Guided modes in periodic slabs: existence and nonexistence

Stephen Shipman
Louisiana State University

Darko Volkov
Worcester Polytechnic Institute

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Abstract. For homogeneous lossless 3D periodic slabs of fixed arbitrary geometry, we characterize guided modes by means of the eigenvalues associated to a variational formulation. We treat robust modes, which exist for frequencies and wavevectors that admit no propagating Bragg harmonics and therefore persist under perturbations, as well as nonrobust modes, which can disappear under perturbations due to radiation loss. We prove the nonexistence of guided modes, both robust and nonrobust, in "inverse" structures, for which the celerity inside the slab is less than the celerity of the surrounding medium. The result is contingent upon a restriction on the width of the slab but is otherwise independent of its geometry.

Keywords: guided mode, periodic slab, photonic crystal, nonexistence in inverse crystals

1 Introduction

The subject of our investigation is the existence and nonexistence of linear scalar waves guided by periodically structured lossless material slabs (Fig. 1). These *guided modes* occur in linear acoustic theory and, in the two-dimensional reduction, in which the structure is invariant in one of the two directions of periodicity, they describe guided polarized electromagnetic fields. Guided modes are characterized by their frequency and Bloch wavevector in the plane of periodicity, and they decay exponentially with distance away from the slab.

We distinguish between two types of guided mode. Those of the first type cannot be destroyed by radiation losses under perturbation because they possess a frequency and wavevector for which no Bragg, or Fourier, harmonics propagate away from the slab (they are all evanescent); we call these *robust guided modes.* Those possessing frequency and wavevector for which some propagating Bragg harmonics exist can be destroyed by radiation loss by "coupling" to these harmonics under perturbation of the structure, frequency, or wavevector. These *nonrobust guided modes* are known to be connected with anomalous scattering behavior in the vicinity of the frequency and wavenumber of the mode.

It is recognized that guided modes as well as the transmission anomalies associated with them will be useful in the design of photonic devices. These phenomena appear in many different photonic structures, and there is a large body of literature devoted to them. We mention just a few references. Anomalous transmission is typically characterized by sharp dips and peaks in the transmission coefficient. An in-depth computational analysis of their relation to leaky modes for slabs that are invariant in the transverse direction is given in Tikhodeev, et. al., [1]. Explicit asymptotic formulas for very general geometry for some types of perturbations have been calculated by Shipman and Venakides [2]. The connection between transmission enhancement and particular types of guided mode on metal films called "surface plasmons" has been studied in a series of papers by several authors; see [3] and [4], for example. An important class of guided modes that we do not treat here consists of those in optical fibers or periodic pillars (see [5], for example). Our present study focuses on the existence and nonexistence of guided modes in lossless dielectric slabs.

The existence of guided modes can be proved using variational principles. Bonnet-Bendhia and Starling [6] treat two-dimensional electromagnetic structures consisting of lossless penetrable and conducting components in which the dielectric coefficient is essentially an arbitrary function. The treatment of nonrobust modes is more delicate because establishing their existence requires proving the vanishing of the propagating Bragg harmonics. The frequencies of these modes, for a given wavevector, are called (as in [6]) *singular frequencies* of the problem of scattering, or diffraction, of plane waves by the slab.

In our study, we consider three-dimensional homogeneous dielectric structures. In this case we are able to prove a nonexistence theorem for *inverse structures*. An inverse structure is one for which the speed of waves, or the celerity, is higher inside the structure than in the surrounding medium. This result is easily understood through the following example. It is simple to calculate fields that are totally internally reflected within an infinite pane of glass surrounded by air. However, if the roles of the air and the glass are switched, such fields no longer exist. A similar result is expected for slabs with more general geometry. This is the content of Theorem 4.1. The result is subject to a restriction on the width of the slab, which depends on the frequency and wavevector; we do not know if this restriction is necessary or if it is only a artifact of our method of proof.

For the existence theory, we include complete proofs in the Appendix to make the work coherent and self-contained and in order to set the context and notation for the proof of the nonexistence result.

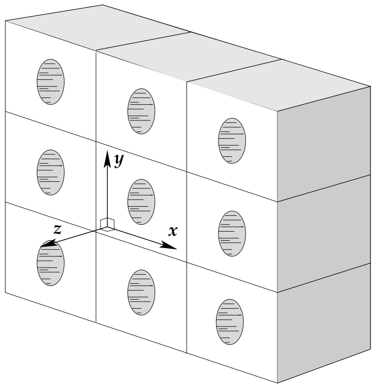


Figure 1: A slab structure periodic in the x and y directions and finite in the z direction.

The governing equation of dynamics is the linear wave equation arising in small-amplitude acoustic theory:

$$\varepsilon \frac{\partial^2}{\partial t^2} w(x,y,z,t) \;=\; \nabla\cdot \frac{1}{\mu}\nabla w(x,y,z,t). \tag{1}$$

The positive material parameters ε and μ depend in general on the position within the slab, but are constant outside of the slab. We will restrict our analysis to slabs in which these parameters are constant inside. The spatial factor $\tilde{u}(x,y,z)$ of a time-harmonic solution

$$w(x,y,z,t) = \tilde{u}(x,y,z)\, e^{-i\omega t}$$

is described by the Helmholtz equation

$$\nabla\cdot \frac{1}{\mu}\nabla \tilde{u}(x,y,z) + \varepsilon\omega^2\, \tilde{u}(x,y,z) \;=\; 0. \tag{2}$$

We are interested in solutions of the Helmholtz equation that are of the pseudoperiodic form

$$\tilde{u}(x,y,z) = u(x,y,z)\, e^{i(\kappa_1 x + \kappa_2 y)}, \quad u \text{ periodic in } x \text{ and } y,$$

in which $u(x,y,z)$ has the same periods as the guiding slab structure. The vector $\boldsymbol{\kappa} = \langle \kappa_1, \kappa_2, 0\rangle$ is known as the *Bloch wavevector*, and the field $\tilde{u}$ is called a Bloch wave. Such a solution to the Helmholtz equation gives rise to a solution of the linear wave equation when multiplied by a harmonic factor in t:

$$w(x,y,z,t) = u(x,y,z) e^{i(\kappa_1 x + \kappa_2 y - \omega t)}.$$

This solution is a plane wave traveling in the direction of the vector $\boldsymbol{\kappa}$ with wave number $|\boldsymbol{\kappa}| = \sqrt{\kappa_1^2 + \kappa_2^2}$, frequency ω, and speed $\omega/|\boldsymbol{\kappa}|$, modulated periodically through multiplication by the factor $u(x,y,z)$.

