# The Large-$N$ Yang-Mills Field on the Plane and Free Noise

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**Abstract.** The large-$N$ limit of the quantum Yang-Mills field over the plane $\mathbb{R}^2$, for the gauge group $U(N)$, is shown to lead to a white noise field in the sense of free probability.

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## INTRODUCTION

In this paper we show that the large-$N$ limit of $U(N)$ quantum gauge theory in two dimensions leads to a 'free' white noise, a concept which we will explain. This is, we hope, a first step towards a more complete stochastic geometry for the two-dimensional Yang-Mills field, where stochasticity is in the sense of free probability.

The study of the large-$N$ limit of $U(N)$ gauge theories grew rapidly in the physics literature since the seminal work of 't Hooft [13]. In the bibliography we cite just a small sample of works by Bralic [2], Douglas [3], D. Gross [6], Kazakov and Kostov [7, 8]; see [11] for a larger bibliography. On the mathematical side, Singer's paper [12] lays out several mathematical challenges in the area. More recent works by Biane [1], Lévy [9] and Xu [17] have greatly clarified the large-$N$ limit of Wilson loop expectation values (see also [11] on this and related questions).

## QUANTUM YANG-MILLS ON THE PLANE

In this section we will present a largely self-contained account of quantum Yang-Mills theory on the plane. The material is adapted and condensed from Driver [4] and Gross et al. [5], to which refer for details and further references.

### $U(N)$ connections on the plane

Gauge fields are described mathematically by connections on principal bundles. We will work entirely in the setting where spacetime is modeled by the Euclidean plane $\mathbb{R}^2$, and the gauge symmetry group is $U(N)$, the group of $N \times N$ complex unitary matrices. In this setting, we can view a gauge field as a 1-form on $\mathbb{R}^2$ with values in the Lie algebra

$$u(N) = i\mathscr{H}_N,$$

where $\mathscr{H}_N$ is the space of all $N \times N$ Hermitian matrices. We will use the inner-product on $\mathscr{H}_N$ given by

$$\langle X, Y \rangle = \mathrm{Tr}(XY).$$

This specifies, naturally, an inner-product on $u(N)$.

Let

$$\mathscr{A}$$

denote the space of all smooth 1-forms on $\mathbb{R}^2$ with values in $u(N)$. Thus, a typical element of $\mathscr{A}$ is a 1-form

$$A = A_x dx + A_y dy$$

where $A_x$ and $A_y$ are smooth functions

$$\mathbb{R}^2 \to u(N).$$

The *curvature* of $A$ is the $u(N)$-valued 2-form given by

$$F^A = dA + A \wedge A = -f^A dx \wedge dy,$$

where the wedge product on the right uses matrix multiplication, $d\sigma$ is the area 2-form on $\mathbb{R}^2$, and

$$f^A : \mathbb{R}^2 \to u(N)$$

may be viewed as the scalar curvature function.

## The Yang-Mills action and gauge transformations

The dynamics of the gauge field theory is governed by the Yang-Mills action functional

$$S_{\mathrm{YM}}(A) = \frac{1}{2g^2} \int_{\mathbb{R}^2} \|f^A\|^2 \, dxdy, \tag{1}$$

where $g$ is to be viewed as a physical 'coupling' constant. Here the integrand uses the inner-product on $u(N)$, and we are using the ordinary Euclidean metric on $\mathbb{R}^2$ for simplicity.

The Yang-Mills action is invariant with respect to the action of the group of gauge transformations on the space of connections. A gauge transformation is, in this framework, a smooth function

$$\phi : \mathbb{R}^2 \to U(N).$$

The set of all such functions is a group $\mathscr{G}$ under pointwise multiplication, and this group acts on the vector space $\mathscr{A}$:

$$\mathscr{A} \times \mathscr{G} \to \mathscr{A} : (A, \phi) \mapsto A^{\phi} = \phi^{-1} A \phi + \phi^{-1} d\phi. \tag{2}$$

The subgroup

$$\mathscr{G}_0$$

consisting of all $\phi$ which have the value $I$ (identity matrix) at the origin $o \in \mathbb{R}^2$ is often more convenient to work with. Since $S_{\text{YM}}$ is invariant under the action of $\mathscr{G}$, it may be viewed as a function on the quotients $\mathscr{A}/\mathscr{G}$ and $\mathscr{A}/\mathscr{G}_o$.

