AN EXACTLY SOLVABLE MODEL FOR NONLINEAR RESONANT SCATTERING

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Abstract of Presentation

The interaction of radiation with an open resonator is known to result in highly sensitive frequency dependence of the response field. This resonant behavior in the linear setting has been analyzed theoretically in a variety of continuous and discrete systems (*e.g.*, [1], [2], [3]). When nonlinearity is introduced into the structure, bistability emerges [4]. This means that the system admits multiple responses to the same exterior source field, at a fixed frequency and intensity. The purpose of this work is to provide an exactly solvable model in which bistability can be demonstrated analytically. Specifically, we study the effects of introducing nonlinearity into a system that, in the linear setting, exhibits resonance due to the perturbation of a system with an embedded eigenvalue.

The Model

A nonlinear resonator is excited through coupling to a point mass in a transmission line at the point $x = 0$, where x denotes the position along the line.

System equations

The system equations are

$$iu_t + u_{xx} = 0 \text{ for } x \neq 0, \text{ (transmission line)} \quad (1)$$

$$y(t) = u(0,t), \quad \text{(point mass on the line)} \quad (2)$$

$$i\dot{y} = \gamma z - (u_x(0^+,t) - u_x(0^-,t)), \quad (3)$$

$$i\dot{z} = \gamma y + E_0 z + \lambda |z|^2 z + \epsilon(t). \quad \text{(resonator)} \quad (4)$$

The field $u(x,t)$ on the transmission line is continuous at the point of coupling to the resonator, $u(0^+,t) = u(0^-,t)$. The field $y(t)$ of the mass at the coupling point on the string is driven by the resonator amplitude $z(t)$ and by the jump of the field gradient u_x at $x=0$ according to (3). The evolution of the resonator amplitude is given by (4). The number E_0 represents the natural frequency of the uncoupled, linearized resonator. The remaining three terms in (4) represent the coupling to the line, the nonlinearity, and an external forcing.

An incident wave $u^{\rm inc}$ of frequency $\omega > 0$ in the transmission line is scattered at the coupling point,

$$u^{\rm inc}(x,t) = Je^{ikx-i\omega t}, \quad k>0, \quad k^2 = \omega. \quad (5)$$

The composite parameter of nonlinearity

$$\mu = \lambda J^2$$

arises naturally in the analysis of the system.

Central Ideas

We are interested in time-harmonic scattering solutions and their stability. These exist due to the modulus in the nonlinear term $\lambda|z|^2 z$. In the analysis of their stability, the temporally localized forcing $\epsilon(t)$ in the resonator serves as a perturbative mechanism.

The system admits a time-harmonic bound state when the $\gamma = 0$ and for no other value of γ. This state consists simply of the resonator in oscillatory motion while the transmission line remains at rest. A nonzero γ destroys the bound state and results in resonance. For $\lambda = 0$, this is the result of the destruction of a linear bound state of an eigenvalue embedded in the continuous spectrum, as the eigenvalue is destroyed by a perturbation.

We discuss two distinguished asymptotic regimes in γ and μ, when both of these parameters are small. In the regime $\gamma^4/\mu \sim C$, the nonlinearity is just large enough to produce multiple responses to harmonic source fields near the resonant frequency E_0 and for very large frequencies. The regime $\gamma^2/\mu \sim C$ is the threshold at which the nonlinearity is large enough to produce multiple responses at all frequencies above $\omega \approx E_0$.

Linear stability analysis about harmonic solutions in the Laplace variable demonstrates the stability of scattering states when only one response is admitted and, in the case of three possible responses, the stability of the states with the largest and smallest response amplitudes and the instability of the intermediate one.

