| + 2\|f\|_{\sup}\mu_{F_n^\perp}\big(D_{-1}(R)^c\big) \end{aligned} \tag{3.19}$$

By the choice of R, the first term on the right is $< \epsilon$, as noted in (3.16). Next, with this R, the second term is $< \epsilon$ when n is large enough. Hence we have the limiting result (3.15). □

Subspaces X and Y, and any of their translates, of a Hilbert space are said to be *perpendicular* if neither is a subspace of the other and they can be split into mutually orthogonal subspaces

$$X = (X\cap Y) + (X\cap Y^\perp), \qquad \text{and} \qquad Y = (X\cap Y) + (X^\perp \cap Y). \tag{3.20}$$

This means that a vector in X (or Y) which is orthogonal to $X\cap Y$ is in fact orthogonal to Y (or X). See Figure 1.

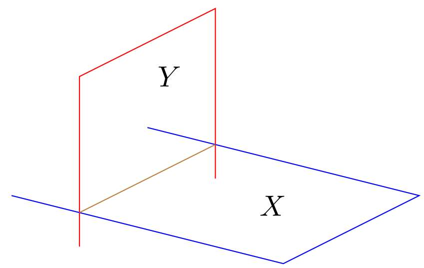


Figure 1. Perpendicular Subspaces

Proposition 3.5. *Consider a hyperplane P in the infinite-dimensional Hilbert space H_0 and a finite-dimensional subspace $F \neq \{0\}$ of H_0. Then P and F are perpendicular if and only if P can be expressed as*

$$P = u + u_0^{\perp},$$

for some non-zero vector $u_0 \in F$ and u is a multiple of u_0. Moreover, $P \cap F$ is a hyperplane within the finite-dimensional space F.

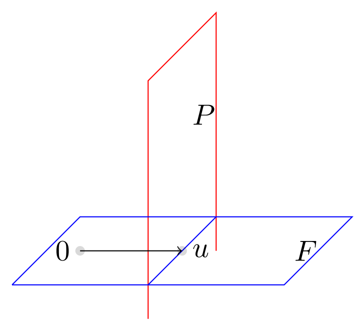


Figure 2. A hyperplane P intersecting a subspace F

Proof. Since P is a hyperplane, it is the translate of a subspace of H_0 of the form $P_0 = u_0^{\perp}$, for some non-zero vector u_0.

Suppose first that P is of the form $u + u_0^{\perp}$, for some non-zero $u_0 \in F$ and $u \in \mathbb{R}u_0$. Let $P_0 = u_0^{\perp}$ be the subspace orthogonal to u_0. Consider any $x \in P_0$; then

$$x = x_F + x_{F^{\perp}},$$

where x_H denotes the orthogonal projection of x onto a closed subspace H. Then

$$\langle x, u_0 \rangle = \langle x_F, u_0 \rangle + \langle x_{F^{\perp}}, u_0 \rangle.$$

In this expression, the left side is 0 because $x \in u_0^{\perp}$, and on the right the second term is 0 because u_0 is on F. Hence x_F is orthogonal to u_0, i.e. $x_F \in P_0$ and hence also $x_{F^\perp}$ is in P_0. This shows that P_0 is the orthogonal direct sum of $P_0 \cap F$ and $P_0 \cap F^\perp$:

$$P_0 = (P_0 \cap F) + (P_0 \cap F^\perp). \tag{3.21}$$

Next, since $u_0 \in F$, we can split F internally as the orthogonal sum of $\mathbb{R}u_0$ and the subspace of F orthogonal to u_0:

$$F = (F \cap u_0^\perp) + \mathbb{R}u_0 = (P_0 \cap F) + (P_0^\perp \cap F). \tag{3.22}$$

Note that P_0, being infinite-dimensional, is not a subspace of F, and F is not a subspace of P_0 because it contains $u_0 \neq 0$ which is orthogonal to P_0. Thus, F and P are perpendicular. Note also that in this case

$$F \cap P = u + (F \cap P_0) = u + (F \cap u_0^\perp) \tag{3.23}$$

wherein u and u_0 are in F, and so $F \cap P$ is a hyperplane within F.