By choosing an appropriate gauge transformation in $\mathscr{G}_o$, any given connection form can be brought to the special form

$$A = A_x dx + 0dy, \tag{3}$$

with $A_x$ vanishing on the $x$-axis. For this reason, we will identify $\mathscr{A}/\mathscr{G}_o$ with the space $\mathscr{A}_x$ of smooth functions $A_x$ on $\mathbb{R}^2$, vanishing on the $x$-axis.

For a connection $A$ whose $y$-component is 0, we have

$$f^A = \frac{\partial A_x}{\partial y}$$

and this leads to an injection

$$\mathscr{A}/\mathscr{G}_o \simeq \mathscr{A}_x \to L^2\left(\mathbb{R}^2\right) \otimes u(N) : A_x \mapsto \frac{\partial A_x}{\partial y} \tag{4}$$

Thus $A_x$ may be recovered as

$$A_x(x,y) = \int_0^y f^A(x,s)\, ds. \tag{5}$$

## The Yang-Mills measure

The quantum theoretic functional integral associated to the Yang-Mills gauge field leads to a 'measure' on $\mathscr{A}$ given formally as

$$e^{-S_{\text{YM}}(A)} DA$$

where $DA$ is the formal 'Lebesgue measure' on $\mathscr{A}$. Passing to the quotient $\mathscr{A}_x$, and changing 'variables' $A \mapsto f^A$ (a linear map), yields a Gaussian measure:

$$d\mu^g_{\text{YM}}(F) = \frac{1}{Z_g} e^{-\|F\|^2_{L^2}/(2g^2)}\, dF, \tag{6}$$

with $F$ running over a linear space of maps $\mathbf{R}^2 \to u(N)$. Technically, this measure actually lives on a Hilbert-Schmidt completion of $L^2(\mathbf{R}^2; u(N))$, and its support does not contain only continuous functions.

## Parallel transport and holonomy

Consider a piecewise smooth path

$$c : [a,b] \to \mathbb{R}^2$$

and a connection form $A$, i.e. a smooth 1-form on $\mathbb{R}^2$ with values in $u(N)$. Associated to this is a path

$$h : [a,b] \to U(N)$$

specified through the differential equation

$$h'(t)h(t)^{-1} = -A\big(c'(t)\big) \qquad \text{with initial condition } h(a) = I. \tag{7}$$

The path $t \mapsto h(t) \in U(N)$ describes *parallel transport* along the path $c$ by the connection $A$.

If $c$ is a loop, then the full transport $h(b)$ is the *holonomy* of $A$ around $c$:

$$h(c;A) \stackrel{\text{def}}{=} h(b). \tag{8}$$

Under gauge transformations the holonomy is conjugated. In particular, if $c$ is a loop based at the origin $o$, and if $\phi \in \mathscr{G}_o$, then

$$h(c;A^{\phi}) = h(c;A)$$

for all $A \in \mathscr{A}$. We will be concerned with holonomies only of such loops. Then, in view of the gauge fixing (3), we may as well focus on connection forms $A$ of the form

$$A = A_x dx,$$

with $A_x$ zero on the $x$-axis. Consider then a smooth path parametrized as

$$[0,b] \to \mathbb{R}^2 : t \mapsto \big(t, y(t)\big)$$

The equation of parallel transport is then

$$h'(t)h(t)^{-1} = -A_x = -\int_0^{y(t)} f^A(t,y)\,dy \tag{9}$$

We can write this as

$$dh(t) = -dM^A(t)h(t), \tag{10}$$

where

$$M^A(t) = \int_0^t \int_0^{y(s)} f^A(s,y)\,dy\,ds, \tag{11}$$

where we used (5).