Time-Harmonic Scattering Fields

Response to harmonic forcing

The *response* of the resonator to the incident field (5) is defined as

$$\mathcal{R} = \left| \frac{z}{J} \right|^2. \quad (6)$$

We derive below the multi-valued response as well as the complex amplitude of the transmitted field, that is, the field $u(x,t)$ for $x>0$. A calculation reveals that the response satisfies a real cubic polynomial equation that involves the parameters γ, μ, ω, and E_0:

$$\mathcal{R}\left(\mu\mathcal{R}+\tfrac{\gamma^2}{\omega+4}-(\omega-E_0)\right)^2+\tfrac{4\gamma^4}{\omega(\omega+4)^2}\mathcal{R}-\tfrac{4\gamma^2}{\omega+4}=0. \tag{7}$$

Evidently, the system admits either one or three responses $\mathcal{R}$ to the incident field $Je^{ikx-i\omega t}$, with two responses at threshold frequencies. The topology of the graph of the multi-valued function $\mathcal{R}=\mathcal{R}(\omega)$ takes two forms depending on the value of the parameters, and we investigate these forms in the limit of small μ and γ, in which resonant phenomena are most pronounced. The passage from one form to the other occurs when $\omega_2=\omega_3$ in Fig. 1, in which $|T/J|^2$ instead of $\mathcal{R}$ is shown for easier viewing. In the graph of $\mathcal{R}$ *vs.* ω, the branches do not cross.

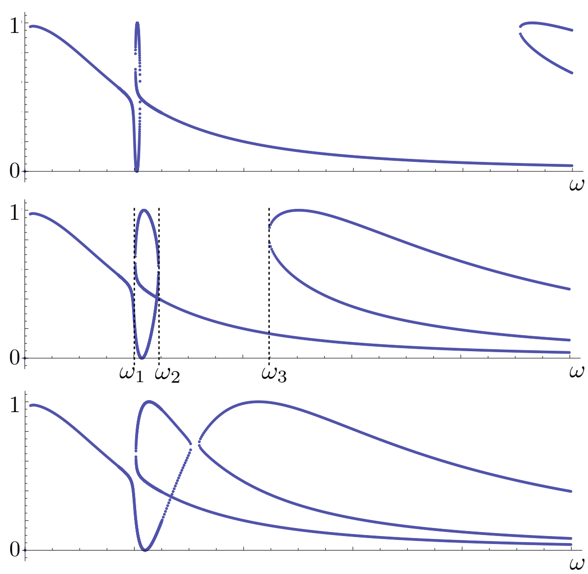

Figure 1: $|T/J|^2$ *vs.* ω showing the frequencies ω_1, ω_2, and ω_3 that separate regions of single and multiple solutions. The top graph is just after emergence of ω_1 and ω_2 and the bottom just after emergence of ω_2 and ω_3.

Specifically, we seek harmonic solutions to the problem of scattering of a sinusoidal wave in the transmission line by the resonator/defect pair, with $\epsilon(t)=0$. These solutions have the form

$$u(x,t)=\begin{cases} Je^{ikx-i\omega t}+Re^{-ikx-i\omega t}, & x<0,\\ Te^{ikx-i\omega t}, & x>0,\end{cases} \tag{8}$$

$$y(t)=Te^{-i\omega t}, \tag{9}$$

$$z(t)=Ae^{-i\omega t}, \tag{10}$$

in which J, R, T, and A are complex constants and $k^2=\omega$. By the continuity of u at $x=0$, $J+R=T$, and it follows that

$$u_x(0^+,t)-u_x(0^-,t)=2ik(T-J)e^{-i\omega t}. \tag{11}$$

Inserting this form into equations (3,4) yields

$$\omega T=\gamma A-2ik(T-J), \tag{12}$$

$$\omega A=E_0A+\gamma T+\lambda|A|^2A, \tag{13}$$

whence

$$\left(\omega-E_0-\frac{\gamma^2}{\omega+2ik}-\lambda|A|^2\right)A=\frac{2ik\gamma J}{\omega+2ik}, \tag{14}$$

$$T=\frac{\gamma A+2ikJ}{\omega+2ik}. \tag{15}$$

The first of these equations results in the polynomial equation (7) for $\mathcal{R}$. By putting $|A|^2=\mathcal{R}J^2$ into (14), for any solution $\mathcal{R}$ of (7), one can solve for A, and then T is determined by (15).