Conversely, suppose P and F are perpendicular. Then F is not a subspace of P translated back to the origin, i.e. of

$$P_0 \stackrel{\text{def}}{=} P - P,$$

and any vector in F orthogonal to $F \cap P_0$ is orthogonal to P_0. Consequently, u_0, any chosen non-zero vector in $P_0^\perp$, is in F. Thus, P is a translate of $u_0^\perp$. Hence P is $u + u_0^\perp$, where u is the point in P closest to 0, and is therefore a multiple of u_0. This shows that P is indeed of the form $u + u_0^\perp$, with $u_0 \in F$ and u a multiple of u_0. □

The following relates the Gauss-Radon transform in infinite dimensions to that for finite-dimensional subspaces by a disintegration process.

Proposition 3.6. *Let P be a hyperplane in H_0, and F a non-zero finite-dimensional subspace of H_0 which is perpendicular to P. Then for any bounded Borel function f on $\mathcal{H}'$ we have*

$$Gf(P) = G_F(f^*)(P \cap F) \tag{3.24}$$

where, on the right, $G_F(f^*)$ is the Gauss-Radon transform of the function f^* on F given by

$$f^*(y) = \int_{\mathcal{H}'} f \, d\mu_{y+F^\perp}.$$

Part of the conclusion here is that f^* is a measurable function on F.

Proof. Let u_0 be a non-zero vector in F orthogonal to the hyperplane P, and let u be the point on $P \cap F$ closest to the origin. Then (see Figure 2, with u_0 along u):

$$P \cap F = u + (u_0^\perp \cap F)$$

Consider first a function of the form $\phi = e^{i\hat{x}}$, where $x \in \mathcal{H}$; then

$$\begin{aligned} G_F(\phi^*)(P \cap F) &= \int_F \left(\int_{\mathcal{H}'} e^{i\hat{x}}\, d\mu_{y+F^\perp} \right) d\mu_{P\cap F}(y) \\ &= \int_F e^{i\langle x,y\rangle_0 - \frac{1}{2}\|x_{F^\perp}\|_0^2}\, d\mu_{u+(u_0^\perp \cap F)}(y) \\ &= e^{-\frac{1}{2}\|x_{F^\perp}\|_0^2} \int e^{i\langle x_F, y\rangle_0}\, d\mu_{u+(u_0^\perp\cap F)}(y) \\ &= e^{-\frac{1}{2}\|x_{F^\perp}\|_0^2 + i\langle x_F,u\rangle_0 - \frac{1}{2}\|x_{u_0^\perp\cap F}\|_0^2} \\ &= e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_{u_0^\perp}\|^2}, \end{aligned} \tag{3.25}$$

where, in the last step, we used the fact that the component of $x_{u_0^\perp}$ orthogonal to $u_0^\perp \cap F$ is simply $x_{F^\perp}$, because $u_0^\perp$ and F are perpendicular so that a vector in $u_0^\perp$ which is orthogonal to $u_0^\perp \cap F$ is just a vector in $u_0^\perp$ orthogonal to F.

Next we observe that

$$\begin{aligned} G\phi(P) &= \int_{\mathcal{H}'} e^{i\hat{x}}\, d\mu_{u+u_0^\perp} \\ &= e^{i\langle x,u\rangle_0 - \frac{1}{2}\|x_{u_0^\perp}\|_0^2} \end{aligned} \tag{3.26}$$

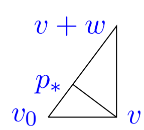
Thus, the result is proved for $\phi = e^{i\hat{x}}$. The rest of the argument is as in the proof of Proposition 3.2. □

Lastly we have a geometric observation:

