

Traces in Two-Dimensional QCD: The Large- N Limit

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ABSTRACT. An overview of mathematical aspects of $U(N)$ pure gauge theory in two dimensions is presented, with focus on the large- N limit of the theory. Examples are worked out expressing Wilson loop expectation values in terms of areas enclosed by loops.

1. Introduction

In this paper we present a short overview of mathematical aspects of the behavior of $U(N)$ pure Yang-Mills gauge theory in two dimensions for large N . We will present a selection of results and their proofs in detail to illustrate the techniques involved.

Interest in the behavior of $U(N)$ Yang-Mills theory for large N exploded after 't Hooft's 1974 paper [48] which showed that only planar Feynman diagrams dominate when the relevant integrals in this theory are expanded in powers of $1/N$. The literature that emerged in this area is too large to list, and our bibliography is only a small selection relevant to this presentation. 't Hooft's work remains a source of new ideas and activity (see, for example, Aharony et al. [1]). For many other aspects of two-dimensional Yang-Mills theory we mention Cordes et al. [9], the papers of Witten [53, 54], and the volumes [11].

We shall begin with a brief introduction to the relevant concepts from Yang-Mills gauge theory. Then we shall describe the quantum field measure for pure Yang-Mills theory on the plane \mathbf{R}^2 , and work out several examples of loop expectation values, illustrating the techniques. We will then describe examples of the large- N limit of Wilson loop expectations in specific cases. In the last section we sketch Singer's [47] proposal for a master field.

2. Yang-Mills Gauge Theory

The classical electromagnetic field is described by a 2-form F on space-time such that the force exerted by the field on an electric charge e moving at velocity v is given by $ei_v F$, where $i_v F$ is the contraction of F on the vector v . Forming the quantum theory for this electric charge moving in the field F leads to Dirac's "magnetic monopole quantization formula" (Dirac [12]), which is an integrality condition on the 2-form F : the integral of $\frac{e}{h}F$ over any closed surface σ should

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be an integer, i.e. $\frac{e}{\hbar} \int_{\sigma} F \in \mathbf{Z}$. Now this is precisely the condition for $-i\frac{e}{\hbar}F$ to be the curvature of a connection form on a principal $U(1)$ -bundle over spacetime ($\hbar = h/(2\pi)$, where h is Planck's constant). Such a connection may be described locally by a 1-form over spacetime; let this 1-form be $i\frac{e}{\hbar}A$. Then

$$F = -dA,$$

and so A is the electromagnetic potential of classical electromagnetic theory. Thus, building a quantum theory for an electric charge coupled to an electromagnetic field leads naturally to a connection $\omega = i\frac{e}{\hbar}A$ on a principal $U(1)$ -bundle, with curvature proportional to the field strength F (for more details on this see [40]). The energy density of the electromagnetic field is known, and taking this as the Hamiltonian of the dynamical system describing the electromagnetic field, the action of the Maxwell field works out to be $-\frac{\epsilon_0 c^2}{2} \int \langle F, F \rangle d\text{vol}$, the integral being over all spacetime. The full Lagrangian of the field together with the wave function ψ of the charged particle involves the term $\frac{e}{\hbar}A\psi$, coupling the electromagnetic potential A to the field ψ , and so e/\hbar is viewed as a 'coupling constant' (to be sure, this term will be defined below, and is slightly different from e/\hbar).

The interaction between the fundamental constituents of matter is governed by the Yang-Mills gauge field. This is analogous to the Maxwell electromagnetic field, but instead of an ordinary 1-form as potential, the potential is a 1-form with values in the Lie algebra of a non-abelian group G such as $SU(3)$. The relationship between physical gauge theories and connections on principal bundles was established by Wu and Yang [55]. They showed that, mathematically, a gauge theory is described by a connection on a principal G -bundle over the spacetime manifold. The curvature of the connection corresponds to the strength of the field. In this paper we shall take the symmetry group G to be $U(N)$, with $N \in \{2, 3, \dots\}$. The Lie algebra $u(N)$ consists of skew-hermitian matrices, and comes equipped with an inner-product

$$\langle X, Y \rangle = \text{tr}(XY^*).$$

Taking spacetime as \mathbf{R}^m , the Yang-Mills field is described by a $u(N)$ -valued connection 1-form A , which is, in analogy with the Maxwell theory, proportional to the field potential. The field strength is then proportional to the curvature F^A , which is the 2-form given by

$$F^A = dA + A \wedge A.$$

Again, analogously to the Maxwell theory, the dynamics of the field is given by the extremals of the Yang-Mills action functional, which we take to be

$$(2.1) \quad -\frac{1}{2g^2} \int \langle F^A, F^A \rangle d\text{vol}$$

where the metric used on the right uses the inner-product on $u(N)$ and the metric on spacetime. For the purposes of quantum theory we will assume that the metric on spacetime is Euclidean (as opposed to Minkowski), and then we take the Yang-Mills action to be

$$(2.2) \quad S_{\text{YM}}(A) = \frac{1}{2g^2} \int \langle F^A, F^A \rangle d\text{vol}$$

The quantum theory for the Yang-Mills field, in the Euclidean setting, involves formal functional integrals of the form

$$\int f(A) d\mu_{\text{YM}}^g(A) = \frac{1}{Z_g} \int_{\mathcal{A}} f(A) e^{-S_{\text{YM}}(A)} DA$$

for suitable functions f on the space \mathcal{A} of all gauge fields potentials A , and Z_g is the formal ‘partition function’ normalizing constant (we have absorbed a factor of \hbar into the coupling constant) given by:

$$(2.3) \quad Z_g = \int_{\mathcal{A}} e^{-\frac{1}{2g^2} \|F^A\|_{L^2}^2} DA.$$

The coupling constant g in the exponent is dimensionless when spacetime is four-dimensional. In the Maxwell theory, keeping track of the coupling constant, the integrand has, in the exponent, the term $\frac{1}{8\pi\alpha} \|F^A\|_{L^2}^2$, where $\alpha = \frac{e^2}{4\pi\epsilon_0 c\hbar} \simeq 1/137$ is the dimensionless Sommerfeld fine-structure constant, and the integral over spacetime uses ct in place of the time coordinate t . If spacetime is two-dimensional, the coupling constant g^2 has the dimension Length^{-2} .