Asymptotics of Response

The change of variable

$$P=\frac{\mu\mathcal{R}}{a},\quad a:=\omega-E_0-\frac{\gamma^2}{\omega+4},$$

transforms equation (7) into

$$P(P-1)^2+\alpha P-\beta=0, \tag{16}$$

in which

$$\alpha:=\frac{4\gamma^4}{\omega\left((\omega-E_0)(\omega+4)-\gamma^2\right)^2}, \tag{17}$$

$$\beta:=\frac{4\gamma^2\mu(\omega+4)^2}{\left((\omega-E_0)(\omega+4)-\gamma^2\right)^3}. \tag{18}$$

The points ω_i in Fig. 1 are characterized by both (16) and the vanishing of the derivative with respect to P of the left-hand side of (16). This pair of conditions can be expressed equivalently as the pair

$$\alpha=-3P^2+4P-1, \tag{19}$$

$$\beta=-2P^3+2P^2. \tag{20}$$

The quantities α and β as functions of P are depicted in Fig. 2 via their graphs as well as in the (α,β)-plane as a curve parameterized by P. Because $\alpha>0$, the values of P must be restricted to the interval $(1/3,2/3)$. The graphs of α and β intersect tangentially at their common root of $P=1$, and both reach their maximal point at $P=2/3$. This latter observation is the reason for the cusp at $(1/3,8/27)$ on the curve $(\alpha(P),\beta(P))$.

The intersection of this parameter-independent curve with the parameter-dependent curve $(\alpha(\omega),\beta(\omega))$ defined by (17,18) determines the frequencies ω_i.

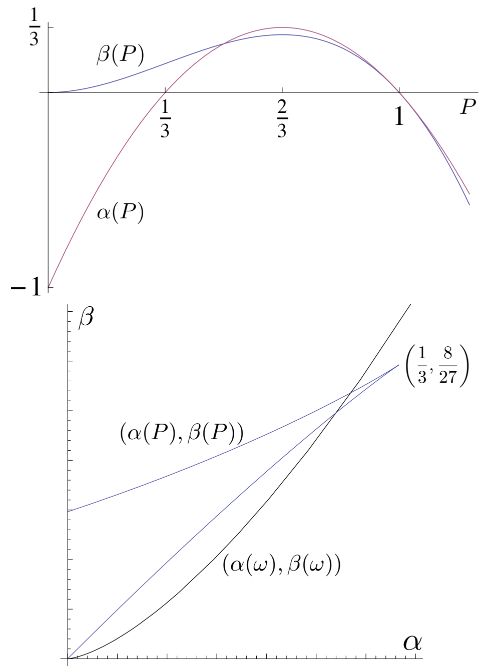

Figure 2: Top: Graphs of $\alpha(P)$ and $\beta(P)$ *vs.* P. Bottom: The parameterized curve $(\alpha(\omega), \beta(\omega))$ for particular values of γ, μ, and E_0, and the parameter-independent curve $(\alpha(P), \beta(P))$.

The ω_1-ω_2 bifurcation

This bifurcation occurs when $(\alpha(\omega), \beta(\omega))$ intersects the cusp. Setting $\alpha = \frac{1}{3}$ and $\beta = \frac{8}{27}$ in (17,18) leads to

$$\gamma^2 = \frac{k(k^2 - E_0)(k^2+4)}{2\sqrt{3}+k}, \tag{21}$$

$$\mu = \frac{16\sqrt{3}(k^2-E_0)^2}{9k(2\sqrt{3}+k)^2}. \tag{22}$$

It is interesting that, for each frequency $\omega = k^2 > E_0$, there are values of γ and μ, satisfying $\gamma^4/\mu = 3\sqrt{3}k^3(k^2+4)^2/16$, for which the ω_1-ω_2 bifurcation occurs at ω. We are interested in the case of small μ and γ, for which the second equation of (21) shows that the bifurcation occurs close to the resonant frequency, E_0. Thus a fourth-power relation between γ and μ emerges,

$$\frac{\gamma^4}{\mu} \sim \frac{3\sqrt{3}}{16} E_0^{3/2}(E_0+4)^2, \quad \text{when } \omega_1 = \omega_2, \quad \mu \to 0.$$

