MMET08: 12th International Conference on Mathematical Methods in Electromagnetic Theory, June 29–July 2, 2008, Odesa, Ukraine, pp. 532–534.

GUIDED MODES AND ANOMALOUS SCATTERING BY A PERIODIC LATTICE

Natalia Ptitsyna, Stephen P. Shipman

Louisiana State University, Department of Mathematics, Baton Rouge, LA 70803, USA
ptitsyna@math.lsu.edu

Abstract – **We study a discrete prototype of anomalous electromagnetic scattering associated with the interaction of guided modes of a periodic scatterer and plane waves incident upon the scatterer.**

I. INTRODUCTION

In a great many applications in the field of photonics, the propagation of electromagnetic waves in periodic slab structures is of central importance. In this paper, we analyze a discrete prototype. Our structure consists of a two-dimentional lattice of uniform masses and spring constants, which models the ambient space, and a periodic one-dimentional lattice, modeling a periodic slab scatterer, which is coupled to the $2D$ lattice along a line. The phenomenon that concerns us is *anomalous transmission of energy across the periodic scatterer*, and specifically, transmission anomalies that result from the interaction of plane-wave sources with guided modes of the scatterer. By transmission anomalies, we refer to sharp peaks and dips in the graph of the transmission coefficient as a function of frequency. A guided mode is a field that is exponentially localized to the scatterer, and the phenomenon of anomalous transmission occurs when such a mode is non-robust. This means that the mode exists at a specific frequency and wave number but ceases to exist at any nearby frequency under a perturbation of the wave number.

Such transmission anomalies have been observed in the scattering of electromagnetic fields by metal films as well as by dielectric slabs ([1], [4]) and appear more generally in problems of scattering of pane waves by a periodic slab structure, whether in electromagnetics, acoustics, elasticity, or lattice dynamics, and for both classical wave equations and Schrödinger equations. We have chosen to analyze a lattice model with Schrödinger equation of evolution because it is a simple prototype that illuminates the phenomenon of anomalous electromagnetic scattering through simple ideas and explicit calculations.

II. MATHEMATICAL FORMULATION

The dynamics of waves in our discrete structure are defined by a Schrödinger equation with generator given by the (minus) discrete uniform Laplacian Ω_2 in the 2D lattice, a periodic discrete Laplacian Ω_1 in the 1D lattice, and a simple coupling Γ between these two systems. The complex fields are denoted by $\{u_{mn}\}_{m,n=-\infty}^{\infty}$ in the 2D lattice and $\{z_n\}_{n=-\infty}^{\infty}$ in the 1D lattice (Figure, left):

$$i\frac{dz_n}{dt} = (\Omega_1 z)_n + (\Gamma u)_n, \tag{1}$$

$$i\frac{du_{mn}}{dt} = (\Gamma^\dagger z)_{mn} + (\Omega_2 u)_{mn}. \tag{2}$$

The coupling operator Γ is defined through its adjoint by

$$(\Gamma^\dagger z)_{mn} = \gamma_n z_n \text{ for } m = 0 \quad \text{and} \quad (\Gamma^\dagger z)_{mn} = 0 \text{ for } m \neq 0, \tag{3}$$

and Ω_1 is given by

$$(\Omega_1 z)_n = -\frac{k_n}{\sqrt{M_n M_{n+1}}} z_{n+1} + \frac{(k_n + k_{n-1})}{M_n} z_n - \frac{k_{n-1}}{\sqrt{M_n M_{n-1}}} z_{n-1}. \tag{4}$$

The masses M_n, spring constants k_n, and coupling constants γ_n are periodic of period N in n. Because of the self-adjointness of Ω_1 and Ω_2 and the use of Γ and $\Gamma^\dagger$ as the coupling operator, the coupled system is conservative.

Replacing the fields $\{u_{mn}\}$ and $\{z_n\}$ by $\{u_{mn}e^{-i\omega t}\}$ and $\{z_n e^{-i\omega t}\}$, we obtain the corresponding equations for harmonic fields,

$$\omega z_n = (\Omega_1 z)_n + (\Gamma u)_n, \tag{5}$$

$$\omega u_{mn} = (\Gamma^\dagger z)_{mn} + (\Omega_2 u)_{mn}. \tag{6}$$

We shall consider pseudo-periodic solutions to the system (5), with Bloch wave number κ in the n-direction. Such fields have finite Fourier decompositions in the n-variable:

$$z_n = \sum_{\ell=0}^{N-1} c_\ell e^{2\pi i \frac{\kappa+\ell}{N} n}, \qquad \begin{cases} u_{mn} = \displaystyle\sum_{\ell=0}^{N-1} (a_\ell^- e^{-2\pi i\theta_\ell m} + a_\ell^+ e^{2\pi i \theta_\ell m}) e^{2\pi i \frac{\kappa+\ell}{N} n}, & m<0, \\ u_{mn} = \displaystyle\sum_{\ell=0}^{N-1} (b_\ell^- e^{-2\pi i\theta_\ell m} + b_\ell^+ e^{2\pi i \theta_\ell m}) e^{2\pi i \frac{\kappa+\ell}{N} n}, & m>0, \end{cases} \tag{7}$$