A gauge transformation is a smooth mapping

$$\phi : \mathbf{R}^m \rightarrow U(N),$$

and it transforms the potential A to a potential A^ϕ given by

$$A^\phi = \phi^{-1}A\phi + \phi^{-1}d\phi.$$

This transformation replaces F^A by $\phi F^A \phi^{-1}$ and so leaves the action S_{YM} invariant. It is this a physical symmetry of the gauge field as a whole. The functions f of interest are invariant under gauge transformations. We will describe them in more detail now.

Let C be a piecewise smooth path in spacetime, parametrized by $s \in [a, b]$. The solution $s \mapsto h(s) \in SU(N)$ of the differential equation

$$(2.4) \quad \frac{dh(s)}{ds} h(s)^{-1} = -A(C'(s)),$$

starting at $h(a) = I$, the identity, is said to describe *parallel transport* along C by the connection A . If C is a loop we have the *holonomy*

$$(2.5) \quad h(C; A) \stackrel{\text{def}}{=} h(b) \in SU(N).$$

It is readily checked that if ϕ is a gauge transformation then

$$h(C; A^\phi) = \phi(o)^{-1} h(C; A) \phi(o),$$

where $o = C(a)$ is the basepoint of the loop C . As a consequence, the *Wilson loop observable*

$$(2.6) \quad \text{tr} \left(h(C; A) \right)$$

is gauge-invariant. In the case of the $U(1)$ Maxwell theory, the differential equation of parallel-transport is readily solved as an exponential, and then, for the loop C , we have $h(C; A) = e^{-i \int_\sigma F^A}$, where σ is the surface with boundary C . Thus $h(C; A)$ is obtained from the flux of the ‘magnetic field’ through C in this case.

The trace in $\text{tr} \left(h(C; A) \right)$ may be replaced by the trace in a general representation of the gauge group to obtain a general Wilson loop expectation. The significance of the holonomy as a measure of curvature goes back, arguably, to the Gauss-Bonnet formula of classical geometry (corresponding to the gauge group

$U(1) \simeq SO(2)$). In the context of gauge theory, the trace of the holonomy became a standard coordinate on gauge fields following Wilson [52]. For more on the determination of a gauge field from traces of holonomies see Lévy [34].

3. Quantum Yang-Mills on \mathbf{R}^2

We work with $U(N)$ gauge fields on two-dimensional spacetime \mathbf{R}^2 . Sometimes we will use Cartesian coordinates (x, y) on the plane.

Every connection 1-form A is gauge-equivalent to one which is 0 in, say, the y -direction:

$$A_x dx + A_y dy = A_x dx + 0 dy.$$

In this case, the curvature is given by

$$F^A = dA + A \wedge A = -f^A dx \wedge dy,$$

where

$$(3.1) \quad f^A = \frac{\partial A_x}{\partial y}.$$

Then the Yang-Mills functional integral measure μ_{YM}^g is informally given by

$$\frac{1}{Z} \delta(A_y) e^{-\|f^A\|_{L^2}^2 / (2g^2)} DA$$

We can then switch ‘variables’ $A \mapsto f^A$, which is linear, as is the constraint that A_y be 0. Thus, in terms of f^A , the measure is Gaussian:

$$(3.2) \quad \frac{1}{Z_g} e^{-\|f\|_{L^2}^2 / (2g^2)} df,$$

with f running over a linear space of maps $\mathbf{R}^2 \rightarrow u(N)$. As is known in probability theory, this measure actually lives on a Hilbert-Schmidt completion of $L^2(\mathbf{R}^2; u(N))$. For the sake of intuitive appearance, we will continue to write \mathcal{A} for this space, and write the Yang-Mills functional measure generally as

$$d\mu_{\text{YM}}^g(A).$$

Note that the measure does not live on the space of *smooth* or even continuous connections, and so the equation (2.4) for parallel-transport needs to be re-interpreted as a stochastic differential equation (see [15, 18] for more; an earlier work examining stochastic parallel transport in this sense is by Albeverio et al. [2]). The path $s \mapsto h(s)$ then *becomes a stochastic process* with values in $U(N)$. Thus, we have, under the Yang-Mills measure, a $U(N)$ -valued stochastic process describing *stochastic parallel transport* along the curve. For a loop, this is the stochastic holonomy. This process is a Brownian motion on $U(N)$, when time is clocked appropriately.

Before proceeding further, let us summarize the essentials of the heat kernel:

- (i) The heat kernel $Q_t(x)$ solves

$$\frac{\partial Q_t(x)}{\partial t} = \frac{1}{2} \Delta Q_t(x),$$

with $Q_0(x) = \delta_I(x)$, the delta function at the identity I on $U(N)$, and Δ is the Laplacian on $U(N)$ for the chosen inner-product on $u(N)$.

(ii) The heat kernel can be expanded as

$$(3.3) \quad Q_t(x) = \sum_R (\dim R) e^{-c_2(R)t/2} \chi_R(x),$$

where the sum is over irreducible representations R of $U(N)$, and $c_2(R)$ is the quadratic Casimir for the representation R .

(iii) The heat kernel is invariant under inverses:

$$Q_t(x^{-1}) = Q_t(x).$$

(iv) The heat kernel satisfies the convolution formula

$$Q_t * Q_s = Q_{t+s}$$

(v) For the standard Brownian motion $t \mapsto B_t$ on $U(N)$, starting at the identity at $t = 0$, the probability density function of B_t with respect to unit-mass Haar measure on $U(N)$ is Q_t .

We shall now summarize the main result for Yang-Mills stochastic parallel transport proved by Driver [15] and by Gross et al. [18]. For technical convenience, the result applies to loops with finitely many self-intersections which are piecewise C^1 , consisting of vertical segments and curves which are graphs of the form $y = y(x)$.

THEOREM 3.1. *If C is a simple closed loop in \mathbf{R}^2 enclosing an area S then the holonomy $h(C)$ is a $U(N)$ -valued random variable whose distribution is given by*

$$Q_{g^2 S}(x) dx,$$

where dx is unit-mass Haar measure on $U(N)$. Moreover, if C_1, \dots, C_N are loops in the plane which enclose non-overlapping regions then $h(C_1), \dots, h(C_N)$ are mutually independent random variables.

Thus, in particular,

$$(3.4) \quad \int_{\mathcal{A}} (\text{tr} h(C; A))^k d\mu_{\text{YM}}^g(A) = \int_{U(N)} (\text{tr}(x))^k Q_{g^2 S}(x) dx,$$

The independence of non-overlapping loops can be used to compute determine the joint distributions of $h(C_1), \dots, h(C_N)$, for any well-behaved family of loops C_1, \dots, C_N . In particular, closed-form expressions can be obtained for integrals

$$(3.5) \quad \int_{\mathcal{A}} \text{tr} \left(h(C_1; A) \right) \dots \text{tr} \left(h(C_m; A) \right) d\mu_{\text{YM}}^g(A)$$

(See, for instance, Driver [15].)