Meanwhile, as this bifurcation takes place, the frequency ω_3 is far away. Indeed, when $\omega_1 = \omega_2$, one finds that, as $\mu \to 0$, α and β tend to zero if $\omega > E_0$ is bounded away from E_0. Thus, $P \to 1$, and, because their graphs intersect tangentially at $P = 1$, the ratio of α and β tends to unity and one obtains finally

$$\omega_3 \sim \frac{C}{\gamma^2} \quad \text{when } \omega_1 = \omega_2, \quad \mu, \gamma \to 0. \tag{23}$$

Here, C is equal to the asymptotic value of γ^4/μ above.

The ω_2-ω_3 bifurcation

Analysis of this bifurcation is a bit more delicate; it occurs when the curve $(\alpha(\omega), \beta(\omega))$ intersects the curve $(\alpha(P), \beta(P))$ tangentially very very near the origin. The main result is the power law

$$\frac{\gamma^2}{\mu} \sim \text{const}, \quad \text{when } \omega_2 = \omega_3, \quad \mu \to 0.$$

In this regime, nonlinearity is much stronger in relation to the coupling than in the regime of the ω_1-ω_2 bifurcation.

Stability of Harmonic Solutions

Let us assume that, for $t \leq 0$, the system is in a harmonic scattering state $(u_h(x,t), y_h(t), z_h(t))$ of the form (8,9,10). The state is perturbed at $t > 0$, by forcing the resonator by a small-amplitude, temporally localized function $\epsilon(t)$. Write the solution to (1–4) as

$$\begin{aligned} u(x,t) &= u_h(x,t) + q(x,t), \\ y(t) &= y_h(t) + \eta(t), \\ z(t) &= z_h(t) + \zeta(t), \end{aligned}$$

in which $q(x,t)$ satisfies an outgoing condition as $|x| \to \infty$. The deviation $(q(x,t), \eta(t), \zeta(t))$ of the solution from the harmonic one vanishes at $t = 0$, as does $\epsilon(t)$, and it satisfies the system

$$iq_t + q_{xx} = 0 \quad \text{for } x \neq 0, \tag{24}$$

$$\eta(t) = q(0,t), \tag{25}$$

$$i\dot{\eta} = \gamma\zeta - (q_x(0^+,t) - q_x(0^-,t)), \tag{26}$$

$$\begin{aligned} i\dot{\zeta} = E_0\zeta + \gamma\eta + \lambda\big(2|z_h|^2\zeta + z_h^2\bar{\zeta} + \\ + 2z_h|\zeta|^2 + \bar{z}_h\zeta^2 + |\zeta|^2\zeta\big) + \epsilon(t). \end{aligned} \tag{27}$$

It is generally understood that, at frequencies for which the system admits only one harmonic solution (for a given system and source-field amplitude J), this field is stable, that is, u relaxes back to the original harmonic solution, or $q(x,t) \to 0$, as $t \to \infty$. At frequencies for which there are three harmonic solutions, one should have bistability, that is, two solutions are stable and one unstable, the unstable one being that with intermediate response amplitude (see, *e.g.*, [4]). Moreover, u should tend to one of the stable scattering solutions as $t \to \infty$.

Toward the end of making these ideas rigorous, we carry out the linear stability analysis about harmonic scattering fields.

Reduction of the system to the resonator

Because of the radiation condition on $q(x,t)$ and the continuity at $x = 0$, q is spatially symmetric, that is, $q(x,t) = q(-x,t)$ and therefore

$$q_x(0^+,t) - q_x(0^-,t) = 2q_x(0^+,t). \quad (28)$$

This can be seen more clearly through the Laplace transform, $is\hat{q} + q_{xx} = 0$, which yields, due to the radiation condition,

$$\hat{q}_x = \begin{cases} -i^{3/2}\sqrt{s}\,\hat{q} & x < 0, \\ i^{3/2}\sqrt{s}\,\hat{q} & x > 0, \end{cases}$$

in which $\arg(i^{3/2}) = 3\pi/4$. Thus,

$$\mathcal{L}\left[q_x(0^+,t) - q_x(0^-,t)\right] = 2i^{3/2}\sqrt{s}\,\hat{\eta} \quad (29)$$

