Resonance Between Bound States and Radiation in Lattices

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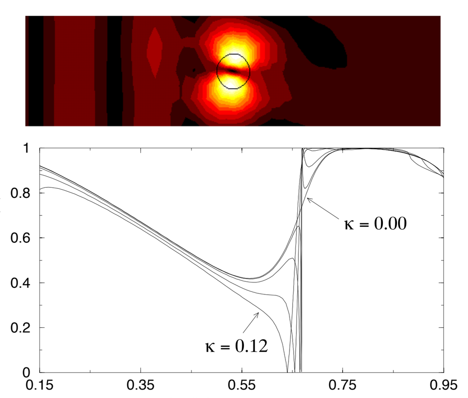
Resonant Interaction Between Bound and Radiating States

Periodic structures, such as crystal lattices or the screen of a microwave oven, possess two properties that make them physically interesting as well as important for the engineering of a wide range of devices:

1) They allow waves (electromagnetic, elastic, acoustic) to propagate only at frequencies in certain "propagation bands". These waves are **radiation states**.

2) A defect in the structure can trap energy at characteristic frequencies by allowing localized oscillations, called **bound states**.

Typically, a bound state is formed at a frequency *outside* a propagation band. But under special conditions, a bound state can be formed at a frequency that is **embedded** in the propagation band. Such states are unstable because they couple with radiation modes when the system is slightly perturbed. This causes **resonant interaction between bound states and radiation**, resulting in field amplification and sensitive dependence of energy transmission across the defect (see figure).



Optical resonance in a waveguide. κ is a perturbation parameter. (Shipman/Venakides 2005)

This study examines this resonant interaction in a 1D lattice of interacting beads. In order to trap energy in a defect at a frequency that allows radiation, **each bead must interact with four neighboring beads** (not just two).

First and Second Neighbor Interactions in a Uniform Lattice

Consider an infinite chain of beads of mass m connected by springs. Each bead is constrained to move vertically and is connected to its nearest neighbors by a spring of tension τ_1 and to its second neighbors by a spring of tension τ_2.

The height $x_n(t)$ of the n^{th} bead is governed by the **equation of motion**

$$m\ddot{x} = -\tau_1(x_n - x_{n-1}) - \tau_1(x_n - x_{n+1}) - \tau_2(x_n - x_{n-2}) - \tau_2(x_n - x_{n+2})$$

In **harmonic motion**, each bead vibrates at the same frequency ω:

$$x_n(t) = U_n e^{-i\omega t}, \qquad U_n = r_n e^{i\theta_n}$$

Inserting this form into the equations of motion gives the fourth-order recursion

$$-\omega^2 m U_n = \tau_1(U_{n+1} + U_{n-1} - 2U_n) + \tau_2(U_{n+2} + U_{n-2} - 2U_n) \tag{1}$$

This system has solutions $U_n = z^n$, where z satisfies the polynomial equation

$$\omega^2 m - 2(\tau_1 + \tau_2) + \tau_1(z + z^{-1}) + \tau_2(z^2 + z^{-2}) = 0 \tag{2}$$

which has four roots z. These roots come in reciprocal pairs $z \in \{z_1, z_1^{-1}, z_2, z_2^{-1}\}$. The general solution for U_n is a linear combination of elementary solutions:

$$U_n = Az_1^n + Bz_1^{-n} + Cz_2^n + Dz_2^{-n}$$

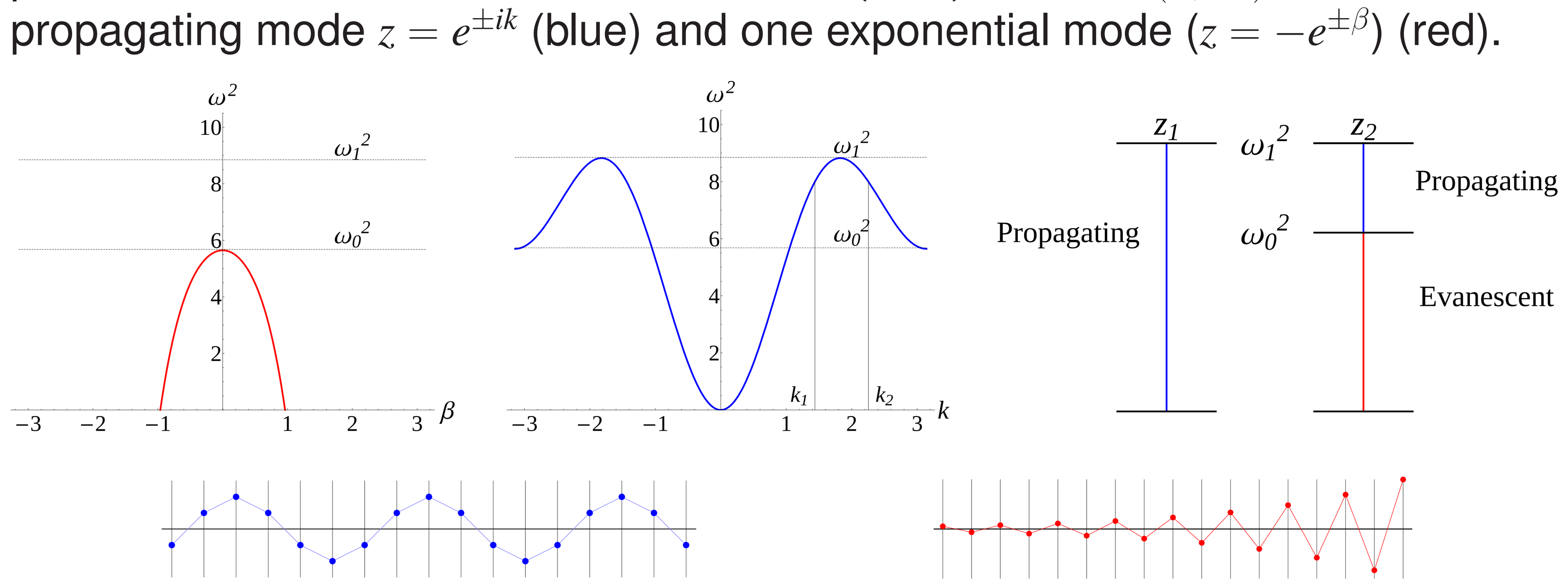
We are particularly interested in the case that $|z| = 1$, that is, $z = e^{ik}$ for k real, because these roots correspond to propagating waves in the lattice:

$$x_n(t) = z^n e^{-i\omega t} = e^{i(kn - \omega t)} \qquad \text{[propagating wave]}$$

There is a **propagation band** of frequencies $(0, \omega_1)$ for which there are such solutions. The **dispersion relation** between *wavenumber* k and *frequency* ω is obtained by putting $z = e^{ik}$ into (2) (blue graph below):

$$\omega^2 \frac{m}{2} = (\tau_1 + \tau_2) - \tau_1 \cos k - \tau_2 \cos 2k \qquad \text{[dispersion relation]}$$

In a subinterval (ω_0, ω_1), there are two propagating modes corresponding to two pairs of wavenumbers $k = \pm k_1$ and $k = \pm k_2$ (blue). For $\omega \in (0, \omega_0)$, there is one propagating mode $z = e^{\pm ik}$ (blue) and one exponential mode ($z = -e^{\pm\beta}$) (red).



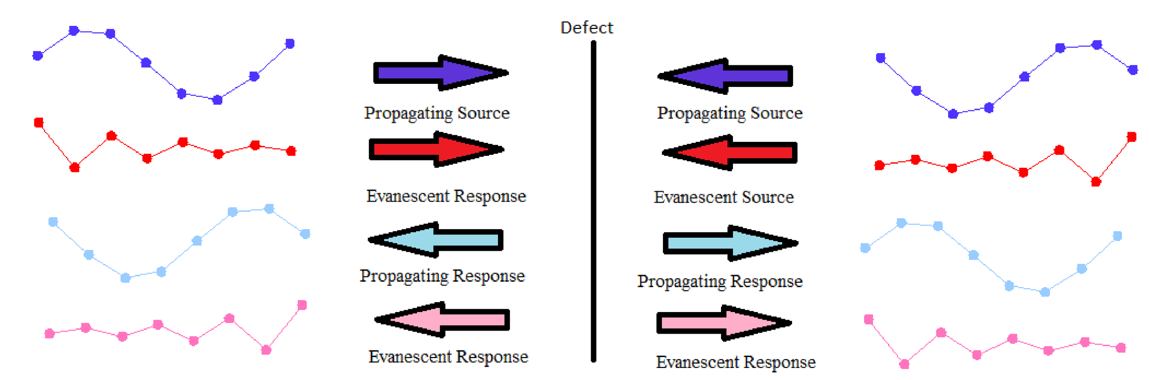
Wave of the form $\mathrm{Re}(e^{ikn})$ — Field of the form $(-e^{\beta})^n$