Fundamental to the structure of Bloch waves is their decomposition in the x and y variables into Fourier harmonics, often called Bragg harmonics, in the regions away from the slab ($|z|$ sufficiently large):

$$u(x,y,z) = \sum_{m,n=-\infty}^{\infty} (c^+_{mn} e^{\nu_{mn} z} + c^-_{mn} e^{-\nu_{mn} z}) e^{i(mx+ny)}.$$

Each element of the sum is a separable solution to the Helmholtz equation, and the coefficients c^+_{mn} and c^-_{mn} differ from one side of the slab to the other. The exponents ν_{mn}, as explained in more detail below, are purely imaginary for a finite number of pairs (m,n), corresponding to the *propagating Fourier harmonics.* For all other pairs, assuming $\nu_{mn} \neq 0$ for all (m,n), this exponent is real, and boundedness of u requires that the coefficients of the exponentially growing components vanish. Thus these pairs correspond to the evanescent harmonics. Assuming then that u is bounded, we can say that *a guided mode is supported by the slab structure if the coefficients of all propagating Fourier harmonics vanish.*

The paper is organized as follows. In section 2, we formulate a precise definition of a guided mode in its strong and weak forms. The role of the vanishing of the radiating Fourier harmonics to exclude radiation losses is made explicit. In section 3, we discuss the existence of sequences of material constants, depending on the geometry of the structure, the wavevector, and the frequency, that admit guided modes in the regime of no radiating Fourier harmonics. We also prove the existence of sequences of material constants for structures symmetric about a plane, depending on the frequency and wave number along the plane of symmetry, for which guided modes that travel parallel to the plane of symmetry exist. All of the proofs are deferred to the Appendix. In section 4, we prove that guided modes cannot exist in "inverse" slab structures; specifically, we show that, under a suitable restriction on their width, slabs with constant μ and ϵ whose value interior to slab is less than its value in the exterior, never admit guided modes. In section 5, we show a few numerical computations of nonrobust guided modes.

2 Mathematical formulation of guided modes

A variational description of guided modes requires truncation of the domain in the z-direction (directed away from the slab) and the introduction of an auxiliary parameter α, serving as the eigenvalue, by which the coefficient ϵ is multiplied within the truncated domain (Fig. 2). The monotonicity of certain eigenvalue sequences $\alpha_j(\epsilon_1)$ with respect to the interior constant ϵ_1 is utilized in the proof of nonexistence of modes in Section 4.

Let $\tilde{\Omega}$ denote a domain in $\mathbb{R}^3$ with C^2 boundary $\partial\tilde{\Omega}$ that is bounded in the z-direction and 2π-periodic in the x- and y-directions (Fig. 2). This means that *i.* There are numbers $z_1 < z_2$ such that

$$z_1 < \inf\{z : (x, y, z) \in \tilde{\Omega}\} < \sup\{z : (x, y, z) \in \tilde{\Omega}\} < z_2.$$

ii. If $(x, y, z) \in \tilde{\Omega}$, then $(x + 2\pi j, y, z) \in \tilde{\Omega}$ and $(x, y + 2\pi j, z) \in \tilde{\Omega}$ for each integer j.

Let $\mathcal{S}$ denote the infinite square cylinder containing one period of $\tilde{\Omega}$:

$$\mathcal{S} = \{(x, y, z) : -\pi < x < \pi,\ -\pi < y < \pi\},$$

and denote by Ω the part of $\tilde{\Omega}$ contained in $\mathcal{S}$, constituting one period of $\tilde{\Omega}$:

$$\Omega = \tilde{\Omega} \cap \mathcal{S} = \{(x, y, z) \in \tilde{\Omega} : -\pi < x < \pi,\ -\pi < y < \pi\}$$

and by Σ the part of $\partial\tilde{\Omega}$ contained in $\mathcal{S}$:

$$\Sigma = \partial\tilde{\Omega} \cap \mathcal{S}.$$

The boundary $\partial\Omega$ of Ω includes Σ and possibly parts of the boundary $\partial\mathcal{S}$ of $\mathcal{S}$. Denote by $\mathcal{R}$ the part of $\mathcal{S}$ between $z = z_1$ and $z = z_2$:

$$\mathcal{R} = \{-\pi < x < \pi,\ -\pi < y < \pi,\ z_1 < z < z_2\}$$

and by $\Gamma = \Gamma_1 \cup \Gamma_2$ the square parts of the boundary of $\mathcal{R}$ parallel to the xy-plane:

$$\begin{aligned}
\Gamma_1 &= \{(x, y, z) : -\pi < x < \pi,\ -\pi < y < \pi,\ z = z_1\},\\
\Gamma_2 &= \{(x, y, z) : -\pi < x < \pi,\ -\pi < y < \pi,\ z = z_2\}.
\end{aligned}$$

We fix outward-pointing normal vectors n to all of the surfaces, as shown in Figure 2. Let the piecewise constant functions ε and μ be defined by

$$\varepsilon(r) = \begin{cases} \alpha\epsilon_1, & r \in \Omega \\ \alpha\epsilon_0, & r \in \mathcal{R} \setminus \Omega \\ \epsilon_0, & r \in \mathcal{S} \setminus \mathcal{R} \end{cases}, \qquad \mu(r) = \begin{cases} \mu_1, & r \in \Omega \\ \mu_0, & r \in \mathcal{S} \setminus \Omega \end{cases}, \tag{3}$$

in which ϵ_0, ϵ_1, α, μ_0, and μ_1 are fixed positive numbers.

We will be considering guided modes in the augmented structure consisting of the flat slab in $\mathbb{R}^3$ filling the space between the planes $z = z_1$ and $z = z_2$ with the periodic structure $\tilde{\Omega}$ embedded in it. For $\alpha = 1$, this augmented structure reduces to $\tilde{\Omega}$ itself. We denote the augmented structure by $\tilde{\Omega}_{\text{aug}}$:

$$\tilde{\Omega}_{\text{aug}} = \text{the augmented structure in Fig. 2 extended periodically to } \mathbb{R}^3.$$

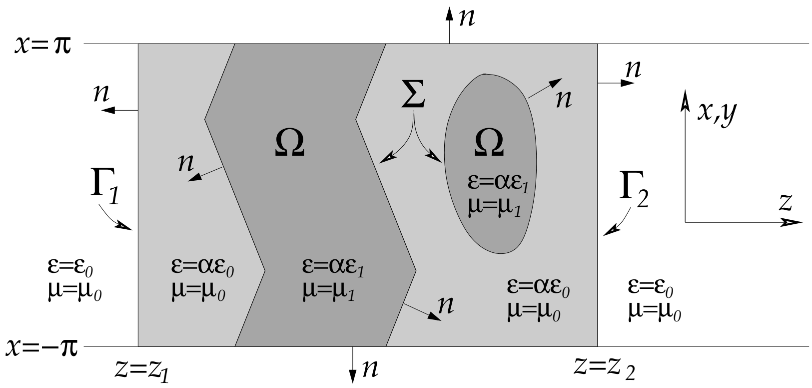


Figure 2: A two-dimensional depiction of one period of a possible augmented three-dimensional slab structure.