Equation (26) now yields a relation between η and ζ,

$$\eta(t) = \gamma\mathcal{L}^{-1}\left[\hat{g}(s)\hat{\zeta}\right](t) = \gamma(g * \zeta)(t), \quad (30)$$

in which

$$\hat{g}(s) = \frac{-i}{s + 2\sqrt{is}} \quad (31)$$

with the branch cut for $\surd$ along $x \leq 0$. The relation (30) allows one to project the system onto the resonator by considering equation (27) for ζ alone.

Linearization about a harmonic solution

Equation (27) is linearized by eliminating the quadratic and cubic terms in ζ and replacing $\zeta(t)$ with the solution $\xi(t)$ of the resulting linear equation. It is convenient to deal with the field $\psi(t) = \xi(t)e^{i\omega t}$. Keeping in mind that $z_h = Ae^{-i\omega t}$, one arrives at the following equation for ψ:

$$i\dot{\psi} + \omega\psi = E_0\psi + \gamma p * \psi + \\ + \lambda\big(2|A|^2\psi + A^2\bar{\psi}\big)\epsilon(t) + e^{i\omega t}, \quad (32)$$

in which $p(t) = g(t)e^{i\omega t}$. In the Laplace variable, this is

$$\big(is + \omega - E_0 - \gamma\hat{p} - 2\lambda|A|^2\big)\hat{\psi} - \lambda A^2\hat{\bar{\psi}} = \hat{\epsilon}|_{s-i\omega}, \quad (33)$$

$$\big(-is + \omega - E_0 - \gamma\hat{\bar{p}} - 2\lambda|A|^2\big)\hat{\bar{\psi}} - \lambda\bar{A}^2\hat{\psi} = \hat{\bar{\epsilon}}|_{s+i\omega}. \quad (34)$$

The second equation is obtained by conjugating the first, replacing s with $\bar{s}$, and then using the rule $\bar{\hat{f}}(s) = \hat{\bar{f}}(\bar{s})$. All quantities are analytic in s within their domain of definition. The determinant of this system is

$$D(s) = 3\lambda^2|A|^4 - 4\lambda|A|^2\big(\omega - E_0 - \gamma(\widehat{\mathrm{Re}\,p})\big) + \\ + (-is + \omega - E_0 - \gamma\hat{\bar{p}})(is + \omega - E_0 - \gamma\hat{p}). \quad (35)$$

In this expression, we must keep in mind that

$$\hat{p}(s) = \frac{-i}{s - i\omega + 2i^{1/2}\sqrt{s - i\omega}},$$

$$\hat{\bar{p}}(s) = \frac{i}{s + i\omega + 2(-i)^{1/2}\sqrt{s + i\omega}}.$$

Thus $D(s)$ has two branch cuts along the half-lines $\{x \pm i\omega, x \leq 0\}$.

Whether $\psi(t)$ grows or decays as $t \to \infty$, that is, the linear stability of the system about a scattering solution, depends on the roots of D. Roots in the right-half s-plane indicate exponential growth, and roots in the left-half plane indicate decay.

In terms of the quantity $P = \mu\mathcal{R}/a = \lambda|A|^2/a$, with $a = \omega - E_0 - \gamma^2/(\omega + 4)$, D has the form

$$D = 3P^2 - 4P\frac{1}{a}\left(\omega - E_0 - \frac{\gamma}{2}(\hat{p} + \hat{\bar{p}})\right) + \\ + \frac{1}{a^2}(\omega - E_0 - is - \gamma\hat{\bar{p}})(\omega - E_0 + is - \gamma\hat{p}). \quad (36)$$

An analysis of the roots of $D(s)$ at the values of P that solve (16) is forthcoming. So far, numerical computations corroborate the expectation that the intermediate solution P of (16) corresponds to an unstable scattering state whereas the lower and upper solutions correspond to stable states, as do solutions in regimes of unique response.

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