Note that $h(t)$ is the holonomy of the loop which travels up from $o$ along the $y$-axis to $(0, y(0))$, then proceeds along the path $s \mapsto \big(s, y(s)\big)$ up to time $t$, then travels down parallel to the $y$-axis till hits $y = 0$, and then returns to the origin along the $x$-axis.

## Stochastic holonomy

Now passing to the quantum theory, we have to consider connections which are in the support of the measure $\mu^g_{\mathrm{YM}}$. These connections are not continuous and so the differential equation (7) is difficult to work with, to say the least. However, as first noted by L. Gross, equation (10) is still sensible when viewed as a *Stratonovich stochastic differential equation*, and is the correct replacement in the stochastic case.

Under the Gaussian measure $\mu^g_{\mathrm{YM}}$, the integral (11) acquires meaning as a $u(N)$-valued Gaussian random variable. More generally, we have a probability space

$$(\Omega, \mathscr{F}, \mu^g_{\mathrm{YM}})$$

and for each $f \in L^2_{\mathrm{real}}(\mathbb{R}^2)$ we have a $u(N)$-valued random variable

$$M(f) : \Omega \to u(N) : \omega \mapsto M(f)(\omega) = \sum_{j=1}^{d} M_j(f)(\omega) iT_j,$$

where $iT_1, ..., iT_d$ is an orthonormal basis of $u(N)$, and each $M_j(f)$ is a mean-0 Gaussian variable satisfying the correlation condition

$$\int M_j(f) M_k(h)\, d\mu^g_{\mathrm{YM}} = g^2 \langle f, h \rangle_{L^2(\mathbb{R}^2)} \tag{12}$$

One should think of $M(f)$ informally as an integral $\int_{\mathbb{R}^2} f(x,y) iF(x,y)\, dxdy$, where $iF$ is the $u(N)$-valued Gaussian field. In particular, if

$$f = 1_S,$$

for some bounded Borel subset $S$ of $\mathbb{R}^2$, then $M(f)$ has independent components $M_j(1_S)$, each Gaussian with mean 0 and variance equal to $g^2$ times the area of $S$.

In particular, we take the random variable $M(t)$, analogous to $M^A(t)$, to be given by evaluating $M$ on the indicator function of the region between the path $[0,t] \to \mathbb{R}^2 : s \mapsto (s, y(s))$, and the $x$-axis. This is Gaussian, mean 0, with each component having variance equal to $g^2$ times the area of the region. Rescaling time by this area (times $g^2$), the stochastic parallel-transport equation is the same as that for Brownian motion on $U(N)$.

We will henceforth work only with 'well-behaved' paths, i.e. piecewise smooth paths in $\mathbb{R}^2$ which are composites of pieces which can be parametrized by the $x$-coordinate and those which rise or fall parallel to the $y$-axis.

Applying these considerations to a simple loop $C$ based at the origin, the stochastic holonomy

$$h(C)$$

is a $U(N)$-valued random variable with distribution

$$Q_{g^2 S}(x) dx,$$

where $dx$ is unit-mass Haar measure on $U(N)$, $S$ is the area enclosed by $C$, and $Q_t(x)$ is the *heat kernel* on $U(N)$.

The heat kernel $Q_t(x)$ solves

$$\frac{\partial Q_t(x)}{\partial t} = \frac{1}{2} \Delta Q_t(x),$$

with $Q_0(x) = \delta_I(x)$, the delta function at the identity $I$ on $U(N)$, and $\Delta$ is the Laplacian on $U(N)$ for the chosen inner-product on $u(N)$. The heat kernel is invariant under inverses:

$$Q_t(x^{-1}) = Q_t(x)$$

and satisfies the convolution formula

$$Q_t * Q_s = Q_{t+s}$$

For the standard Brownian motion $t \mapsto B_t$ on $U(N)$, starting at the identity at $t = 0$, the probability density function of $B_t$ with respect to unit-mass Haar measure on $U(N)$ is $Q_t$.