When referring to $\tilde{\Omega}_{\rm aug}$, the material constants (3), repeated periodically, are tacitly assumed.

Given a frequency ω and Bloch wavevector $\boldsymbol{\kappa} = \langle \kappa_1, \kappa_2, 0\rangle$, we seek solutions $\tilde{u} = \tilde{u}(x,y,z)$ of the Helmholtz equation (2) in $\mathcal{S}$ with $\boldsymbol{\kappa}$-pseudoperiodic boundary conditions in x and y, that is,

$$\tilde{u}(\pi,y,z) = e^{2\pi i\kappa_1}\tilde{u}(-\pi,y,z), \quad \partial_n\tilde{u}(\pi,y,z) = -e^{2\pi i\kappa_1}\partial_n\tilde{u}(-\pi,y,z), \tag{4}$$

$$\tilde{u}(x,\pi,z) = e^{2\pi i\kappa_2}\tilde{u}(x,-\pi,z), \quad \partial_n\tilde{u}(x,\pi,z) = -e^{2\pi i\kappa_2}\partial_n\tilde{u}(x,-\pi,z). \tag{5}$$

The minus signs arise because the normal vector n to $\partial\mathcal{S}$ is always taken to point out of $\mathcal{S}$. Such a solution can be extended to a $\boldsymbol{\kappa}$-pseudoperiodic solution in $\mathbb{R}^3$, that is, retaining $\tilde{u}$ to denote this extension,

$$\tilde{u}(x,y,z) = e^{i(\kappa_1 x+\kappa_2 y)}u(x,y,z),$$

in which $u(x,y,z)$ is 2π-periodic in x and y. It is convenient to work with the function u, which has periodic boundary conditions in $\mathcal{S}$:

$$u(\pi,y,z) = u(-\pi,y,z), \quad \partial_n u(\pi,y,z) = -\partial_n u(-\pi,y,z), \tag{6}$$

$$u(x,\pi,z) = u(x,-\pi,z), \quad \partial_n u(x,\pi,z) = -\partial_n u(x,-\pi,z). \tag{7}$$

The Helmholtz equation (2) for $\tilde{u}$ is equivalent to the following modified equation for the periodic factor u:

$$(\nabla + i\boldsymbol{\kappa})\cdot \frac{1}{\mu}\,(\nabla + i\boldsymbol{\kappa})\,u + \varepsilon\omega^2 u \;=\; 0. \tag{8}$$

In the precise formulation of a guided mode below, Condition 2.2, we make clear the implied behavior of a solution to this equation at the surfaces of discontinuity of ε and μ, namely, Σ and Γ.

We take $\boldsymbol{\kappa} = \langle \kappa_1, \kappa_2, 0 \rangle$ to lie in the first symmetric Brillouin zone pertaining to our structure, which is 2π-periodic in x and y, that is

$$-1/2 \leq \kappa_1 < 1/2 \quad \text{and} \quad -1/2 \leq \kappa_2 < 1/2.$$

In the intervals $(-\infty, z_1)$ and (z_2, ∞), every periodic solution u of (8) is equal to a superposition of Fourier harmonics:

$$\begin{aligned} u(x,y,z) &= \sum_{m,n=-\infty}^{\infty} (a_{mn}^{+} e^{\nu_{mn} z} + a_{mn}^{-} e^{-\nu_{mn} z}) e^{i(mx+ny)}, \quad z < z_1, \\ u(x,y,z) &= \sum_{m,n=-\infty}^{\infty} (b_{mn}^{+} e^{\nu_{mn} z} + b_{mn}^{-} e^{-\nu_{mn} z}) e^{i(mx+ny)}, \quad z > z_2, \end{aligned} \tag{9}$$

in which

$$\nu_{mn}^2 = -\epsilon_0 \mu_0 \omega^2 + (m+\kappa_1)^2 + (n+\kappa_2)^2, \tag{10}$$

provided that $\nu_{mn}^2 \neq 0$ for all integer pairs (m,n). If $\nu_{mn}^2 = 0$ for some pair (m,n), then its contribution to the sum (9) must be replaced by

$$\begin{array}{ll} (a_{mn}^{+} + a_{mn}^{-} z) e^{i(mx+ny)}, & z < z_1, \\ (b_{mn}^{+} z + b_{mn}^{-}) e^{i(mx+ny)}, & z > z_2. \end{array} \tag{11}$$

For a finite number of pairs (m,n) we have $\nu_{mn}^2 < 0$, and we take $\operatorname{Im} \nu_{mn} > 0$; these correspond to the harmonics in (9) whose two terms have constant modulus and oscillate as functions of z. We denote this set of *propagating harmonics* by $\mathcal{P}$:

$$\mathcal{P} = \left\{ (m,n) \in \mathbb{Z}^2 : \nu_{mn}^2 < 0 \right\}. \qquad \text{(propagating Fourier harmonics)} \tag{12}$$

We call the harmonics of the form (11), for which $\nu_{mn}^2 = 0$, the *linear harmonics*. We denote the union of the linear and propagating harmonics by $\tilde{\mathcal{P}}$:

$$\tilde{\mathcal{P}} = \left\{ (m,n) \in \mathbb{Z}^2 : \nu_{mn}^2 \leq 0 \right\}. \quad \text{(linear and propagating Fourier harmonics)} \tag{13}$$

For a generic set of parameters ϵ_0, μ_0, α, $\boldsymbol{\kappa}$, and ω, there are no linear harmonics, that is, $\tilde{\mathcal{P}} = \mathcal{P}$. For all pairs such that $\nu_{mn}^2 > 0$ we take $\operatorname{Re} \nu_{mn} > 0$; these correspond to the exponential harmonics. We require the solution u to be bounded, so that

$$a_{mn}^{-} = 0 \quad \text{and} \quad b_{mn}^{+} = 0 \quad \text{for all linear and exponential harmonics} \tag{14}$$

to exclude unbounded growth as $|z| \to \infty$. The harmonics that are exponentially decaying as $|z| \to \infty$ are called the decaying harmonics, or *evanescent harmonics*.