In the stochastic approach, parallel-transport along a curve describes a Brownian motion on $U(N)$, and the heat kernel arises as the density of the Brownian motion's location at any time.

In the early physics literature, loop expectation values were calculated using Feynman diagram expansions (as used by 't Hooft [48]). The role of the convolution formula of the heat kernel (though not called as such) was stressed by Migdal [37]. We cite more references for the physics literature in the next section. On the mathematical side, Klimek and Kondracki [31] developed the Yang-Mills functional integral measure for \mathbf{R}^2 using the loop expectation values known in the physics literature. They also noted that the loop expectation values involve the heat kernel on the gauge group. These expressions were first proved rigorously by Driver [15]

from the Yang-Mills measure. Gross et al. [18] also computed loop expectation values in a rigorous framework in a way most closely connected with the approach of Bralic [8] in the physics literature. The Yang-Mills functional integral over compact surfaces was used to determine loop expectation values by Fine [16, 17], and the measure was constructed in [43, 44, 45] and the loop expectation values derived rigorously. Going the other direction, Lévy [32, 33] constructed the Yang-Mills measure using the loop expectation values.

4. Wilson Loop Expectation Values: Examples

We will work out some examples of Wilson loop expectations, expressing the expectation values in terms of areas of regions marked out in the plane by the loops.

To start with, consider a simple closed loop C in the plane, enclosing an area S . As we have noted before, the expected value of the trace of the holonomy $h(C)$ around C is

$$\int_{U(N)} \text{tr}(x) Q_{g^2 S}(x) dx.$$

We will work with the normalized trace

$$(4.1) \quad \text{tr}_N = \frac{1}{N} \text{tr}.$$

It will be convenient to introduce the scaled coupling constant

$$(4.2) \quad \tilde{g}^2 = g^2 N.$$

We will write the Yang-Mills expectation value as

$$(4.3) \quad \langle f \rangle \stackrel{\text{def}}{=} \int f d\mu_{\text{YM}}^g.$$

However, it will be convenient to use $\langle \cdot \rangle$ for other averages as well. Thus, we will also write

$$(4.4) \quad \langle \text{tr}_N(U) \rangle = \int_{U(N)} \text{tr}_N(x) Q_{g^2 S}(x) dx,$$

viewing U as a $U(N)$ -valued random variable with density $Q_{g^2 S}$.

From the character-expansion of the heat kernel given by (3.3) it may be seen that

$$(4.5) \quad \langle \text{tr}_N h(C) \rangle = e^{-\tilde{g}^2 S/2}.$$

However, let us verify this by a method which will be useful for more complicated loop expectation values. The Lie algebra $u(N)$ is given by

$$u(N) = i\mathcal{H}_N,$$

where \mathcal{H}_N is the space of $N \times N$ hermitian matrices, and, for hermitian matrices A and B we have

$$\langle iA, iB \rangle_{u(N)} = \text{tr} \left((iA)(iB)^* \right) = \text{tr}(AB) = \langle A, B \rangle_{\mathcal{H}_N},$$

where, on the right, we have the inner-product on the real vector space \mathcal{H}_N . Let

$$(4.6) \quad E_1, \dots, E_D$$

be an orthonormal basis of the space of $N \times N$ hermitian matrices:

$$(4.7) \quad \text{tr}(E_a E_b) = \delta_{ab},$$

where δ_{ab} is 1 if $a = b$, and 0 otherwise.

We enumerate a few useful properties of these matrices:

LEMMA 4.1. *Let $(E_a)_{ij}$ be the ij -component of the matrix E_a . Then for any $N \times N$ matrices X and Y*

$$(4.8) \quad \sum_a \operatorname{tr}(XE_a)\operatorname{tr}(E_aY) = \operatorname{tr}(XY)$$

$$(4.9) \quad \sum_a (E_a)_{ij}(E_a)_{kl} = \delta_{il}\delta_{jk}$$

$$(4.10) \quad \sum_a E_a^2 = NI,$$

$$(4.11) \quad \sum_a \operatorname{tr}(XE_aYE_a) = \operatorname{tr}(X)\operatorname{tr}(Y),$$

where the sums are over $a \in \{1, \dots, D\}$.

PROOF. Equation (4.8), in the case when X and Y are hermitian, is a direct consequence of $\{E_1, \dots, E_D\}$ being an orthonormal basis of the space of $N \times N$ hermitian matrices. Since any arbitrary $N \times N$ complex matrix X is a complex-linear combination of hermitian matrices ($X + X^*$ and $i(X - X^*)$) and since (4.8) is complex linear in X and Y , the identity holds for all complex matrices X and Y . Then (4.9) follows on taking $X = E_{ji}$, the matrix whose only non-zero entry is 1 in the (j, i) position, and $Y = E_{lk}$. Then (4.10) follows immediately, and (4.11) also follows readily. \square

Now let us work out the one-loop expectation value $\langle \operatorname{tr}_N h(C) \rangle$ for a simple loop C enclosing an area S . Using the heat kernel property and integration-by-parts we observe first that

$$(4.12) \quad \frac{\partial \langle \operatorname{tr}_N h(C) \rangle}{\partial S} = \int_{U(N)} \operatorname{tr}_N(x) \frac{\partial Q_{g^2 S}(x)}{\partial S} dx = \frac{g^2}{2} \int \Delta_x \operatorname{tr}_N(x) Q_{g^2 S}(x) dx$$

Recall that the Laplacian is given by

$$(4.13) \quad \Delta = \sum_{a=1}^D \partial_{(iE_a)}^2,$$

and so,

$$(4.14) \quad \Delta \operatorname{tr}_N(x) = \sum_{a=1}^D \partial_{iE_a} \operatorname{tr}_N(xiE_a) = - \sum_{a=1}^D \operatorname{tr}_N(xE_a^2) = -N \operatorname{tr}_N(x).$$

Thus, recalling the notation $\tilde{g}^2 = g^2 N$, we have

$$(4.15) \quad \frac{\partial \langle \operatorname{tr}_N h(C) \rangle}{\partial S} = -\frac{\tilde{g}^2}{2} \langle \operatorname{tr}_N h(C) \rangle$$

Clearly, $\langle \operatorname{tr}_N h(C) \rangle$ equals 1 when $S = 0$, it follows that

$$(4.16) \quad \langle \operatorname{tr}_N h(C) \rangle = e^{-\tilde{g}^2 S/2}.$$

Now we shall work out the more general expectation value:

$$(4.17) \quad \langle \operatorname{tr}_N h(C)^{k_1} \dots \operatorname{tr}_N h(C)^{k_n} \rangle$$

following the elegant strategy of Xu [56].