The following result (Gross et al. [5] and Driver [4]) summarizes the main facts about stochastic holonomy under the Yang-Mills measure:

**Theorem 0.1** *If $C$ is a simple closed loop in $\mathbf{R}^2$ enclosing an area $S$ then the holonomy $h(C)$ is a $U(N)$-valued random variable whose distribution is given by*

$$Q_{g^2S}(x)dx,$$

*where $dx$ is unit-mass Haar measure on $U(N)$. Moreover, if $C_1, \ldots, C_N$ are loops in the plane which enclose non-overlapping regions then $h(C_1), \ldots, h(C_N)$ are mutually independent random variables.*

Thus, in particular,

$$\int (\mathrm{tr} h(C))^k \, d\mu^g_{\mathrm{YM}} = \int_{U(N)} (\mathrm{tr}(x))^k \, Q_{g^2S}(x)\, dx, \tag{13}$$

The large-$N$ limit of (13) has been studied by Biane [1], Xu [17], the author [11], and, more extensively, by Lévy [9].

## LIMITING FORM OF THE WHITE NOISE

As we have seen before, the curvature of the gauge field connection form is described, in a suitable gauge, as a $u(N)$-valued Gaussian white noise process on the plane. In this section we will determine the limiting form of this process, as $N \to \infty$ and $g^2N$ is kept constant

$$\tilde{g}^2 = Ng^2.$$

### Notions from Algebraic Probability

Here we shall summarize some notions from algebraic probability theory. For an extensive account we refer to the excellent monograph by Nica and Speicher [10] (note, however, that our terminology is slightly different).

Consider a complex algebra $\mathscr{A}$, with unit element 1, and an *involution*

$$\mathscr{A} \to \mathscr{A} : a \mapsto a^*$$

satisfying

$$(ab)^* = b^* a^*$$

and

$$a^{**} = a$$

for all $a, b \in \mathscr{A}$. Two examples to keep in mind are :

(E1) $\mathscr{A}$ is the set of all complex-valued random variables, on some probability space, with finite moments of all orders, under pointwise operations;

(E2) $\mathscr{A}$ is the algebra of complex $N \times N$ matrices with the involution being given by the adjoint.

An algebraic probability 'measure' on $\mathscr{A}$ is a linear map

$$\phi : \mathscr{A} \to \mathbb{C}$$

satisfying

$$\phi(1) = 1$$

and

$$\phi(aa^*) \geq 0 \quad \text{for all } a \in \mathscr{A}.$$

We will call $\mathscr{A}$, equipped with $\phi$, an *algebraic probability space*.

In example (E1) above, we can take $\phi$ to be given by the expected value, and in (E2) by the trace normalized $N^{-1}\mathrm{Tr}$.

If $a \in \mathscr{A}$ is a normal element, i.e. it commutes with $a^*$, and there is a unique Borel measure $\mu_a$ on $\mathbb{C}$ such that

$$\phi\big(P(a, a^*)\big) = \int_{\mathbb{C}} P(z, \bar{z})\, d\mu_a(z)$$

holds for all polynomials $P(z, w)$ in two variables, them $\mu_a$ is called the *distribution* of $a$.

Consider subalgebras $\mathscr{B}_1, ..., \mathscr{B}_n$ of $\mathscr{A}$, each closed under the involution and containing the unit element 1. These subalgebras are said to be mutually *free* if for any $b_1, ..., b_m \in \mathscr{A}$, for which each $\phi(b_i)$ is 0, we have

$$\phi(b_1 ... b_m) = 0$$

whenever consecutive $b_i$ belong to different $\mathscr{B}_j$ (more precisely, $b_1 \in \mathscr{B}_{j(1)}, ..., b_m \in \mathscr{B}_{j(m)}$ with $j(1) \neq j(2) \neq ... \neq j(m)$).

In particular, we say that elements $a_1, ..., a_m \in \mathscr{A}$ are free if the unital $*$-closed algebras generated by these elements are mutually free.