The energy conservation law holds for solutions of the Helmholtz equation. This means that the the time-averaged energy flux through in $\mathcal{S}$ through planes parallel to the xy-plane is independent of z. Only the propagating harmonics contribute to this energy, and equating its values through Γ_1 and Γ_2 gives

$$\sum_{(m,n)\in\mathcal{P}} \nu_{mn} \left(|a^+_{mn}|^2 - |a^-_{mn}|^2\right) = \sum_{(m,n)\in\mathcal{P}} \nu_{mn} \left(|b^+_{mn}|^2 - |b^-_{mn}|^2\right). \tag{15}$$

A *guided mode* u, which we will define precisely in definition 2.3, is a nonzero solution of the Helmholtz equation with exponential decay as $|z| \to \infty$. If u satisfies the condition (14) of boundedness as well as the vanishing of the linear and propagating harmonics, that is, $a^+_{mn} = a^-_{mn} = b^+_{mn} = b^-_{mn} = 0$ for all $(m,n) \in \tilde{\mathcal{P}}$, then u has exponential decay as $|z| \to \infty$, and its periodic extension to $\mathbb{R}^3$ is a guided mode. In the generic case that there are no linear harmonics, that is, $\nu^2_{mn} \neq 0$ for each (m,n), we may characterize guided modes by the condition that

$$a^-_{mn} = 0 \quad \text{and} \quad b^+_{mn} = 0 \qquad \text{for all } (m,n) \qquad (\text{if } \tilde{\mathcal{P}} = \mathcal{P}). \tag{16}$$

Indeed, (16) and (15) together imply in this case the *vanishing of all propagating harmonics as well as all exponentially growing harmonics.* In the general case, in which linear harmonics may exist, we must augment this condition to exclude these harmonics explicitly:

$$\begin{aligned} a^-_{mn} = 0 \quad \text{and} \quad b^+_{mn} = 0 \qquad &\text{for all} \quad (m,n),\\ a^+_{mn} = 0 \quad \text{and} \quad b^-_{mn} = 0 \qquad &\text{if} \quad \nu^2_{mn} = 0. \end{aligned} \tag{17}$$

The reason for treating the case of no linear harmonics specially is that the condition (16) is simple. In fact, it is equivalent to the condition that u obey the following Dirichlet-to-Neumann map on Γ defined in terms of the Fourier coefficients of u restricted to Γ_1 and Γ_2:

$$\begin{array}{lll} \text{If} & u|_{\Gamma_1} = \sum a_{mn} e^{i(mx+ny)} & \text{and} \quad u|_{\Gamma_2} = \sum b_{mn} e^{i(mx+ny)}, \\ \text{then} & \partial_n u|_{\Gamma_1} = -\sum \nu_{mn} a_{mn} e^{i(mx+ny)} & \text{and} \quad \partial_n u|_{\Gamma_2} = -\sum \nu_{mn} b_{mn} e^{i(mx+ny)}, \end{array} \tag{18}$$

in which the sum is over all integer pairs (m,n). This motivates the definition of an operator B on functions defined on Γ, which will enable us to give a precise formulation of a guided mode. For the differential, or strong, formulation, this Dirichlet-to-Neumann map only needs to be defined for twice differentiable functions in $\mathcal{R} \setminus \Sigma$ whose first derivative is continuous up to Γ. We give a refined definition that will accommodate the variational, or weak, formulation, which includes functions in $H^1(\mathcal{R})$, that is, functions belonging to

$L^2(\mathcal{R})$ that possess weak first derivatives also belonging to $L^2(\mathcal{R})$. For such functions, a restriction to Γ is well defined as a function in the fractional Sobolev space $H^{1/2}(\Gamma)$, but a normal derivative is not well defined. The Dirichlet-to-Neumann map is replaced by a bounded operator B from $H^{1/2}(\Gamma)$ to its dual space $H^{-1/2}(\Gamma)$, which coincides with the Dirichlet-to-Neumann map (18) when restricted to twice differentiable functions with continuous derivatives up to Γ.

Definition 2.1 (Dirichlet-to-Neumann map B) *Let $f \in H^{1/2}(\Gamma)$ be given, and represent f as $f = (f^1, f^2)$ according to the decomposition $H^{1/2}(\Gamma) = H^{1/2}(\Gamma_1) \oplus H^{1/2}(\Gamma_2)$. Put $\hat{f}_{mn} = (\hat{f}^1_{mn}, \hat{f}^2_{mn})$ where $\hat{f}^{1,2}_{mn}$ are the Fourier coefficients of $f^{1,2}$. Note that $\nu_{mn}\hat{f}_{mn} = (\nu_{mn}\hat{f}^1_{mn}, \nu_{mn}\hat{f}^2_{mn}) \in H^{-1/2}(\Gamma_1) \oplus H^{-1/2}(\Gamma_2) = H^{-1/2}(\Gamma)$, and define Bf through its Fourier coefficients by putting*

$$(\widehat{Bf})_{mn} = \nu_{mn}\hat{f}_{mn}. \qquad \textit{(definition of B)} \tag{19}$$

We use integral notation to denote the action of the function $Bf \in H^{-1/2}(\Gamma)$ on $g \in H^{1/2}(\Gamma)$, and this action is concretely expressed through the Fourier coefficients of f and g:

$$\int_\Gamma (Bf)g = \sum_{m,n=-\infty}^{\infty} \nu_{mn}(\hat{f}^1_{mn}\hat{g}^1_{mn} + \hat{f}^2_{mn}\hat{g}^2_{mn}). \tag{20}$$

The action of Bf restricted to Γ_j, for $j = 1, 2$ is given by

$$\int_{\Gamma_j} (Bf)g = \sum_{m,n=-\infty}^{\infty} \nu_{mn}\hat{f}^j_{mn}\hat{g}^j_{mn}.$$

If all the ν_{mn} are positive, that is, if $\tilde{\mathcal{P}} = \emptyset$, then B is a positive operator, that is, for each $f \in H^{1/2}(\mathcal{R})$, $\int_\Gamma (Bf)\bar{f} > 0$.

We are now ready to state the condition that allows a precise definition of a guided mode. Condition 2.2 makes precise the behavior of a solution to the Helmholtz equation (8) at the surfaces of discontinuity of the functions ε and μ (equation 3) and enforces the exponential decay through the Dirichlet-to-Neumann operator B. The condition (17) necessary when linear harmonics are present is stated separately for that case (equation 21).