First let us compute the Laplacian of the integrand. In the following we will often work with sequences $\underline{k} \in \{0, 1, 2, \dots\}^{\{1, 2, \dots\}}$ which are eventually 0, i.e. have the form

$$\underline{k} = (k_1, \dots, k_r, 0, 0, \dots).$$

It will be notationally convenient not to distinguish between the finite sequence (k_1, \dots, k_r) and $(k_1, \dots, k_r, 0, 0, \dots)$. It will also be convenient to set up a definition here, stating certain operators I, II, and III which will appear in our study of the differential equation satisfied by Wilson loop expectation values.

DEFINITION 4.2. We denote by V_k the vector space of all $(f_{\underline{k}})_{\underline{k}}$ with $\underline{k} \in \{0, 1, 2, \dots\}^{\{1, 2, \dots\}}$ having the fixed value k for $|\underline{k}| = \sum_j k_j$:

$$(4.18) \quad V_k = \mathbf{C}^{\{\underline{k}: |\underline{k}|=k\}}$$

On this space we will consider the operators I, II, and III, specified as follows:

$$(4.19) \quad \mathbf{I}f = f$$

and

$$(4.20) \quad \mathbf{II}f = \sum_{j=1}^r \mathbf{II}_j f,$$

where

$$(4.21) \quad (\mathbf{II}_j f)_{\underline{k}} = k_j \sum_{s=1}^{k_j-1} f_{(k_1, \dots, k_j, s, k_j-s, \dots, k_r)},$$

and

$$(4.22) \quad (\mathbf{III}f)_{\underline{k}} = \sum_{1 \leq l < m \leq r} k_l k_m f_{(k_1, \dots, k_l, \dots, k_m, \dots, k_r, k_l+k_m)}$$

Thus the expectation values

$$(4.23) \quad W_N(C)_{\underline{k}} = \langle \text{tr}_N(h(C)^{k_1}) \dots \text{tr}_N(h(C)^{k_r}) \rangle,$$

for fixed $k = |\underline{k}|$, form the components of a vector

$$\overrightarrow{W_N(C)} \in V_k.$$

LEMMA 4.3. For any positive integer k , let $f : U(N) \rightarrow V_k$ be the function with components $f_{\underline{k}}$ specified through

$$f_{\underline{k}}(U) = \prod_{j \geq 1} \text{tr}_N(U^{k_j}),$$

where $\underline{k} = (k_1, \dots, k_r, 0, 0, \dots) \in \{0, 1, \dots, k\}^{\{1, 2, \dots\}}$, has $|\underline{k}| \stackrel{\text{def}}{=} \sum_j k_j = k$. Then

$$(4.24) \quad \Delta f = -N \left[k\mathbf{I} + \mathbf{II} + \frac{2}{N^2} \mathbf{III} \right] f,$$

with I, II, III being the operators specified in Definition 4.2.

PROOF. First note that

$$(4.25) \quad \partial_{iE_a} \text{tr}_N(U^k) = \sum_{s=1}^k \text{tr}_N(U^{s-1}(U_i E_a)U^{k-s}) = k \text{tr}_N(U^k i E_a),$$

and so

$$(4.26) \quad \begin{aligned} \Delta \text{tr}_N(U^k) &= -k \sum_{s=1}^k \sum_a \text{tr}_N(U^s E_a U^{k-s} E_a) \\ &= -Nk \sum_{s=1}^k \text{tr}_N(U^s) \text{tr}_N(U^{k-s}) \quad (\text{by (4.11)}) \\ &= -N \left[k \text{tr}_N(U^k) + k \sum_{s=1}^{k-1} \text{tr}_N(U^s) \text{tr}_N(U^{k-s}) \right]. \end{aligned}$$

This explains the terms I_f and II_f in (4.24). The remaining terms in (4.24) arise from a sum over all ordered pairs (l, m) , with $1 \leq l < m \leq r$, of

$$2 \sum_{a=1}^D k_l \text{tr}_N(U^{k_l} i E_a) k_m \text{tr}_N(U^{k_m} i E_a) = -2 \frac{k_l k_m}{N} \text{tr}_N(U^{k_l} U^{k_m}),$$

thus explaining the term III_f . \square

Now we can determine the differential equation satisfied by the Wilson loop expectation value

$$(4.27) \quad W_N(C)_{\underline{k}} = \langle \text{tr}_N(h(C)^{k_1}) \dots \text{tr}_N(h(C)^{k_r}) \rangle,$$

for $\underline{k} = (k_1, \dots, k_r, 0, 0, \dots)$, and solve it. Recall our notation

$$(4.28) \quad W_N(C)_{\underline{k}} = \langle \text{tr}_N(h(C)^{k_1}) \dots \text{tr}_N(h(C)^{k_r}) \rangle,$$

for fixed $k = |\underline{k}|$.

THEOREM 4.4. *If S is the area enclosed by the simple loop C then*

$$(4.29) \quad \frac{\partial \overrightarrow{W}_N(C)}{\partial S} = -\frac{\tilde{g}^2}{2} \left[k\mathbf{I} + \mathbf{II} + \frac{2}{N^2} \mathbf{III} \right] \overrightarrow{W}_N(C).$$

Hence

$$(4.30) \quad \overrightarrow{W}_N(C) = e^{-\frac{\tilde{g}^2 S}{2} (k\mathbf{I} + \mathbf{II} + \frac{2}{N^2} \mathbf{III})} \mathbf{1},$$

where $\mathbf{1}$ is the vector in V_k with all entries equal to 1.