Proposition 3.7. *Let K_0 be a closed convex subset of the Hilbert space H_0, and v a point outside K_0. Then there is a finite-dimensional subspace $F_0 \subset H_0$ containing v, and a sequence of finite-dimensional subspaces $F_n \subset H_0$ with $F_0 \subset F_1 \subset \cdots$ and $\cup_{n\geq 0} F_n$ dense in H_0 such that $v + F_n^\perp$ is disjoint from K_0 for each $n \in \{1, 2, 3, ...\}$. Moreover, v lies outside the orthogonal projection $\mathrm{pr}_{F_n}(K_0)$ of K_0 onto F_n:*

$$v \notin \mathrm{pr}_{F_n}(K_0) \qquad \text{for all } n \in \{1, 2, 3, ...\}. \tag{3.27}$$

Proof. Since K_0 is closed and convex in the Hilbert space H_0, there is a unique point $v_0 \in K_0$ closest to v. Then the hyperplane through v orthogonal to the vector $u_0 = v - v_0$, i.e. the hyperplane $v + u_0^\perp$, does not contain any point of K_0. For, otherwise, there would be some $w \in u_0^\perp$ with $v + w$ in K_0, and

then in the right angled triangle 

formed by the points v_0, v, and $v + w$ (which has a right angle at the point v) there would be a point p_* on the hypotenuse, joining v_0 and $v + w$, and hence lying in the convex set K, which would be closer to v than is v_0.

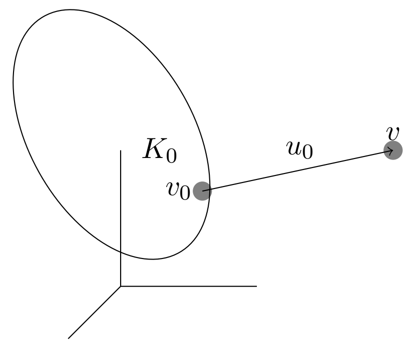


FIGURE 3. A compact convex set K_0 and a point v outside it.

Let F_0 be the subspace of H_0 spanned by the vectors v_0 and u_0; note that $v \in F_0$. Now choose an orthonormal basis $u_1, u_2, ...$ of the closed subspace $F_0^{\perp}$, and let

$$F_n = F_0 + \text{linear span of } u_1, ..., u_n. \tag{3.28}$$

This gives an increasing sequence of finite-dimensional subspaces whose union contains F_0 as well as all the vectors u_n, and hence is dense in H_0. Next observe that

$$v + F_n^{\perp} \subset v + F_0^{\perp} \subset v + u_0^{\perp}.$$

As noted before, $v + u_0^{\perp}$ is disjoint from K_0, and so $v + F_n^{\perp}$ is disjoint from K_0.

Since the hyperplane $v + u_0^{\perp}$ is precisely the set of points in H_0 whose inner-product with u_0 equals $\langle u_0, v\rangle$, it follows that no point in K_0 has inner-product with u_0 equal to $\langle u_0, v\rangle$. In particular, the orthogonal projection of K_0 on F_n cannot contain v, for if a point p in K_0 projected orthogonally onto F_n produced v then its inner-product $\langle u_0, p\rangle$ with u_0 would be the same as $\langle u_0, v\rangle$. This proves (3.27). □

4. The Support Theorem

We continue with the framework set up in the preceding sections, and turn now to our main result. Thus, H_0 is a separable, real Hilbert space, $\mathcal{H}$ a dense subspace equipped with a nuclear space structure, and $\mathcal{H}'$ its dual, having the strong and weak topologies for which the notions of sequential continuity coincide.

Theorem 4.1. *Suppose that $f : \mathcal{H}' \to \mathbb{R}$ is a bounded, sequentially continuous function on $\mathcal{H}'$. Let K_0 be a closed, bounded, convex subset of H_0. If the Gauss-Radon transform of f is 0 on hyperplanes which do not intersect K_0 then f is 0 on the complement of K_0 in H_0.*

Proof. We work with a point $v \in H_0$ outside K_0.