Condition 2.2 (Strong condition for a guided mode) *Let u be a twice differentiable function in $\mathcal{S} \setminus (\Sigma \cup \Gamma)$ with continuous value and first derivative up to $\partial\mathcal{S}$, Σ, and Γ. Denote by $\partial_n u_\pm$ the values of the normal derivative of u on Σ and Γ, where the $+$-sign refers to the side toward the direction of the normal vector n. If $\nu^2_{mn} \neq 0$ for all (m, n), then u satisfies the strong condition for a guided mode provided* *i.* $(\nabla + i\boldsymbol{\kappa})^2 u + \mu_0\epsilon_0\omega^2 u = 0$ *in* $\mathcal{S}\setminus\mathcal{R}$,

ii. $(\nabla + i\boldsymbol{\kappa})^2 u + \alpha\mu_0\epsilon_0\omega^2 u = 0$ *in* $\mathcal{S}\setminus\Omega$,

iii. $(\nabla + i\boldsymbol{\kappa})^2 u + \alpha\mu_1\epsilon_1\omega^2 u = 0$ *in* Ω,

iv. u *is continuous in* $\mathcal{S}$,

v. $\partial_n u_+ = \partial_n u_- = -Bu$ *on* Γ,

vi. $\mu_1(\partial_n u_+ + (i\boldsymbol{\kappa}\cdot n)u) = \mu_0(\partial_n u_- + (i\boldsymbol{\kappa}\cdot n)u)$ *on* Σ,

vii. $u(-\pi, y, z) = u(\pi, y, z)$ *and* $\partial_n u(-\pi, y, z) = -\partial_n u(\pi, y, z)$,

viii. $u(x, -\pi, z) = u(x, \pi, z)$ *and* $\partial_n u(x, -\pi, z) = -\partial_n u(x, \pi, z)$.

If $\nu_{mn} = 0$ *for some* (m, n)*, then for each such pair we require, in addition, that the corresponding Fourier coefficient of* u *be zero on* Γ*:*

$$(2\pi)^2 (u|_{\Gamma_j})\widehat{\ }_{mn} = \int_{\Gamma_j} u(x, y, z) e^{-i(mx+ny)} = 0, \quad j = 1, 2. \qquad ((m, n) \in \tilde{\mathcal{P}}\setminus\mathcal{P}) \tag{21}$$

Definition 2.3 (Guided mode) *A* guided mode *in the augmented periodic slab structure* $\tilde{\Omega}_{aug}$ *is the pseudoperiodic extension to* $\mathbb{R}^3$ *of a function of the form*

$$u(x, y, z)e^{i(\kappa_1 x + \kappa_2 y - i\omega t)},$$

in which u *satisfies the strong Condition 2.2.*

It is possible to restrict analysis to the region $\mathcal{R}$, for if we omit the condition (i) and the first equality in (v), then a function in $\overline{\mathcal{R}}$ (the closure of $\mathcal{R}$) satisfying the remaining conditions can be extended in a unique way to $\mathcal{S}$ such that (i) is satisfied simply by declaring

$$\begin{aligned} u(x, y, z) &= \sum_{m,n=-\infty}^{\infty} a_{mn} e^{\nu_{mn}(z-z_1)} e^{i(mx+ny)}, \quad z \le z_1, \\ u(x, y, z) &= \sum_{m,n=-\infty}^{\infty} b_{mn} e^{-\nu_{mn}(z-z_2)} e^{i(mx+ny)}, \quad z \ge z_2, \end{aligned} \tag{22}$$

in which a_{mn} are the Fourier coefficients of $u|_{\Gamma_1}$ and b_{mn} are the Fourier coefficients of $u|_{\Gamma_2}$, for this function satisfies the condition $Bu = \partial_n u_+ = \partial_n u_-$ on Γ.

We will need a variational formulation for guided modes. The appropriate function space is the periodic subspace $H^1_{\text{per}}(\mathcal{R})$ of the Sobolev space $H^1(\mathcal{R})$ of functions in $L^2(\mathcal{R})$ with weak gradients in $L^2(\mathcal{R})$: $H^1_{\text{per}}(\mathcal{R})$ is the subspace of functions $f \in H^1(\mathcal{R})$ satisfying

$f(-\pi, y, z) = f(\pi, y, z)$ and $f(x, -\pi, z) = f(x, \pi, z)$, where the boundary values of f are well defined by a bounded trace operator to $H^{1/2}(\partial\mathcal{S})$. $H^1_{\mathrm{per}}(\mathcal{R})$ is a Hilbert space, retaining the same inner product as $H^1(\mathcal{R})$:

$$(u, v)_{H^1(\mathcal{R})} = \int_{\mathcal{R}} (u\bar{v} + \nabla u \nabla \bar{v})\,.$$

In referring to the trace of f on Γ, we will be more precise and denote the trace operator by $T : H^1(\mathcal{R}) \to H^{1/2}(\Gamma)$, so that the restriction of f to Γ is denoted by Tf.

Condition 2.4 (Weak condition for a guided mode, first form) *A function $u \in H^1_{per}(\mathcal{R})$ satisfies the weak condition for a guided mode provided*

$$\int_{\mathcal{R}} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa})\, u \cdot (\nabla - i\boldsymbol{\kappa})\, \bar{v} + \frac{1}{\mu_0} \int_{\Gamma} (BTu)(T\bar{v}) - \int_{\mathcal{R}} \varepsilon\omega^2 u\bar{v} \;=\; 0 \quad \textit{for all} \quad v \in H^1_{per}(\mathcal{R}). \tag{23}$$

In case $\nu^2_{mn} = 0$ for any pair (m, n), it is required additionally that $(\widehat{Tf})_{mn} = 0$.

The sesquilinear form in Condition 2.4 is conjugate-symmetric in $H^1_{\mathrm{per}}(\mathcal{R})$ if and only if $\mathcal{P} = \emptyset$. We introduce a subspace X in which it is always conjugate-symmetric, and, in fact, positive, namely, the subspace of functions whose traces on Γ have vanishing Fourier coefficients for $(m, n) \in \tilde{\mathcal{P}}$.

$$(\widehat{Tf})_{mn} = 0 \quad \text{for} \quad (m, n) \in \tilde{\mathcal{P}} \qquad \text{(defining condition for } f \in X\text{)},$$

or, equivalently,

$$\int_{\Gamma_1} (Tf) e^{-i(mx+ny)} = \int_{\Gamma_2} (Tf) e^{-i(mx+ny)} = 0 \quad \text{for} \quad (m, n) \in \tilde{\mathcal{P}} \quad \text{(condition for } f \in X\text{)}.$$

X is closed under the norm of $H^1(\mathcal{R})$. In order to exclude linear and propagating Fourier harmonics from the extension to all of $\mathcal{S}$ of a function f in X, that has well defined normal derivatives on Γ, *it must also be demanded that the normal derivative have vanishing Fourier coefficients for $(m, n) \in \tilde{\mathcal{P}}$.* We give an alternate variational formulation of guided modes in the weak Condition 2.5.