PROOF. For the derivative of $W_N(\overrightarrow{C})$ we have

$$\begin{aligned}
\frac{\partial W_N(\overrightarrow{C})}{\partial S} &= \int_{U(N)} f(U) \frac{\partial Q_{g^2 S}(U)}{\partial S} dU \\
&= \frac{g^2}{2} \int_{U(N)} f(U) \Delta Q_{g^2 S}(U) dU \\
&= \frac{g^2}{2} \int_{U(N)} \Delta f(U) Q_{g^2 S}(U) dU \\
&= -\frac{g^2 N}{2} \left\langle \left[k\mathbf{I} + \mathbf{II} + \frac{2}{N^2} \mathbf{III} \right] f(U) \right\rangle \quad (\text{by Lemma 4.3}) \\
&= -\frac{\tilde{g}^2}{2} \left[k\mathbf{I} + \mathbf{II} + \frac{2}{N^2} \mathbf{III} \right] W_N(C).
\end{aligned}$$

This is an ordinary differential equation with constant coefficients, and with initial value of $W_N(\overrightarrow{C})$ being $\mathbf{1}$ when $S = 0$. Hence the solution is (4.30). \square

We have followed the strategy of Xu [56] in determining the loop expectation values (however, our expressions in Theorem 4.4 differ slightly from those in [56]). Biane [7] used the character expansion of the heat kernel, along with the exact form of characters of $U(N)$, to determine $W_N(C^k)$ in closed form as a polynomial in S .

Next consider the loop

$$C_1 C_2,$$

where C_1 and C_2 are the loops shown in Figure 1. The inner loop C_1 encloses an area S_1 , and between C_1 and the outer loop C_2 lies an area S_2 .

Then

$$(4.31) \quad W_N(C_1 C_2) = \langle \text{tr}_N h(C_1 C_2) \rangle$$

is given by

$$\begin{aligned}
W_N(C_1 C_2) &= \int_{U(N)} \text{tr}_N(xy) Q_{g^2 S_2}(xy^{-1}) Q_{g^2 S_1}(y) dx dy \\
&= \int_{U(N)} \text{tr}_N(xy^2) Q_{g^2 S_2}(x) Q_{g^2 S_1}(y) dx dy \\
&= \langle \text{tr}_N(UV^2) \rangle,
\end{aligned}$$

where U and V are independent $U(N)$ -valued random variables with densities (with respect to unit-mass Haar measure) given by $Q_{g^2 S_2}$ and $Q_{g^2 S_1}$, respectively.

It will be useful to also bring in

$$(4.32) \quad W_N(C_1, C_2) = \langle \text{tr}_N h(C_2) \text{tr}_N h(C_1) \rangle$$

which is given by

$$\begin{aligned}
W_N(C_1, C_2) &= \int_{U(N)} \text{tr}_N(x) \text{tr}_N(y) Q_{g^2 S_2}(xy^{-1}) Q_{g^2 S_1}(y) dx dy \\
&= \langle \text{tr}_N(UV) \text{tr}_N(V) \rangle.
\end{aligned}$$

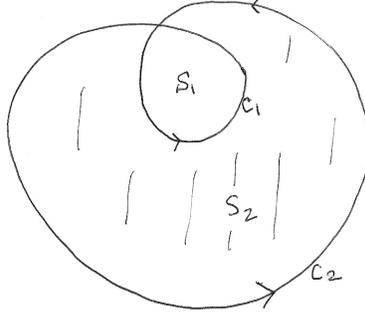


FIGURE 1. A loop passing inside itself.

Using the same strategy as in Lemma 4.3, we have (taking U and V as variables on $U(N)$ now):

$$\begin{aligned}
 \Delta_U \text{tr}_N(UV^2) &= -N \text{tr}_N(UV^2) \\
 \Delta_V \text{tr}_N(UV^2) &= -2N [\text{tr}_N(UV^2) + \text{tr}_N(UV)\text{tr}_N(V)] \\
 (4.33) \quad \Delta_U \text{tr}_N(UV)\text{tr}_N(V) &= -N \text{tr}_N(UV)\text{tr}_N(V) \\
 \Delta_V [\text{tr}_N(UV)\text{tr}_N(V)] &= -2N \left[\text{tr}_N(UV)\text{tr}_N(V) + \frac{1}{N^2} \text{tr}_N(UV^2) \right]
 \end{aligned}$$

Let us verify, as an example, the last of these identities. As before, let E_1, \dots, E_D be an orthonormal basis of the $N \times N$ hermitian matrices; then iE_1, \dots, iE_D form an orthonormal basis of $u(N)$, and, with ∂_{iE_a} denoting the derivative in direction of iE_a , we have

$$\begin{aligned}
 (4.34) \quad \partial_{iE_a}^2 [\text{tr}_N(UV)\text{tr}_N(V)] &= \partial_{iE_a} [\text{tr}_N(UViE_a)\text{tr}_N(V) + \text{tr}_N(UV)\text{tr}_N(ViE_a)] \\
 &= \text{tr}_N(UViE_a iE_a)\text{tr}_N(V) + \text{tr}_N(UViE_a)\text{tr}_N(ViE_a) \\
 &\quad + \text{tr}_N(UViE_a)\text{tr}_N(ViE_a) + \text{tr}_N(UV)\text{tr}_N(ViE_a iE_a).
 \end{aligned}$$

Now summing over a , and using the trace-splitting identities from Lemma 4.1, we have

$$\begin{aligned}
\Delta_V[\mathrm{tr}_N(UV)\mathrm{tr}_N(V)] &= -N\mathrm{tr}_N(UV)\mathrm{tr}_N(V) - \frac{1}{N}\mathrm{tr}_N(UV^2) \\
(4.35) \quad &\quad - \frac{1}{N}\mathrm{tr}_N(UV^2) - N\mathrm{tr}_N(UV)\mathrm{tr}_N(V) \\
&= -2N \left[\mathrm{tr}_N(UV)\mathrm{tr}_N(V) + \frac{1}{N^2}\mathrm{tr}_N(UV^2) \right],
\end{aligned}$$

which is the expression claimed earlier.

Then, again with the method used before, we have

$$(4.36) \quad \frac{\partial}{\partial S_2} \begin{bmatrix} W_N(C_1 C_2) \\ W_N(C_1, C_2) \end{bmatrix} = -\frac{\tilde{g}^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} W_N(C_1 C_2) \\ W_N(C_1, C_2) \end{bmatrix}$$

and

$$(4.37) \quad \frac{\partial}{\partial S_1} \begin{bmatrix} W_N(C_1 C_2) \\ W_N(C_1, C_2) \end{bmatrix} = -\tilde{g}^2 \begin{bmatrix} 1 & 1 \\ \frac{1}{N^2} & 1 \end{bmatrix} \begin{bmatrix} W_N(C_1 C_2) \\ W_N(C_1, C_2) \end{bmatrix}.$$