In Proposition 3.7 we constructed a sequence of finite-dimensional subspaces $F_n \subset H_0$ containing v, with $F_1 \subset F_2 \subset \cdots$ such that $\cup_{n\geq 1} F_n$ is dense in H_0 and $v + F_n^\perp$ is disjoint from K_0 for every positive integer n. Recall briefly how this was done. First we chose a point $u \in K_0$ closest to v, we set

$$u_0 = v - u,$$

then we chose $u_1, u_2, ...$ an orthonormal basis of $u_0^\perp$, set F_0 to be the linear span of v and u_0, and took F_n to be the linear span of $v, u_0, u_1, ..., u_n$. We showed that the hyperplane

$$v + u_0^\perp$$

is disjoint from K_0, and that v is outside the orthogonal projection $\mathrm{pr}_{F_n}(K_0)$ of K_0 onto F_n.

Let $f^*_{F_n}$ be the function on F given by

$$f^*_{F_n}(x) = \int f \, d\mu_{x+F_n^\perp}$$

We will show that this is 0 whenever x lies outside $\mathrm{pr}_{F_n}(K_0)$. In particular, it will follow that $f^*_{F_n}(v)$ is 0 for all $n \in \{1, 2, 3, ...\}$, and so, by the limiting result of Proposition 3.4,

$$f(v) = \lim_{n\to\infty} f^*_{F_n}(v) = 0. \tag{4.1}$$

We drop the subscript n in the following, and work with a finite dimensional space $F \subset H_0$ for which

$$v \in F, \text{ but } v \notin \mathrm{pr}_F(K_0). \tag{4.2}$$

Let P' be a hyperplane within the subspace F. Then

$$P' = P \cap F,$$

where P is the hyperplane in H_0 given by

$$P = P' + F^\perp,$$

which is perpendicular to F. If P' is disjoint from $\mathrm{pr}_F(K_0)$ then P is disjoint from K_0, because any point in $P \cap K_0$ projects by pr_F to a point which is both P' and $\mathrm{pr}_F(K_0)$. Recall now the disintegration formula (3.24) says:

$$Gf(P) = G_F(f^*_F)(P'). \tag{4.3}$$

By hypothesis, this is 0 if P is disjoint from K_0, i.e. if P' is disjoint from $\mathrm{pr}_F(K_0)$. Now observe that $\mathrm{pr}_F(K_0)$ is a convex, compact subset of F (compactness follows because K_0, being convex, closed and bounded is weakly compact and hence any finite-dimensional projection is compact). Hence, by the finite-dimensional support theorem, f^*_F is 0 outside $\mathrm{pr}_F(K_0)$. In particular, $f^*_F(v)$ is 0, which is what was needed to complete the proof. □