In example (E1), with $\mathscr{A}$ the algebra of bounded random variables on a probability space and $\phi$ the expectation value, there is also the traditional probabilistic notion of *independence*. Consider $*$-closed, unital subalgebras $\mathscr{B}_1, ..., \mathscr{B}_n$, and assume, furthermore, that every bounded Borel function of each element of $\mathscr{B}_i$ is in $\mathscr{B}_i$, for all $i \in \{1,...,n\}$. Then $\mathscr{B}_1,...,\mathscr{B}_n$ are *independent* if for any $b_1,...,b_m \in \mathscr{A}$, for which each $\phi(b_i)$ is 0, we have

$$\phi(b_1...b_m) = 0$$

whenever the $b_i$ all belong to different $\mathscr{B}_j$, i.e. $b_1 \in \mathscr{B}_{j(1)},...,b_m \in \mathscr{B}_{j(m)}$, with $j(1),...,j(m)$ distinct in $\{1,...,n\}$. To compare this with the standard definition of independence, take $b_j$ to be $1_{B_j} - \mathbb{E}(1_{B_j})$ for measurable sets $B_j$.

Finally, we come to the important notion of convergence. Suppose $\mathscr{A},\mathscr{A}_1,\mathscr{A}_2,...$ is a sequence of algebraic probability spaces. Consider normal elements $a_j \in \mathscr{A}_j$, for all $j \geq 1$, and a normal element $a \in \mathscr{A}$, and suppose

$$\lim_{N\to\infty} \phi\big(P(a_N,a_N^*)\big) = \phi\big(P(a,a^*)\big)$$

for all polynomials $P$ in two variables (note that $\phi$ on the left depends on $N$, and could more properly be denoted $\phi_N$, but we will use the unadorned $\phi$ as much as possible to denote the trace functional in any context). Then we say that the sequence $(a_N)_{N\geq 1}$ *converges in distribution* to $a$:

$$a_N \xrightarrow{\mathrm{d}} a. \tag{14}$$

More generally, suppose, for each $N$, we have normal elements $a_{N,1},...,a_{N,m} \in \mathscr{A}_N$, and we also have $a_1,...,a_m \in \mathscr{A}$. We say that $(a_{N,1},...,a_{N,m})$ converges in distribution to $(a_1,...,a_m)$, as $N \to \infty$, if

$$\lim_{N\to\infty} \phi\,\big(P\big(a_{N,1},a_{N,1}^*,...,a_{N,m},a_{N,m}^*\big)\big) = P\big(a_1,a_1^*,...,a_m,a_m^*\big)$$

for all polynomials $P$ in $2m$ non-commuting variables.

## Free Limit of the White Noise Process

For $N \in \{1,2,...\}$, let $\mathscr{A}_N$ be the algebra of all complex $N\times N$ random matrices $a$ (on some probability space) such that each entry of $a$ has finite moments of all orders; for $a \in \mathscr{A}_N$ let

$$\phi(a) = \mathbb{E}\big(\mathrm{Tr}_N(a)\big) \tag{15}$$

where

$$\mathrm{Tr}_N = \frac{1}{N}\mathrm{Tr}$$

Then $(\mathscr{A}_N,\phi)$ is an algebraic probability space. It is a combination of the two basic examples (E1) and (E2) described earlier.

Consider a real separable Hilbert space $H$. In the application we have, $H$ is $L^2_{\mathrm{real}}(\mathbb{R}^2)$. Suppose now that we have a probability space and for each $N \in \{1,2,...\}$, and $f \in H$, a random Hermitian $N\times N$ matrix

$$F(f) \in \mathscr{A}_N,$$

satisfying the following conditions:

(i) $F(f)$ is a random Hermitian matrix;

(ii) $F(f)$ depends linearly on $f$;

(iii) for $f \neq 0$, the random variable $F(f)$ on $\mathscr{H}_N$ has density proportional to

$$e^{-\mathrm{Tr}(T^2)/(2g^2\|f\|^2)} = e^{-N\frac{\mathrm{Tr}(T^2)}{2\tilde{g}^2\|f\|^2}} \tag{16}$$

with $T$ running over $\mathscr{H}_N$, the space of $N \times N$ Hermitian matrices.