where θ_ℓ is the horizontal component of a two-dimensional wave vector in the 2D lattice, determined by the dispersion relation for the operator Ω_2

$$\omega = 4 - 2\cos(2\pi\theta_\ell) - 2\cos(2\pi\phi_\ell), \tag{8}$$

and $\phi_\ell = (\kappa+\ell)/N$. Those values of ℓ for which θ_ℓ is real correspond to propagating Fourier harmonics (diffractive orders), and those values of ℓ for which θ_ℓ is imaginary correspond to evanescent harmonics. The square $[0,1]\times[0,8]$, which consists of real pairs (κ,ω) with κ in the Brillouin zone $[0,1]$ of wave numbers and ω in the spectrum $[0,8]$ of Ω_2, is divided into regions according to the number of propagating Fourier harmonics (Figure, middle).

The scattering problem consists of prescribing a plane-wave source field (incident on the scatterer from the left) $u_{mn}^{\text{inc}} = e^{2\pi i \frac{\kappa+\bar\ell}{N} n} e^{2\pi i \theta_{\bar\ell} m}$ for some fixed value of $\ell = \bar\ell$ for which θ_ℓ is real and solving for the total field $\{z_n\}, \{u_{mn}\}$. One requires that the difference of the total field and the incident field be outgoing. The problem can be reduced to a system of equations for the Fourier coefficients c_ℓ, a_ℓ^-, and b_ℓ^+,

$$B(\kappa,\omega)\overrightarrow{X} = \overrightarrow{F}, \tag{9}$$

where B is a $3N\times3N$ matrix, the vector $\overrightarrow{F}$ represents the source field (with coefficients $a_\ell^+ = \delta_{\ell\bar\ell}$), and the vector $\overrightarrow{X}$ represents the reflected and the transmitted fields a_ℓ^- and b_ℓ^+ (necessarily $b_\ell^- = 0$ by our choice of source field).

III. Guided Modes and Anomalous Transmission

A (generalized) guided mode is a nonzero solution of the sourceless equation

$$B(\kappa,\omega)\overrightarrow{X} = \mathbf{0}, \tag{10}$$

in which ω and κ are in general complex-valued. The pairs (κ,ω) for which such a solution exists satisfy the dispersion relation $\det(B(\kappa,\omega))=0$.

Let us illustrate the phenomenon of anomalous transmission near parameters (κ_0,ω_0) of a true guided mode through a numerical computation of scattering for a structure in which $N=2$, in the (κ,ω)-regime in which one of the Fourier harmonics is propagating and the other is evanescent (see the Figure). We use a and b to denote the coefficients of the propagating harmonic in the reflected and transmitted fields, respectively. The conditions for existence of a generalized guided mode reduce to

$$\frac{(\bar\gamma_1-\bar\gamma_0)}{(\bar\gamma_0+\bar\gamma_1)}\left(\frac{2}{M_1}-\frac{2}{M_0}\right) - \frac{\bar\gamma_0\bar\gamma_1(\gamma_0+\gamma_1)}{(\bar\gamma_0+\bar\gamma_1)i\sin(2\pi\theta_1)} + 2\omega - \frac{2}{M_0} - \frac{2}{M_1} - \frac{4\cos(\pi\kappa)}{\sqrt{M_0M_1}} = 0, \tag{11}$$

$$\frac{(\bar\gamma_1-\bar\gamma_0)}{(\bar\gamma_0+\bar\gamma_1)}\left(2\omega - \frac{2}{M_0} - \frac{2}{M_1} + \frac{4\cos(\pi\kappa)}{\sqrt{M_0M_1}}\right) + \frac{\bar\gamma_0\bar\gamma_1(\gamma_1-\gamma_0)}{(\bar\gamma_0+\bar\gamma_1)i\sin(2\pi\theta_1)} + \frac{2}{M_1} - \frac{2}{M_0} = 0, \tag{12}$$

where we take $\kappa\in[0,1/2)$ and $\omega\in(2-2\cos(\pi\kappa), 2+2\cos(\pi\kappa))$ and $\sin(2\pi\theta_1) = i\sqrt{(2-\frac{\omega}{2}+\cos(\pi\kappa))^2-1}$. This region is shown in the Figure, and the dotted curve in that region is a graph of Re (ω) in the dispersion relation. The solid dot shows an isolated pair (κ_0,ω_0) on this curve for which ω is real. This is the only point on this part of the dispersion curve for which the generalized mode is a true guided mode (decaying exponentially away from the 1D lattice), for we can prove the following analogue of Theorem 5 in [3]: *If (κ,ω) satisfies $\det(B(\kappa,\omega))=0$ and κ is real, then Im $\omega\le 0$, with equality if and only if the associated generalized guided mode decays exponentially away from the 1D lattice (in which case it is a true guided mode).* This means that the guided mode at (κ_0,ω_0) is nonrobust with respect to perturbations of κ from κ_0.