Condition 2.5 (Weak condition for a guided mode, second form) *A function $u \in H^1_{per}(\mathcal{R})$ that possesses a normal derivative on Γ satisfies the weak condition for a guided mode provided that $u \in X$ and*

$$\textit{i.} \quad \int_{\mathcal{R}} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa})\, u \cdot (\nabla - i\boldsymbol{\kappa})\, \bar{v} + \frac{1}{\mu_0} \int_{\Gamma} (BTu)(T\bar{v}) - \int_{\mathcal{R}} \varepsilon\omega^2 u\bar{v} \;=\; 0 \quad \textit{for all} \quad v \in X,$$

ii. $(\partial_n u|_\Gamma)\widehat{\ }_{mn} = 0$ *for all* $(m,n) \in \tilde{\mathcal{P}}$.

We prove in Theorem 2.7 that Conditions 2.2, 2.4, and 2.5 are all equivalent. In particular, a function in $H^1_{\text{per}}(\mathcal{R})$ that satisfies 2.4 is in fact regular and satisfies the other two conditions, and part (i) of 2.5 actually implies the existence of a normal derivative on Γ.

In section 3, we will show the existence of a sequence of relations between α and ϵ_1, for each choice of ω and $\boldsymbol{\kappa}$, that describe all of the pairs (α, ϵ_1) that support a solution of part (i) of Condition 2.5. Because these solutions are in X, the coefficients in their Fourier expansion for all $(m,n) \in \mathcal{P}$ (equation 9) satisfy

$$|a^+_{mn}| - |a^-_{mn}| = 0 \quad \text{and} \quad |b^+_{mn}| - |b^-_{mn}| = 0,$$

implying the vanishing of energy flux in the z-direction.

Part (ii) of Condition 2.5 indicates that guided modes typically do not exist in the $(\boldsymbol{\kappa}, \omega)$-regime of propagating or linear harmonics ($\tilde{\mathcal{P}} \neq \emptyset$) due to this extra condition that each of these harmonics must satisfy. The vanishing of this finite number of harmonics must be accomplished through the tuning of other parameters of the structure. In particular, if the structure is symmetric about the yz-plane and $\boldsymbol{\kappa} = (0, \kappa_2, 0)$ or it is symmetric about the xz-plane and $\boldsymbol{\kappa} = (\kappa_1, 0, 0)$, then the functions satisfying part (i) of Condition 2.5 are symmetric or antisymmetric. We focus on structures with symmetry about the yz-plane. Ω is symmetric about the yz-plane if

$$(x, y, z) \in \Omega \implies (-x, y, z) \in \Omega.$$

In this case, the antisymmetric solutions to (i) also satisfy (ii) for all $(m,n) \in \tilde{\mathcal{P}}$ with m even. Thus, if there is only one propagating mode $(0,0)$ and the rest are evanescent, then Condition 2.5 is satisfied in full and the solutions therefore represent *nonrobust guided modes* traveling parallel to the plane of symmetry. These modes are nonrobust because, under a general perturbation of κ_1 or the structure itself, the $(0,0)$ harmonic, which is not evanescent, is no longer guaranteed to vanish.

We make the formulation for an antisymmetric nonrobust mode in a symmetric structure precise in Condition 2.6 and Theorem 2.7. For this, we introduce the orthogonally complementary subspaces X^{sym} and X^{ant} of X to treat the case that Ω is symmetric about the yz-plane, $\kappa_1 = 0$, and $\tilde{\mathcal{P}} \neq \emptyset$.

$$\begin{aligned} X^{\text{sym}} &= \{v \in X : v(x,y,z) = v(x,y,z) \text{ a.e. in } \mathcal{R}\}, \\ X^{\text{ant}} &= \{v \in X : v(x,y,z) = v(-x,y,z) \text{ a.e. in } \mathcal{R}\}. \end{aligned}$$

It is straightforward to verify that X^{sym} and X^{ant} are orthogonal in the usual H^1 and L^2 inner products on X and with respect to the sesquilinear form on the left-hand side of part (i) of Condition 2.6:

$$X = X^{\text{sym}} \oplus X^{\text{ant}}.$$

Condition 2.6 (Weak condition for a nonrobust guided mode) *Suppose that Ω is symmetric about the yz-plane and that $\kappa_1 = 0$. A function $u \in H^1_{per}(\mathcal{R})$ satisfies the weak condition for an antisymmetric nonrobust mode provided $u \in X^{ant}$ and*

i. $$\int_{\mathcal{R}} \frac{1}{\mu}\,(\nabla + i\boldsymbol{\kappa})\,u \cdot (\nabla - i\boldsymbol{\kappa})\,\bar{v} + \frac{1}{\mu_0}\int_\Gamma (BTu)(T\bar{v}) - \int_{\mathcal{R}} \varepsilon\omega^2 u\bar{v} \;=\; 0 \quad \text{for all } v \in X^{ant},$$

ii. $\tilde{\mathcal{P}} \neq \emptyset$ *and* $(\partial_n u|_\Gamma)\widehat{\ }_{mn} = 0$ *for all* $(m,n) \in \tilde{\mathcal{P}}$ *with* m *odd.*

Theorem 2.7 (Equivalence of strong and weak conditions)

i. Let u satisfy Condition 2.2. Then the restriction of u to $\mathcal{R}$ is in $H^1_{per}(\mathcal{R})$ and satisfies Condition 2.4.

ii. Let u satisfy Condition 2.4. Then u can be extended to a twice differentiable function in $\mathcal{S} \setminus (\Sigma \cup \Gamma)$ with continuous value and first derivative up to $\partial\mathcal{S}$, Σ, and Γ. This extension satisfies Condition 2.2, and u is in X and satisfies Condition 2.5.

iii. If u satisfies Condition 2.5, then u satisfies Condition 2.4.

iv. If u satisfies Condition 2.6, then u satisfies Condition 2.5 (for $\kappa_1 = 0$).

3 Existence of guided modes

The theoretical development presented in this section is in essence that followed in [6]. Nevertheless, we feel that complete proofs are necessary to ensure that consistency and mathematical rigor is observed. The proofs are given in the Appendix.

Define the following sesquilinear forms in $H^1_{\text{per}}(\mathcal{R})$:

$$A(u,v) \;=\; \int_{\mathcal{R}} \frac{1}{\mu}\,(\nabla + i\boldsymbol{\kappa})\,u \cdot (\nabla - i\boldsymbol{\kappa})\,\bar{v} + \frac{1}{\mu_0}\int_\Gamma (BTu)(T\bar{v}), \tag{24}$$

$$\ell(u,v) \;=\; \int_{\mathcal{R}\setminus\Omega} \epsilon_0\omega^2 u\bar{v} + \int_\Omega \epsilon_1\omega^2 u\bar{v}. \tag{25}$$

Notice that A depends on $\boldsymbol{\kappa}$, ω, ϵ_0, μ_0, and μ_1 (the dependence on ω and ϵ_0 is through B—see equations (19) and (10)) and ℓ depends on ω, ϵ_0, and ϵ_1; neither of them depends on α.