Solving, we have,

$$(4.38) \quad \begin{bmatrix} W_N(C_1 C_2) \\ W_N(C_1, C_2) \end{bmatrix} = e^{-\frac{\tilde{g}^2}{2}(S_2+2S_1)} \begin{bmatrix} \cosh(\tilde{g}^2 S_1/N) & -N \sinh(\tilde{g}^2 S_1/N) \\ -\frac{1}{N} \sinh(\tilde{g}^2 S_1/N) & \cosh(\tilde{g}^2 S_1/N) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In particular,

$$(4.39) \quad W_N(C_1 C_2) = e^{-\frac{\tilde{g}^2}{2}(S_2+2S_1)} (\cosh(\tilde{g}^2 S_1/N) - N \sinh(\tilde{g}^2 S_1/N)).$$

In the early physics literature, a set of differential equations for Wilson loop expectations in terms of the areas S_i was developed by Makeenko and Migdal [35] (see Singer [47] for a succinct statement of some of these equations in the present context). The latter approach was used extensively by Migdal [37, 38, 39], Kazakov [29], Kazakov and Kostov [27, 28]. The paper by Bralic [8] has a detailed treatment of loop expectation values in terms of areas, with numerous examples tabulated (in particular, the expression (4.39) is listed).

5. The Limit as $N \rightarrow \infty$

The behavior of large matrices has been of interest the context of nuclear physics at least since Wigner's work [50, 51], which presented the celebrated semi-circle law on the distribution of eigenvalues of large hermitian matrices (for more on large random matrices see Mehta's book [36]; see also works by Guionnet, for example [22] and Johansson [26]).

In the context of gauge theory, it was 't Hooft's work [48] which launched the investigation of the large- N limit of $U(N)$ gauge theories. As noted earlier, the coupling constant g^2 for the Maxwell electromagnetic field is the fine structure constant $\simeq 1/137$. For the strong force of QCD, with non-abelian gauge group, an expansion in positive powers of g^2 is not feasible. In [48], $U(N)$ gauge theory integrals were expanded in powers of $1/N$, as opposed to the coupling constant, but with $\tilde{g}^2 = g^2 N$ fixed.

Recall the formal expression

$$d\mu_{\mathrm{YM}}^g(A) = Z_g^{-1} e^{-\frac{1}{2g^2} \|f^A\|_{L^2}^2} DA.$$

Keeping $\tilde{g}^2 = g^2 N$ fixed, we view this as

$$Z_g^{-1} e^{-\frac{N}{2\tilde{g}^2} \|f^A\|_{L^2}^2} DA.$$

One can then consider expanding integrals with respect to this measure in powers of $1/N$.

We work with pure Yang-Mills theory in the plane. Having already worked out several loop expectation values explicitly, we can determine their $N \rightarrow \infty$ behavior readily.

If C_1, \dots, C_r are loops in the plane, we write

$$W_N(C_1, \dots, C_r) = \langle \text{tr}_N h(C_1) \dots \text{tr}_N h(C_r) \rangle,$$

and

$$W_\infty(C_1, \dots, C_r) = \lim_{N \rightarrow \infty} W_N(C_1, \dots, C_r).$$

The simplest case is that of a simple closed loop C enclosing an area S . We have seen that

$$W_N(C) = e^{-\frac{\tilde{g}^2}{2} S},$$

which is independent of N , and so

$$(5.1) \quad W_\infty(C) = e^{-\frac{\tilde{g}^2}{2} S}.$$

For the loop $C_1 C_2$ shown in Figure 1, we have, on using the expectation value formula (4.39),

$$(5.2) \quad W_\infty(C_1 C_2) = e^{-\frac{\tilde{g}^2}{2} (S_1 + 2S_2)} (1 - \tilde{g}^2 S_1).$$

Next consider a simple loop C enclosing an area S . Recall the notation (4.27)

$$W_N(C)_{\underline{k}} = \langle \text{tr}_N (h(C)^{k_1}) \dots \text{tr}_N (h(C)^{k_r}) \rangle,$$

for $\underline{k} = (k_1, \dots, k_r, 0, 0, \dots)$, and the corresponding vector

$$\overrightarrow{W_N(C)}$$

whose components are $W_N(C)_{\underline{k}}$ for the fixed value k of $|\underline{k}| = \sum_{j \geq 1} k_j$.

We had computed an explicit expression for $\overrightarrow{W_N(C)}$ in Theorem 4.4. Using this for the operator Π in the simple case of a one-component vector) we can now determine the $N \rightarrow \infty$ limit of $\overrightarrow{W_N(C)}$. The following result was proven by Xu [56] and by Biane [7].

THEOREM 5.1. *Let C be a simple loop enclosing an area S . Then*

$$(5.3) \quad \overrightarrow{W_\infty(C)} \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \overrightarrow{W_N(C)}$$

exists and is given by

$$(5.4) \quad \overrightarrow{W_\infty(C)} = e^{-\frac{\tilde{g}^2 S}{2} (k\mathbf{I} + \Pi)} \mathbf{1},$$

where, in the exponent, \mathbf{I} is the identity matrix and Π is the matrix given in (4.20), and $\mathbf{1}$ is the vector with all components equal to 1. Moreover,

$$(5.5) \quad \lim_{N \rightarrow \infty} \left\langle \prod_{j=1}^r \text{tr}_N (h(C)^{k_j}) \right\rangle = \prod_{j=1}^r \lim_{N \rightarrow \infty} \langle \text{tr}_N (h(C)^{k_j}) \rangle.$$

Furthermore, for $k \in \{1, 2, \dots\}$,

$$(5.6) \quad W_\infty(C^k) = e^{-k\frac{\tilde{g}^2}{2}S} P_k(\tilde{g}^2 S),$$

where $P_k(x)$ is a polynomial of degree $k - 1$ satisfying

$$(5.7) \quad \frac{dP_k(x)}{dx} = -\frac{k}{2} \sum_{j=1}^{k-1} P_j(x) P_{k-j}(x),$$

with $P_k(0) = 1$, and $P_1(x) = 1$.

PROOF. Letting $N \rightarrow \infty$ in (4.30) yields the existence of $\lim_{N \rightarrow \infty} \overrightarrow{W_N}(C)$ and the expression (5.4). Differentiating with respect to S gives:

$$(5.8) \quad \frac{d\overrightarrow{W_\infty}(C)}{dS} = -\frac{\tilde{g}^2}{2} (k\mathbf{I} + \mathbf{II}) \overrightarrow{W_\infty}(C),$$

where the operator \mathbf{II} is as in (4.20).