Scattering by a Defect

Now suppose that **the masses at $\mathbf{n = -1, 0, 1}$ are defective.**

Oscillatory sources to the right and left of the defect emit both propagating and evanescent (exponentially decaying) waves, and these are scattered by the defect, being partially reflected and partially transmitted. The position of the beads is a superposition of evanescent sources, propagating sources and scattered field evanescent responses, and propagating responses.

$$U_n = \begin{cases} U_n = J_{rp}e^{ikn} + V_{lp}e^{-ikn} + J_{re}z^n + V_{le}z^{-n} & n \le 0\,, \\ U_n = J_{lp}e^{-ikn} + V_{rp}e^{ikn} + J_{le}z^{-n} + V_{re}z^n & n \ge 0\,. \end{cases} \qquad (-1 < z < 0) \tag{3}$$



The eight coefficients must satisfy four linear conditions:

Matching of solutions at $n = 0$

Balance of forces (the analog of (1)) at $n = 1$, $n = 0$, and $n = 1$

The system is too large to fit on this poster, but it can be put in matrix form

$$B\,\vec{V} = A\,\vec{J} \qquad \text{[scattering problem]} \tag{4}$$

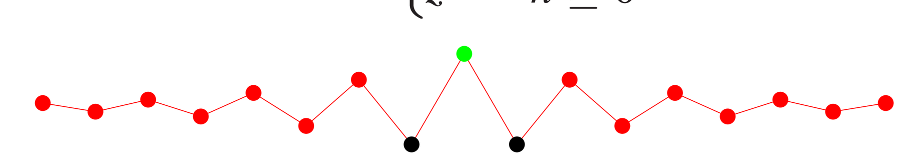
in which A and B are 4×4 matrices, $\vec{J}$ are the incoming source fields, and $\vec{V}$ are the outgoing scattered fields:

$$\vec{J} = [J_{rp}, J_{lp}, J_{re}, J_{le}]^T \quad \text{[incoming source]} \qquad \vec{V} = [V_{lp}, V_{rp}, V_{le}, V_{re}]^T \quad \text{[outgoing scattered]}$$

Embedded Bound States: Trapping Energy at a Defect

In a propagation band, it is difficult to trap energy at a defect because it is possible for energy to radiate out at that frequency and a purely evanescent field is thus unstable. However, under specific conditions, energy can be trapped in a field (a bound state) that is exponentially decaying as $n \to \pm\infty$.

$$U_n = \begin{cases} z^{-n} & n \le 0 \\ z^n & n \ge 0 \end{cases} \qquad (-1 < z < 0)$$

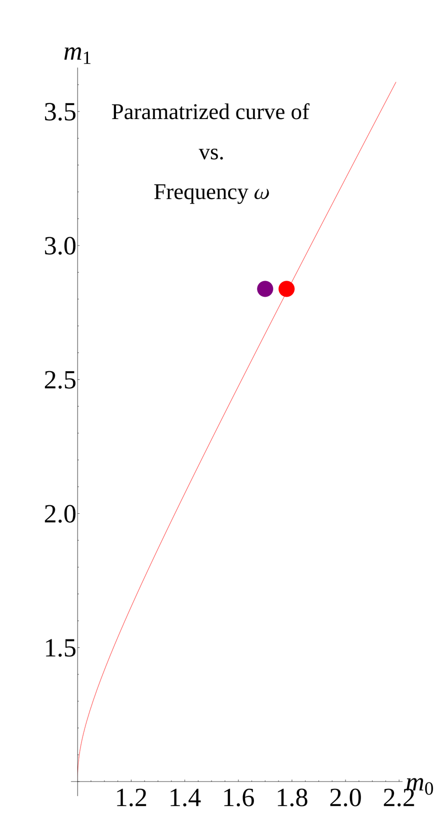


[bound state]

Such a field must satisfy the balance-of-force equations analogous to (1) at $n = -1, 0, 1$. Assuming the masses at $n = -1$ and $n = 1$ are equal ($m_{-1} = m_1$), this results in expressions for m_1 and m_0 as functions of ω (and all the other fixed parameters):

$$m_0 = \tau_1(2z - 2) + \tau_2(2z^2 - 2)$$
$$m_1 = m_{-1} = \tau_1(z + z^{-1} - 2) + \tau_2(z^2 - 1) \qquad (z = z(\omega)) \tag{5}$$

These equations parameterize a curve (m_0, m_1) as a function of ω (right). Each point on this curve corresponds to a bound state. A typical perturbation of either mass from one of these pairs destroys the bound state. A point such as the purple one is very close to the curve, and so it corresponds to a system that can almost trap energy. When an incident wave impinges upon the defect, resonant amplification causes the scattered field to resemble the bound state. This resonance is analyzed in the following panel.