A function $u \in H^1_{\text{per}}(\mathcal{R})$ satisfies the weak condition for a guided mode if and only if $A(u,v) - \alpha\ell(u,v) = 0$ for each $v \in H^1_{\text{per}}(\mathcal{R})$. This is equivalent to the condition that u is an eigenfunction of the map $H^1_{\text{per}}(\mathcal{R}) \to H^1_{\text{per}}(\mathcal{R})^* :: u \mapsto \overline{A(u,\cdot)}$ (the asterisk denotes the dual space) with eigenvalue α in the sense that

$$A(u,\cdot) = \ell(\alpha u,\cdot).$$

If the set of propagating harmonics is empty ($\mathcal{P} = \emptyset$), then A is conjugate-symmetric, that is $A(u,v) = \overline{A(v,u)}$ for all $u, v \in H^1_{\text{per}}(\mathcal{R})$. Otherwise, it is not due to the purely imaginary values of ν_{mn} in the definition (2.1) of B for $(m,n) \in \mathcal{P}$. In X, both A and ℓ are conjugate-symmetric, and therefore the eigenvalues α are real.

Define the Rayleigh quotient by

$$J(u) = \frac{A(u,u)}{\ell(u,u)} = \frac{\int_{\mathcal{R}} \frac{1}{\mu} |\,(\nabla + i\boldsymbol{\kappa})\,u|^2 \;+\; \frac{1}{\mu_0} \int_\Gamma (BTu)(T\bar{u})}{\epsilon_0\omega^2 \int_{\mathcal{R}\setminus\Omega} |u|^2 \;+\; \epsilon_1\omega^2 \int_\Omega |u|^2}. \tag{26}$$

Recall that the operator B depends on κ, ω, ϵ_0, and μ_0, but not on ϵ_1. The dependence of $J(u)$ on ϵ_1 comes only in the second term of ℓ. For an exposition of the role the Rayleigh quotient in the theory of eigenvalues of elliptic operators, the reader may refer to Jost ([7], §8.5) or Gould ([8], Ch. II); a more brief discussion is found in Gilbarg/Trudinger ([9], §8.12).

Theorem 3.1 (Eigenvalue sequences) *There exists a sequence of real numbers (eigenvalues) $\{\alpha_j\}_{j=0}^\infty$ and functions (eigenfunctions) $\{\psi_j\}_{j=0}^\infty$ such that*

i. $0 < \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_j \leq \ldots,$

ii. $\alpha_j \to \infty \quad \textit{as} \quad j \to \infty,$

iii. $A(\psi_j, v) = \alpha_j \ell(\psi_j, v) \quad \textit{for all} \quad v \in X,$

iv. if $A(\psi, v) = \alpha\ell(\psi, v)$ for all $v \in X$, then there is an integer j such that $\alpha = \alpha_j$ and $\psi \in span\{\psi_k : \alpha = \alpha_k\}$,

v. the sequence $\{\psi_j\}_{j=0}^\infty$ is an orthonormal Hilbert-space basis for $L^2(\mathcal{R}, \ell)$.

The eigenvalues and eigenfunctions arise from the process of successive minimization of the Rayleigh quotient:

$$\alpha_j = \inf_{u \in X_j, u \neq 0} J(u) = J(\psi_j), \quad \psi_j \in X_j,$$

in which

$$X_j = \{v \in X : A(\psi_k, v) = 0 \textit{ for } k = 0, \ldots, j-1\}.$$

If Ω is symmetric about the yz-plane and $\kappa_1 = 0$, then $\{\psi_j\}_{j=0}^{\infty}$ is the union of two nondecreasing sequences $\{\psi_j^{sym}\}_{j=0}^{\infty}$ and $\{\psi_j^{ant}\}_{j=0}^{\infty}$ from X^{sym} and X^{ant}, respectively. We denote the associated sequences of eigenvalues by $\{\alpha_j^{sym}\}_{j=0}^{\infty}$ and $\{\alpha_j^{ant}\}_{j=0}^{\infty}$. The symmetric and antisymmetric eigenfunctions and associated eigenvalues arise from minimization of the Rayleigh quotient over X^{sym} and X^{ant}:

$$\alpha_j^{ant} = \inf_{u\in X_j^{ant}, u\neq 0} J(u) = J(\psi_j^{ant}), \quad \psi_j \in X_j^{ant},$$

in which

$$X_j^{ant} = \{v \in X^{ant} : A(\psi_k^{ant}, v) = 0 \text{ for } k = 0, \dots, j-1\}.$$

Lemma 3.2 *The eigenvalues α_j are continuous strictly decreasing functions of ϵ_1, and $\alpha_j \to 0$ as $\epsilon_1 \to \infty$. Similarly, the α_j are continuous strictly decreasing functions of μ_1, and $\alpha_j \to 0$ as $\mu_1 \to \infty$.*

Using this Lemma, let us parse Theorem 3.1 in a form in which α is fixed, so that the functions ε and μ given by (3) outside of the domain Ω are fixed. For the argument, let us also fix μ and denote the dependence of α_j on ϵ_1 by $\alpha_j(\epsilon_1)$. Since $\alpha_j(\epsilon_1) \nearrow \infty$ as $j \to \infty$ and because of Lemma 3.2, we may put

$$N_\alpha = \min\{j : \lim_{\epsilon\to 0} \alpha_j(\epsilon) > \alpha\}$$

and define implicitly a nondecreasing sequence of material parameters $\epsilon_1 = \{E_j(\alpha)\}_{j=N_\alpha}^{\infty}$ and a sequence of corresponding functions $\{\phi_j(\alpha)\}_{j=N_\alpha}^{\infty}$ that satisfy

$$\alpha_j(E_j(\alpha)) = \alpha, \qquad \phi_j(\alpha) = \psi_j(E_j(\alpha)).$$

Because the functions $\alpha_j(\epsilon_1)$ are continuous and strictly decreasing in ϵ, the values $E_j(\alpha)$ are defined uniquely, and, as functions of α, are continuous and strictly decreasing. By Theorem 3.1, these sequences satisfy

$$A(\phi_j(\alpha), v) = \alpha \ell_{E_j(\alpha)}(\phi_j(\alpha), v) \qquad \text{for all} \qquad v \in X.$$

If ω and $\boldsymbol{\kappa}$ are such that the medium exterior to $\mathcal{R}$ admits no propagating or linear Fourier harmonics, that is, if $\tilde{\mathcal{P}} = \emptyset$, then each $\phi_j(\alpha)$ can be extended to all of $\mathcal{S}$ as a guided mode, for the second part of 2.5 requiring vanishing of all of these harmonics is vacuously satisfied.