The equation (5.8) can be solved more explicitly than the exponential expression (5.4). Following the idea of Xu [56], let $w_l(S)$ satisfy the differential equation

$$(5.9) \quad \frac{dw_l}{dS} = -\frac{\tilde{g}^2}{2} l \sum_{j=1}^l w_j w_{l-j} = -\frac{\tilde{g}^2}{2} \left(l w_l + l \sum_{j=1}^{l-1} w_j w_{l-j} \right),$$

with initial condition $w_l(0) = 1$, for $l \in \{1, 2, \dots\}$. Let w be the vector with components

$$w_{\underline{k}} = w_{k_1} \dots w_{k_r},$$

for $\underline{k} = (k_1, \dots, k_r, 0, 0, \dots)$, with fixed $k = \sum_{j \geq 1} k_j$. Using the product rule for differentiation and the expression for \mathbf{II} , we can then readily check that (5.9) implies that the vector w satisfies the same first order linear differential equation (5.8) as $\overrightarrow{W_\infty}(C)$:

$$(5.10) \quad \frac{dw}{dS} = -\frac{\tilde{g}^2}{2} (k\mathbf{I} + \mathbf{II}) w,$$

and has the same initial value $w(0) = \mathbf{1}$. Therefore,

$$\overrightarrow{W_\infty}(C) = w.$$

This proves the freeness property (5.5). Note that, in particular, for every positive integer k , we have

$$w_k = W_\infty(C^k).$$

Next, from the differential equation (5.9), it follows that the function P_k specified through (5.6) satisfies the differential equation (5.7). Since we already know that $W_\infty(C) = e^{-\frac{\tilde{g}^2}{2}S}$, it follows that $P_1(x) = 1$. It is also clear that $P_k(0) = 1$ since $W_\infty(C^k)$ equals 1 if C encloses 0 area. Inductively, $P_k(x)$ is a polynomial of degree $k - 1$. \square

The polynomials P_k were identified via the Laguerre polynomials in Singer [47]. In fact,

$$(5.11) \quad P_k(x) = \frac{1}{k} L_{k-1}^1(kx),$$

where L_n^1 is the associated Laguerre polynomial:

$$(5.12) \quad L_n^1(x) = \sum_{m=0}^n \binom{n+1}{n-m} \frac{(-x)^n}{m!}.$$

Thus,

$$(5.13) \quad W_\infty(C^k) = e^{-k \frac{\tilde{g}^2 S}{2}} \frac{1}{k} L_{k-1}^1(k \tilde{g}^2 S).$$

The method of Biane [7] leads directly to the explicit formulae for the polynomials $L_{k-1}^1(kx)$ by using the characters of $U(N)$.

Consider the ‘exponential generating function’ $F(\lambda, x)$ specified by the formal power series

$$(5.14) \quad F(\lambda, \tilde{g}^2 S) = \sum_{k \geq 1} e^{\lambda k} e^{\frac{\tilde{g}^2 S}{2} k} W_\infty(C^k).$$

The differential equation (5.7) then implies that F solves:

$$(5.15) \quad F \frac{\partial F}{\partial \lambda} + \frac{\partial F}{\partial x} = 0.$$

Equations related to this appear in Biane [7], and, earlier, in the physics literature (Paffuti and Rossi [41], for instance, in the context of the recurrence relation specified through the right side of (5.9)).

Note that because of the inversion-invariance $Q_t(x^{-1}) = Q_t(x)$ of the heat kernel,

$$(5.16) \quad \langle W_N(C^k) \rangle = \langle W_N(C^{-k}) \rangle,$$

for all integers $k \in \mathbf{Z}$.

The large- N limit of the Wilson loop variables lead to free random variables. We have seen only a hint of this in the preceding results and observations. This ‘asymptotic freeness’ is developed in the papers of Biane [7] and Xu [56].

For results concerning the large- N limit on surfaces (instead of the plane) see Baez and Taylor [5].

6. The Master Field

A master field for the $N = \infty$ Yang-Mills measure could be thought of as a ‘ $U(\infty)$ -gauge field’ A_∞ such that

$$(6.1) \quad \text{tr}_\infty(h(C; A_\infty)) = W_\infty(C) = \lim_{N \rightarrow \infty} \langle \text{tr}_N h(C) \rangle$$

for all nice-enough loops C . The left side in (6.1) is only formally meaningful at this point.

We shall first describe a simple setting for the master field, and then describe Singer’s method [47] for the master field on \mathbf{R}^2 .

Let us first consider one fixed loop C . We use notation and results from the preceding section.

The holonomy $h(C) \in U(N)$ is a unitary matrix, and so $\text{tr}_N h(C)^k$ is the average of the eigenvalues of $h(C)^k$, for any integer $k \in \mathbf{Z}$. Since these eigenvalues all lie on $S^1 = \{z \in \mathbf{C} : |z| = 1\}$, we have

$$W_N(C^k) = \langle \text{tr}_N(h(C)^k) \rangle = \int_{S^1} z^k d\nu_{\tilde{g}^2 S, N}(z),$$

for some probability measure $\nu_{\tilde{g}^2 S, N}$ on the unit circle S^1 . This sequence is non-negative in the sense that for any $a_1, \dots, a_m \in \mathbf{C}$ we have

$$(6.2) \quad \sum_{j=1}^m a_k \bar{a}_j W_N(C^{k-j}) = \left\langle \text{tr}_N \left| \sum_{j=1}^m a_j h(C^j) \right|^2 \right\rangle \geq 0,$$

where, for a matrix A , we have $|A|^2 = AA^*$, whose trace is $\sum_{jk} |A_{jk}|^2 \geq 0$, and we have used the fact that

$$h(C^{-k}) = h(C^k)^*,$$

by unitarity of $h(C)$. Letting $N \rightarrow \infty$ then shows that the numbers $W_\infty(C^k)$, for $k \in \mathbf{Z}$, form a non-negative sequence as well:

$$(6.3) \quad \sum_{j,k=1}^m a_k \bar{a}_j W_\infty(C^{k-j}) \geq 0.$$

Then, by the Herglotz theorem (see Helson[23, page 40]), there is a Borel measure $\nu_{\tilde{g}^2 S}$ on S^1 such that, for every integer k ,

$$(6.4) \quad W_\infty(C^k) = \int_{S^1} z^k d\nu_{\tilde{g}^2 S}(z).$$

For $k = 0$ the left side is 1, and so $\nu_{\tilde{g}^2 S}$ is a probability measure. We have also the exponential generating function (considered earlier in (5.14))

$$(6.5) \quad F(\lambda, \tilde{g}^2 S) = \sum_{k \geq 1} (e^{\lambda + \tilde{g}^2 S/2})^k W_\infty(C^k) = \int_{S^1} \frac{az}{1-az} d\nu_{\tilde{g}^2 S}(z),$$

where $a = e^{\lambda + \tilde{g}^2 S/2}$ is assumed to have $|a| < 1$.