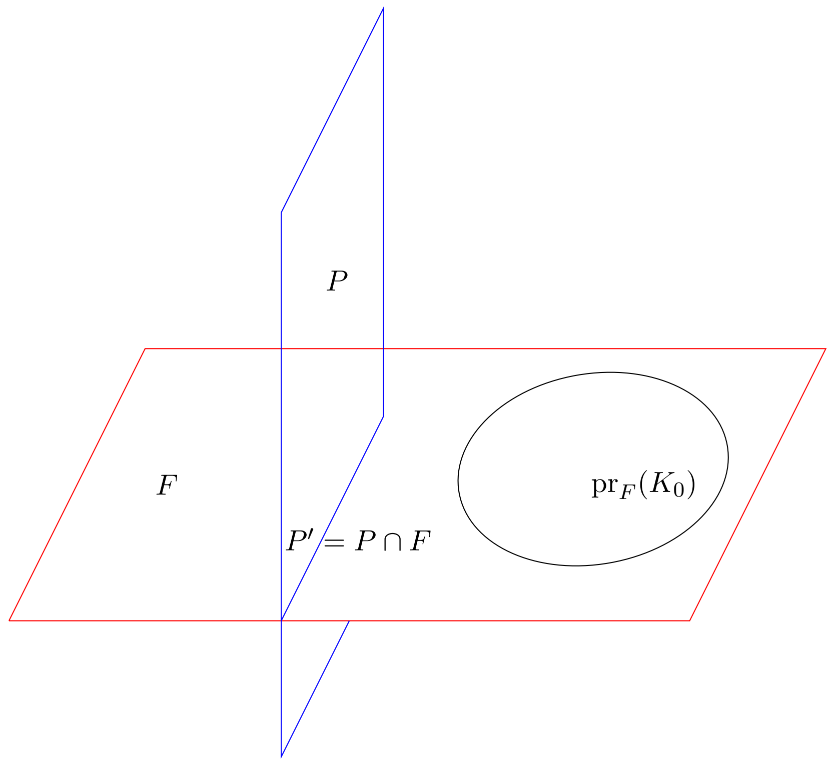


FIGURE 4. The projection of a compact set K_0 on F, disjoint from a hyperplane $P' = P \cap F$ in P

In the preceding result, the requirement that the convex set K_0 be closed and bounded is too stringent. The proof shows that it would suffice to assume, for instance, that there is an orthonormal basis $v_0, v_1, v_2, ..$ of H_0 such that the orthogonal projection of K_0 onto the linear span of $v_0, ..., v_n$ is compact for each $n \in \{0, 1, 2, ...\}$. Furthermore, the conclusion of the result can also be strengthened: clearly if f is zero off K_0 then it is also zero at all points in $\mathcal{H}'$ which are limits of sequences in $H_0 \cap K_0^c$.

APPENDIX

We will work out here a construction of Gaussian measure for affine subspaces of a Hilbert space. The existence of such measures can be obtained by direct appeal to the well-know theorem of Minlos but we provide a detailed construction. The following result, adapted from [6], explains how to construct Gaussian measures on subspaces of the infinite-dimensional space $\mathbb{R}^{\mathbb{P}}$, where

$$\mathbb{P} = \{1, 2, 3, ...\}.$$

Theorem A-1. *Let H_0 be a real, separable, infinite-dimensional Hilbert space, F a closed subspace of H_0, and v a vector in F. Let $\mathbb{R}^{\mathbb{P}}$ be the infinite product $\mathbb{R}^{\{1,2,\dots\}}$, equipped with the σ-algebra generated by the coordinate projection maps $\mathbb{R}^{\mathbb{P}} \to \mathbb{R} : (x_j)_{j\geq 1} \mapsto x_k$. Then there is a unique measure $\mu_{u+F^\perp}$ on $\mathbb{R}^{\mathbb{P}}$ and a linear map $H_0 \to L^2(\mathbb{R}^{\mathbb{P}}, \mu_{u+F^\perp}) : y \mapsto \hat{y}$ such that, for all $y \in H_0$:*

$$\int e^{i\hat{y}}\, d\mu_{u+F^\perp} = e^{i\langle u,y\rangle_0 - \frac{1}{2}\|y_\perp\|_0^2}, \tag{A-1}$$

where $y_\perp$ is the orthogonal projection of y onto $F^\perp$. In particular, taking F to be $\mathbb{R}u_0$, for some non-zero vector $u_0 \in H_0$, there is a unique measure $\mu_{u+u_0^\perp}$ on $\mathbb{R}^{\mathbb{P}}$ satisfying

$$\int e^{i\hat{y}}\, d\mu_{u+u_0^\perp} = e^{i\langle y,u\rangle_0 - \frac{1}{2}\|y_\perp\|_0^2}, \tag{A-2}$$

where u is any multiple of u_0, and $y_\perp$ is the projection of y orthogonal to u_0.