Note that $F(f)$ is an $N \times N$ matrix, and occasionally, to stress the role of $N$, we may denote it as $F_N(f)$. We also state separately that

> If $f$ and $h$ are orthogonal in $H$ then the entries of the matrix $F_N(f)$ are independent, as random variables, of the entries of $F_N(h)$.

For a Hermitian matrix $T$, we have

$$T_{ab} = S_{ab} + iA_{ab},$$

where $S$ is a real symmetric matrix, and $A$ a real skew-symmetric matrix. Then

$$\mathrm{Tr}(T^2) = \sum_{a=1}^{N} T_{aa}^2 + 2 \sum_{1 \leq a < b \leq N} (S_{ab}^2 + A_{ab}^2)$$

The density factor (16) is then

$$\prod_{a=1}^{N} e^{-T_{aa}^2/(2g^2\|f\|^2)} \prod_{1 \leq a < b \leq N} e^{-S_{ab}^2/(g^2\|f\|^2)} e^{-A_{ab}^2/(g^2\|f\|^2)}$$

which means that the random variables corresponding to the matrix entries $T_{aa}$, $S_{ab}$, $A_{ab}$, for $a < b$, are independent mean-0 Gaussians, and

$$\begin{aligned} \mathbb{E}(T_{aa}^2) &= \frac{\tilde{g}^2\|f\|^2}{N}, \\ \mathbb{E}(S_{ab}^2) &= \frac{\tilde{g}^2\|f\|^2}{2N} \\ \mathbb{E}(A_{ab}^2) &= \frac{\tilde{g}^2\|f\|^2}{2N} \end{aligned} \tag{17}$$

for all $1 \leq a < b \leq N$. Here, and always, $\mathbb{E}$ denotes the expectation value.

A celebrated result of Wigner [16, Equation (4)] (with sketch proof in the earlier paper Wigner [15, pp. 552-557]) says that if $T$ is a random Hermitian $N \times N$ matrix with density proportional to

$$e^{-N\mathrm{Tr}(T^2)/2} \tag{18}$$

then, for any $p \in \{0, 1, 2, \ldots\}$,

$$\lim_{N \to \infty} \mathbb{E}\left[\mathrm{Tr}_N(T^{2p})\right] = \frac{1}{p+1}\binom{2p}{p} \tag{19}$$

and, note again that $T$ is an $N \times N$ matrix. The odd moments are zero by symmetry of the Gaussian distribution. The moments on the right in (19) are those of the standard *semi-circular distribution*

$$\frac{1}{2\pi}\sqrt{4-x^2}\,dx \qquad x \in [-2,2]. \tag{20}$$

The semicircular distribution of radius $r > 0$ has density

$$\frac{2}{\pi r^2}\sqrt{r^2-x^2}\,dx \qquad x \in [-r,r]. \tag{21}$$

From the Wigner semi-circular law (19) we conclude that

$$\lim_{N\to\infty} \phi\left(F_N(f)^{2p}\right) = \tilde{g}^{2p}\|f\|^{2p}\frac{1}{p+1}\binom{2p}{p} \tag{22}$$

Now let $f_1,\dots,f_m \in H$ be orthogonal vectors. Then the random matrices

$$F_N(f_1),\dots,F_N(f_m)$$

are mutually independent in the sense that each entry of the matrix $F_N(f_j)$ is independent of each entry of $F_N(f_k)$ for $j \neq k$.

We will now use a powerful extension, due to Voiculescu [14, Theorem 2.2], of the Wigner semi-circle law: if $A_{N,1},\dots,A_{N,m}$ are independent Hermitian $N \times N$ random matrices, each with Gaussian distribution given through (18), then $(A_{N,1},\dots,A_{N,m})$ converges in distribution to $(a_1,\dots,a_m)$ where $a_1,\dots,a_m$ are mutually free elements in some algebraic probability space with each $a_j$ having the standard semicircular distribution (20).