The powers C^k of the loop C , with k running over the integers, form a group ($\simeq \mathbf{Z}$) under composition of loops. This group has a unitary representation on the Hilbert space $L^2(S^1, \nu_{\tilde{g}^2 S})$ given by

$$h_\infty(C^k) = M_z^k,$$

where M_z the multiplication operator specified by

$$(M_z f)(w) = wf(w).$$

In the case of a finite-dimensional group G , an element $g \in G$ specifies (up to gauge equivalence) a connection on a principal G -bundle over the circle C with g as holonomy around the loop C . Thus we can formally view $h_\infty(C)$ as the holonomy around C of a connection A_∞ on a bundle over C with structure group being the unitaries on $L^2(S^1, \nu_{\tilde{g}^2 S})$. Using the ‘vacuum vector’ $1 \in L^2(\nu_{\tilde{g}^2 S})$ we have a corresponding trace operation on operators B commuting with the representation of $\{C^k\}_{k \in \mathbf{Z}}$,

$$\text{tr}_\infty(B) = \langle B1, 1 \rangle.$$

Then

$$(6.6) \quad W_\infty(C^k) = \text{tr}_\infty(h(C^k; A_\infty)).$$

We turn, finally, to a construction of the master field for the plane \mathbf{R}^2 outlined by Singer [47]. Elements of the construction are formal. View the set Ω_o of loops, with a fixed basepoint o , as a group under composition of paths with backtracks erased. Then

$$K_N(C, C') = W_N(C\bar{C}'),$$

is a non-negative kernel in the sense that for any finite collection of loops C and associated complex numbers a_C we have

$$\sum_{C, C'} K_N(C, C') a_C \bar{a}_{C'} \geq 0.$$

The reason this holds is that $W_N(C)$ arises from a genuine measure, the Yang-Mills measure for the plane, and the argument is essentially the same as seen in (6.2). Then, just as we applied the Herglotz theorem in the preceding discussion to the commutative group $\{C^k\}_{k \in \mathbf{Z}}$, we shall use the Gelfand-Naimark-Segal (GNS) construction for the non-commutative group Ω_o .

Define the kernel K_∞ by

$$(6.7) \quad K_\infty(C, C') = \lim_{N \rightarrow \infty} K_N(C, C') = W_\infty(C\bar{C}').$$

This, being the limit of K_N , is also a non-negative definite Hermitian bilinear form on the set of functions f on Ω_o of finite support:

$$\langle f, g \rangle_K = \sum_{C, C'} K_\infty(C, C') f(C) \overline{g(C')}.$$

Then we obtain a Hilbert space \mathcal{H} by completing this space with the corresponding norm followed by quotienting with the null space (note that each δ_C has norm 1). Observe that, just as K_N , the kernel K_∞ is left and right invariant under the action of Ω_o , i.e.

$$K_\infty(C_1 C, C_2 C) = K_\infty(C_1, C_2) = K_\infty(C C_1, C C_2).$$

This implies that the operator U_C given by

$$(U_C f)(C') = f(C^{-1} C')$$

is unitary, and it is then clear that $C \mapsto U_C$ is a unitary representation of Ω_o on the Hilbert space \mathcal{H} . Again, by analogy with the case of holonomies of $U(N)$ -bundles, we formally think of a connection A_∞ on the principal bundle over the plane with structure group given by the unitaries of \mathcal{H} , having the holonomies

$$(6.8) \quad h(C; A_\infty) = U_C.$$

(The bundle and connection are described in more detail below.)

For a bounded operator B on \mathcal{H} , commuting with the operators U_C , define the trace $\text{tr}_\infty(B)$ by

$$\text{tr}_\infty(B) = \langle B \delta_o, \delta_o \rangle,$$

where δ_o is the ‘vacuum state’, the function equal to 1 on the constant loop at o (the identity of Ω_o) and 0 elsewhere. Then we have the desired relation:

$$(6.9) \quad \text{tr}_\infty(h(C; A_\infty)) = W_\infty(C).$$

Let us take a look at the bundle and connection that lead to the holonomies $h(C; A_\infty)$, following the method of Singer [47]. Consider the space \mathcal{P}_o of paths emanating from o , with the natural endpoint projection $\pi : \mathcal{P}_o \rightarrow \mathbf{R}^2$ as a principal Ω_o -bundle over \mathbf{R}^2 . The action of Ω_o on \mathcal{P}_o is given by pre-composition (with backtracks erased). This principal bundle has a natural connection \tilde{A}_∞ : if $x : [a, b] \rightarrow \mathbf{R}^2$ is a path and $p \in \mathcal{P}_o$ ends at $x(a)$, then define the path p_t to be p followed by $x|_{[a, t]}$, for $t \in [a, b]$. Then $t \mapsto p_t$ gives parallel-transport for \tilde{A}_∞ by

definition of this connection. Then the parallel-transport of the path $p \in \pi^{-1}(o)$ around a loop C based at o is simply the composite path pC . Thus,

$$h(C; \tilde{A}_\infty) = C$$

for all loops C based at o . The representation $C \mapsto U_C$ then produces a principal bundle with structure group generated by the unitaries U_C , and the connection \tilde{A}_∞ goes over to a connection A_∞ . The holonomy $h(C; A_\infty)$ is then U_C , as required in (6.8).

We refer to Singer [47] for more ideas and results on the master field, including possible extensions to higher dimensions.

7. Concluding Remarks

Though our discussion has focused on the large N limit itself, the expansion in powers of $1/N$ is of at least as great interest. It may seem odd to expand in such powers, particularly as the physically relevant values of N are low integers such as 2 and 3, but there are more traditional areas of quantum physics where such expansions have provided great insight and utility (I am thankful to S. Rajeev for pointing this out to me). For instance, in a classic work, Hylleraas [24] presented a useful expansion of the ground state energy of Helium, and ions of other atoms having low atomic number Z , in powers of Z^{-1} (see also Bethe and Salpeter [6] for more).

In the context of Yang-Mills over compact surfaces, the $1/N$ -expansion of the QCD partition function was explored and related to string theory in the works of Gross [19], Gross and Taylor [21, 20], and Baez and Taylor [5]. (Aspects of the exact relationship with string theory have been re-examined by Zelditch [57].) A q -deformed version of this is explored by de Haro et al. [10].

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