The special case $F = \{0\}$ gives the standard Gaussian measure μ on $\mathbb{R}^{\mathbb{P}}$.

By a *hyperplane* in the Hilbert space H_0 we shall mean a translate of a closed subspace of codimension one, i.e. a subset of the form $P = u + u_0^\perp$ with $u, u_0 \in H_0$ and $u_0 \neq 0$. The vector u is uniquely determined by P if we require that it be a multiple of u_0 and then it is the point on P closest to the origin.

We may take the *Gauss-Radon transform* Gf of a function f on $\mathbb{R}^{\mathbb{P}}$ to be the function which associates to each hyperplane P in H_0 the integral of f over P:

$$Gf(P) = \int f\, d\mu_P \tag{A-3}$$

Proof. Fix an orthonormal basis $e_1, e_2, \dots$ in H_0. For $y \in H_0$ of the form $y_1e_1 + \cdots y_me_m$, with m any positive integer and $(y_1, \dots, y_m) \in \mathbb{R}^m$, let $\hat{y}$ be the function on $\mathbb{R}^{\mathbb{P}}$ given by

$$\hat{y}(x) = \sum_{j=1}^{m} y_jx_j,$$

for all $x = (x_j)_{j\geq 1} \in \mathbb{R}^{\mathbb{P}}$.

For any integer $N \geq 1$ consider the Gaussian measure μ_N on $\mathbb{R}^N$ specified through its characteristic function $\hat{\mu}_N$ given by

$$\begin{aligned}\hat{\mu}_N(y_1, \dots, y_N) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^N} e^{i\sum_{r=1}^N y_rx_r}\, d\mu_N(x) \\ &= e^{i\left\langle u, \sum_{r=1}^N y_re_r\right\rangle - \frac{1}{2}\sum_{r,s=1}^N y_ry_s\langle (e_r)_\perp, (e_s)_\perp\rangle},\end{aligned} \tag{A-4}$$

for all $(y_1, \dots, y_N) \in \mathbb{R}^N$. Observe that

$$\hat{\mu}_{N+1}(y_1, \dots, y_N, 0) = \hat{\mu}_N(y_1, \dots, y_N) \tag{A-5}$$

From the consistency property (A-5) of the measures μ_N we conclude by the Kolmogorov extension theorem that there is a measure $\mu_{u+F^\perp}$ on $\mathbb{R}^{\mathbb{P}}$ for which

$$\int_{\mathbb{R}^{\mathbb{P}}} e^{i\hat{y}} d\mu_{u+F^\perp} = \hat{\mu}_N(y_1, ..., y_N) \tag{A-6}$$

holds for every $y \in H_0$ of the form $y = y_1 e_1 + \cdots + y_N e_N$. From the expression for $\hat{\mu}_N$ in (A-4) we then have the desired formula

$$\int_{\mathbb{R}^{\mathbb{P}}} e^{i\hat{y}}\, d\mu_{u+F^\perp} = e^{i\langle y,u\rangle_0 - \frac{1}{2}\|y_\perp\|_0^2}, \tag{A-7}$$

for all y in the linear span H_{00} of the vectors $e_1, e_2, ...$ Thus, with respect to the probability measure $\mu_{u+F^\perp}$, the random variable $\hat{y}$, for $y \in H_{00}$, is Gaussian with mean $\langle v, y\rangle$ and variance $\|y_\perp\|_0^2$, and so

$$\|\hat{y}\|^2_{L^2(\mu_{u+F^\perp})} = |\langle u, y\rangle_0|^2 + \|y_\perp\|_0^2 \le (\|u\|_0^2 + 1)\|y\|_0^2. \tag{A-8}$$

Thus, $y \mapsto \hat{y}$ is continuous as a map $H_{00} \to L^2(\mu_{u+F^\perp})$, and so extends to a continuous linear map $H_0 \to L^2(\mu_{u+F^\perp})$.