We conclude then that

$$(F_N(f_1),\dots,F_N(f_m)) \xrightarrow{\text{d}} (f'_1,\dots,f'_m) \tag{23}$$

where $f'_1,\dots,f'_m$ are mutually free elements in some algebraic probability space and each $f'_j$ is semicircular with radius $2\tilde{g}\|f_j\|$ (if $f_j$ is 0 then $f'_j$ is 0).

Clearly, we would like one algebraic probability space to which all the elements of the form $f'$ belong. Fortunately, this is possible, by means of the *full Fock space*:

$$\mathscr{F}(H) = \bigoplus_{n\geq 0} H_c^{\otimes n}, \tag{24}$$

where $H_c$ is the complexification of $H$, and the 0-th tensor power is simply $\mathbb{C}$. The element 1 in $\mathbb{C}$, viewed as an element of $\mathscr{F}(H)$ is called the *vacuum vector* and we will denote it as $\mathbf{1}$. The $C^*$-algebra

$$B\big(\mathscr{F}(H)\big)$$

of all bounded operators on the Hilbert space $\mathscr{F}(H)$, is an algebraic probability space when equipped with the trace functional

$$\phi(A) = \langle A\mathbf{1},\mathbf{1}\rangle \quad \text{for } A \in B\big(\mathscr{F}(H)\big). \tag{25}$$

Now for any $f \in H$ let $c(f)$ be the *creation operator* defined on $\mathscr{F}(H)$ by

$$c(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n$$

for all $n \in \{1, 2, ...\}$ and all $f_1, ..., f_n \in H$, and

$$c(f)\mathbf{1} = f.$$

The *annihilation operator* $a(f)$ is the adjoint

$$a(f) = c(f)^*,$$

and acts by

$$a(f)(f_1 \otimes \cdots \otimes f_n) = \langle f_1, f \rangle f_2 \otimes \cdots \otimes f_n.$$

Then (see [10, Corollary 7.17]) the self-adjoint element

$$b(f) = a(f) + c(f)$$

is semicircular with radius $2\|f\|$; moreover, if $f_1, ..., f_m$ are orthogonal vectors in $H$ then $b(f_1), ..., b(f_m)$ are mutually free.

The mapping

$$H \to B\big(\mathscr{F}(H)\big) : f \mapsto b(f)$$

may be thought of as a 'free noise' process, analogous the white noise process of standard probability theory.

Combining all our observations, we see that the algebraic probability space $B\big(\mathscr{F}(H)\big)$, with the trace functional $\phi$ from (25), is the appropriate 'limit' of the spaces $(\mathscr{A}_N, \phi)$. To state this more precisely, let

$$F_\infty(f) = \tilde{g}b(f) \qquad \text{for } f \in H. \tag{26}$$

Then, for any orthogonal $f_1, ..., f_m \in H$, we have

$$\big(F_N(f_1), ..., F_N(f_m)\big) \xrightarrow{\mathrm{d}} \big(F_\infty(f_1), ..., F_\infty(f_m)\big) \tag{27}$$

as $N \to \infty$, keeping $\tilde{g}$ fixed.

We can remove the orthogonality assumption on the vectors $f_1, ..., f_m$. For non-zero vectors $f_1, ..., f_m$, we can always express them as real linear combinations of a suitable set of orthonormal vectors. Since both $F_N$ and $b(\cdot)$ are real-linear, (27) continues to hold.

## CONCLUDING REMARKS

In this paper we have shown how the $U(N)$ quantum Yang-Mills measure for the plane $\mathbb{R}^2$ yields a free noise process as $N \to \infty$. It would be of interest to relate this limit to the approach described by Singer [12]. It should also be possible to connect the free-noise process to the large-$N$ limit of Wilson loop expectations described in the physics literature as well as the mathematical works of Biane [1], Lévy [9], and Xu [17]. Results concerning the large-$N$ limit of Wilson loop expectations are reviewed and developed in [11]. There are numerous outstanding mathematical challenges concerning the $N = \infty$ theory, as well as the limiting behavior as $N \to \infty$, that may be taken up.

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