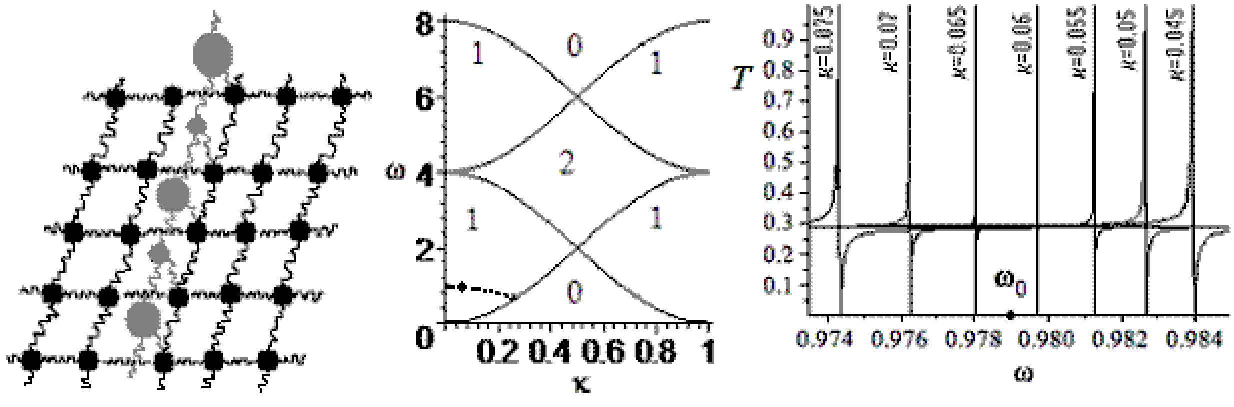

Figure. *Left:* Coupled lattices. *Middle:* The number of propagating harmonics for $N = 2$. The dashed line is the real part of the complex dispersion relation in the regime of one propagating harmonic for $M_0 = 2$, $M_1 = 1$, $\gamma_0 = 1$ and $\gamma_1 = 7$. The solid dot is an isolated point (κ_0, ω_0) at which ω_0 is real. *Right:* Transmission coefficient for various values of κ near κ_0.

The explicit nature of the dispersion relation makes it possible to obtain results concerning existence of guided modes. In particular, in the case of period 2, one can prove that if $\gamma_0 = \gamma_1$ or $\gamma_0 = -\gamma_1$, then there exist no true guided modes.

The right-hand graphs in the Figure show that a sharp resonance emanates from the guided-mode frequency ω_0 as the wave number κ is perturbed from κ_0. The anomaly widens as κ becomes larger. Using analytic perturbation theory as in [4], we derive a rigorous asymptotic formula for the anomaly that is independent of the period N and, in fact, describes transmission anomalies in a very general setting of scattering by periodic slab structures. This analysis is described in another work [5].

The graph of transmission indicates that the transmitted energy reaches a maximum of 100% and a minimum of 0% on the anomaly. This has also been observed in numerical calculations of the transmission of energy of electromagnetic fields across a lossless periodic dielectric slab [5]. In that case, a proof of this feature is still unavailable. In our discrete model, we can prove that the transmission anomaly does indeed reach 100% and 0% as a function of ω for values of κ in a neighborhood of κ_0. This holds if the γ_i are real and $N = 2$, assuming the generic conditions

$$\frac{\partial \det(B)}{\partial \omega}(\kappa_0, \omega_0) \neq 0, \quad \frac{\partial a}{\partial \omega}(\kappa_0, \omega_0) \neq 0, \quad \frac{\partial b}{\partial \omega}(\kappa_0, \omega_0) \neq 0. \tag{13}$$

The proof involves first explicit calculations that reduce the dispersion relation to a real-valued function of real pairs (κ, ω), and then an application of the Implicit Function Theorem.

ACKNOWLEDGMENT

The work of both authors was supported by NSF grant DMS-0505833, and N. Ptitsyna was supported by Louisiana State Board of Regents Student Travel Grant LEQSF(2005-2007)-ENH-TR-21.

REFERENCES

[1] T. W. Ebbesen, H. J. Lezec, H. F. Ghaemi, T. Thio, and P. A. Wolff, "Extraordinary optical transmission through sub-wavelength hole arrays," *Letters to Nature*, vol. 391, pp. 667–669, 1998.

[2] Natalia Ptitsyna, Ph.D. Dissertation in preparation, Louisiana State University, 2008.

[3] Stephen P. Shipman and Stephanos Venakides, "Resonance and bound states in photonic crystal slabs," *SIAM J. Appl. Math.*, vol. 64, no. 1, pp. 322–342, 2003.

[4] Stephen P. Shipman and Stephanos Venakides, "Resonant transmission near non-robust periodic slab modes," *Phys. Rev. E*, vol. 71, no. 1, pp. 026611-1–10, 2005.

[5] Natalia Ptitsyna, Stephen P. Shipman, and Stephanos Venakides, "Fano Resonance of Waves in Periodic Slabs," Proceedings of MMET, Odessa, Ukraine, 2008