Resonant Scattering Between Bound and Propagating States

Let us now combine the phenomena of scattering and trapped energy.

Question: *What happens when source fields are scattered by a defect near parameters (frequency, masses) of bound state?*

Suppose the masses $m_{-1} = m_1$ and m_0 and the frequency ω are adjusted to trap energy according to (5). Denote these special parameters by (m_0^*, m_1^*, ω^*). The bound state is of the form (3) with no incoming field $\vec{J} = 0$ and evanescent outgoing field $\vec{V}_0 = [0, 0, 1, 1]$ (no energy propagates). Thus,

$$B\,\vec{V}_0 = 0 \qquad [\,B = B(m_0^*, m_1^*, \omega^*)\,]$$

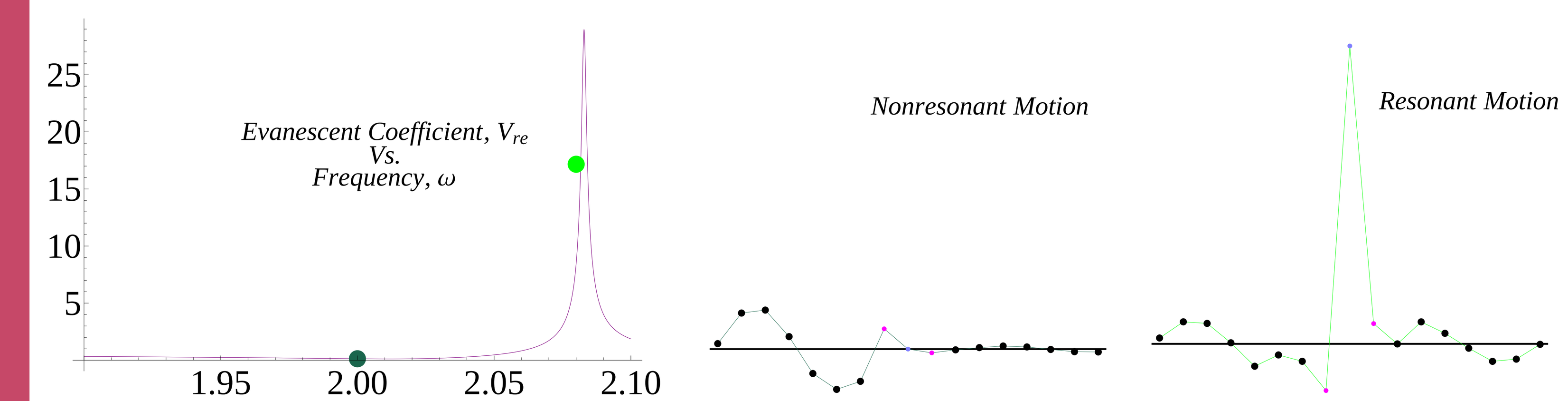
***This means* 0 *is an eigenvalue of the matrix* $B(m_0^*, m_1^*, \omega^*)$.**

If the eigenvalue is simple, the matrix $B(m_0, m_1, \omega)$ has an eigenvalue $\lambda(m_0, m_1, \omega)$ with eigenvector $\vec{V}(m_0, m_1, \omega)$ that are analytic functions of the parameters (m_0, m_1, ω) near (m_0^*, m_1^*, ω^*):

$$B(m_0, m_1, \omega)\vec{V}(m_0, m_1, \omega) = \lambda(m_0, m_1, \omega)\vec{V}(m_0, m_1, \omega) \tag{6}$$

The vector $\lambda\vec{V}$ can be equated with $A\vec{J}$ in the scattering problem (4), meaning that $\vec{V}$ is the field that is scattered by the source field $\vec{J} = A^{-1}\lambda\vec{V}$.