Finally, since both sides in (A-7) are continuous in $y \in H_0$, the relation holds for all $y \in H_0$. □

Fix any sequence of positive real numbers λ_n satisfying

$$1 \le \lambda_1 < \lambda_2 < \cdots, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty. \tag{A-9}$$

For any $w \in \mathbb{R}^{\mathbb{P}}$, and any $p \in \mathbb{Z}$, define

$$\|w\|_p = \left\{\sum_{n=1}^{\infty} \lambda_n^{2p} w_n^2\right\}^{1/2}. \tag{A-10}$$

Then, with the setting as in the preceding Theorem,

$$\int \sum_{n=1}^{\infty} \lambda_n^{-2} \hat{e}_n \, d\mu_{u+F^\perp} \le \sum_{n=1}^{\infty} \lambda_n^{-2}(\|u\|_0^2 + 1) < \infty.$$

Hence, the integrand on the left is finite almost everywhere, i,e,

$$\mu_{u+F^\perp}[w \in \mathbb{R}^{\mathbb{P}} \,:\, \|w\|_{-1} < \infty] = 1. \tag{A-11}$$

Let H_p be the subspace of H_0 consisting of all $x \in H_0$ for which $\|x\|_p$ is finite, where

$$\|x\|_p \stackrel{\text{def}}{=} \left\{\sum_{n=1}^{\infty} \lambda_n^{2p} \langle x, e_n\rangle_0^2\right\}^{1/2}. \tag{A-12}$$

Then H_p is a Hilbert space with the inner-product given by

$$\langle x, y\rangle_p = \sum_{n=1}^{\infty} \lambda_n^{2p} \langle x, e_n\rangle_0 \langle e_n, y\rangle_0.$$

The inclusion map $H_p \to H_{p-1}$ is readily checked to be Hilbert-Schmidt, with Hilbert-Schmidt norm square given by $\sum_{n=1}^{\infty} \lambda_n^{-2}$, for every $p \in \mathbb{P}$. Let

$$\mathcal{H} = \bigcap_{p \in \{0,1,2,\ldots\}} H_p \tag{A-13}$$

Then $\mathcal{H}$ is dense in each H_p. Equip $\mathcal{H}$ with the topology consisting of all unions of translates of all open balls of the form $\{x \in \mathcal{H} : \|x\|_p < \epsilon\}$, with p running over $\{0, 1, 2, \ldots\}$ and ϵ over $(0, \infty)$. Then $\mathcal{H}$ is a locally convex topological vector space.

The dual $\mathcal{H}'$ is the set of all continuous linear functionals on $\mathcal{H}$:

$$x' : \mathcal{H} \to \mathbb{R} : x \mapsto \langle x', x \rangle = \hat{x}(x').$$

It is the union of the increasing family of spaces $\mathcal{H}'_p$, of linear functionals on $\mathcal{H}$ which are continuous with respect to the norm $\|\cdot\|_p$. We identify $\mathcal{H}'$ with a subset of $\mathbb{R}^{\mathbb{P}}$ by the injective map:

$$\mathcal{H}' \to \mathbb{R}^{\mathbb{P}} : x' \mapsto \big(\hat{e}_1(x'), \hat{e}_2(x'), \ldots\big)$$

The image of $\mathcal{H}'_p$ under this mapping is precisely the set of all $w \in \mathbb{R}^{\mathbb{P}}$ for which $\|w\|_{-p}$ is finite, the image of $\mathcal{H}'$ is the union of these images and hence is a measurable subset of $\mathbb{R}^{\mathbb{P}}$. We transfer the measure $\mu_{u+F^\perp}$ over to $\mathcal{H}'$, to the σ-algebra of subsets of $\mathcal{H}'$ generated by the maps $\hat{e}_j$; from (A-11) we see that $\mu_{u+F^\perp}$ is a probability measure.

This completes the discussion for Proposition 2.1.

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