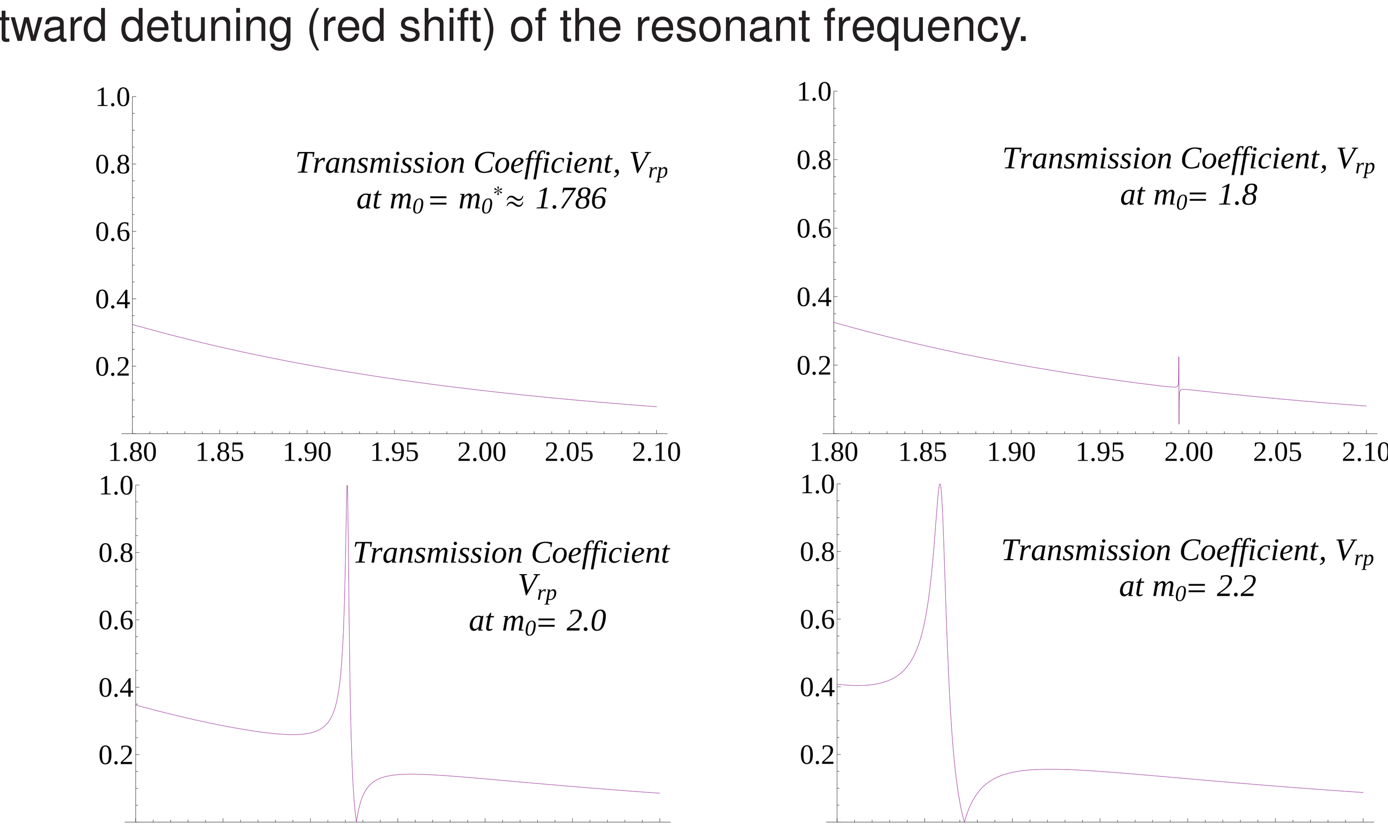
Answer: *Because λ is small near (m_0^*, m_1^*, ω^*), the response field is large compared to the source. This is the meaning behind resonance. So resonance occurs when a system admitting an embedded bound state is perturbed.*



The graph on the left shows the V_{re} part of the outgoing field with incoming field $\vec{J} = [1, 0, 0, 0]^T$ as a function of ω for a system that traps energy at $(m_0, m_1, \omega) = (m_0^*, m_1^*, \omega^*) \approx (1.786, 2.8381, 2)$, except that m_0 has been perturbed from 1.786 to 1.6. Equation (6) shows that the response is very large near (m_0^*, m_1^*, ω^*) since $\lambda(m_0^*, m_1^*, \omega^*) = 0$.

As the perturbed system vibrates near the frequency ω^*, the system resonates because its parameters are near those which can trap energy.

Graphs of the propagating transmission coefficient V_{rp} *vs.* frequency ω below show sharp anomalies near the frequency ω^* of the bound state that emerge as the mass m_0 is perturbed from the special mass m_0^*. One also observes a leftward detuning (red shift) of the resonant frequency.



Acknowledgments

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