In the case that Ω is symmetric about the yz-plane, $\boldsymbol{\kappa} = \langle 0, \kappa_2, 0\rangle$, and there is only one propagating Fourier harmonic, namely $(0,0)$, the antisymmetric eigenfunctions $\phi_j^{\text{ant}}(\epsilon_1)$ satisfy Condition 2.6 and therefore give rise to nonrobust guided modes.

We summarize these results in the following theorem.

Theorem 3.3 (Existence of guided modes) *For each $\alpha > 0$, there exists a sequence $\{E_j(\alpha)\}_{j=N_\alpha}^{\infty}$ of real numbers and a sequence $\{\phi_j(\alpha)\}_{j=N_\alpha}^{\infty}$ of functions from X such that*

i. *for each $\alpha > 0$, $0 < E_0(\alpha) \le E_1(\alpha) \le \cdots \le E_j(\alpha) \le \ldots$,*

ii. *for each $\alpha > 0$, $E_j(\alpha) \to \infty$ as $j \to \infty$,*

iii. *for each integer $n \ge 0$, $E_j(\alpha)$ is a strictly decreasing function of α, and $E_j(\alpha) \to \infty$ as $\alpha \to 0$.*

iv. *$\phi_j(\alpha)$ satisfies part (i) of the second form of the weak Condition 2.5 for guided modes.*

If ω and $\boldsymbol{\kappa}$ are such that the medium exterior to $\mathcal{R}$ admits no propagating or linear Fourier harmonics, that is, if $\tilde{\mathcal{P}} = \emptyset$, then for each α and each j, the function $\phi_j(\alpha)$ satisfies both conditions of Condition 2.5. In particular, it can be extended into $\mathcal{S}$ to a function that satisfies 2.2, and gives rise to a guided mode

$$\psi_j(\alpha)(x, y, z)e^{i(\kappa_1 x + \kappa_2 y - \omega t)}$$

in the augmented slab structure defined by the functions (3)*.*

If Ω is symmetric about the yz-plane, $\boldsymbol{\kappa} = (0, \kappa_2, 0)$, and there is only one propagating Fourier harmonic $(0, 0)$, the rest being evanescent, then for each α and j the function $\psi_j^{ant}(\alpha)$ satisfies Condition 2.5. In particular, it can be extended into $\mathcal{S}$ to a function that satisfies 2.2, giving rise to a nonrobust guided mode traveling parallel to the yz-plane:

$$\psi_j^{ant}(\alpha)(x, y, z)e^{(\kappa_2 y - i\omega t)}.$$

An analogous statement holds if Ω is symmetric about the xz-plane.

There are two special cases of interest:

i. Taking $\alpha = 1$ gives the unaugmented structure $\tilde{\Omega}$, as the material properties ε and μ in $\mathcal{S} \setminus \mathcal{R}$ and $\mathcal{R} \setminus \Omega$ coincide:

$$\varepsilon(r) = \begin{cases} \epsilon_0, & r \in \mathcal{S} \setminus \Omega \\ \epsilon_1, & r \in \Omega \end{cases} \quad \text{and} \quad \mu(r) = \begin{cases} \mu_0, & r \in \mathcal{S} \setminus \Omega \\ \mu_1, & r \in \Omega \end{cases}.$$

Theorem 3.3 gives a sequence of constants $\epsilon_1 = E_j(1)$ for which a guided mode exists, provided the vanishing of all linear and propagating harmonics.

ii. Fixing $\epsilon_1 = \epsilon_0$ and $\mu_1 = \mu_0$ corresponds to the case of a slab with no periodic structure, having uniform material properties in $\mathcal{R}$:

$$\varepsilon(x, y, z) = \begin{cases} \alpha\epsilon_0, & z \in \mathcal{R} \\ \epsilon_0, & z \notin \mathcal{R} \end{cases} \quad \text{and} \quad \mu(x, y, z) = \mu_0. \tag{27}$$

We analyze this instructive case explicitly in subsection 4.1.

4 Nonexistence of guided modes in inverse structures

An "inverse structure" is one in which the speed of light is higher than the speed of light in the surrounding medium. This means that $\epsilon_1\mu_1 < \epsilon_0\mu_0$. Using the sequences of eigenvalues constructed in section 3, we prove that certain inverse structures cannot support guided modes, robust or nonrobust. Specifically, we fix $\mu_1 = \mu_0 > 0$ arbitrarily and take $0 < \epsilon_1 < \epsilon_0$. Our statement requires an additional restriction on the width of the slab that depends on the material parameters, frequency, and wavevector (33). We do not know if this restriction arises only as a consequence of our method of proof or if it is truly necessary.

It is known that under a certain restrictive geometric condition, guided modes do not exist in inverse structures. Theorem 3.5 of [6], extended to three-dimensional structures amounts to the following conditions for homogeneous slabs:

i. The surface Σ of the slab has two sides, given by $z = f_1(x, y) \leq 0$ and $z = f_2(x, y) \geq 0$, where the common domain of f_1 and f_2 is a subset of the square $\{-\pi \leq x, y \leq \pi\}$.

ii. $\epsilon_1\mu_1 \leq \epsilon_0\mu_0$, that is, the celerity inside the slab is greater than that outside the slab (as for a film of air within a ceramic matrix).

The first condition is a severe restriction. It excludes, among other types of structures, periodic arrays of ellipses that do not have a major axis parallel to the z-axis and structures that some line parallel to the z-axis intersects in more than one segment.

In addition, it is shown in [6], Theorem 4.1 that, for a given wavevector $\boldsymbol{\kappa}$, the set of frequencies for which a guided mode exists is greater than $|\boldsymbol{\kappa}|/n_+$, where n_+ is the maximum value of $\epsilon\mu$. It follows that, if $\epsilon_1\mu_1 < \epsilon_0\mu_0$, then robust guided modes do not exist.

Our approach to proving the nonexistence of both types of modes for $\mu_1 = \mu_0$ is first to compute explicitly the eigenvalues $\alpha_j(\epsilon_0)$, corresponding to the case $\epsilon_1 = \epsilon_0$, in which the slab $\tilde{\Omega}_{\text{aug}}$ has no genuine periodicity and then to use the restriction on the width (33) to prove that the eigenvalues are all greater than or equal to 1. Finally, since the eigenvalues are decreasing as a function of ϵ_1, we observe that $\alpha_j(\epsilon_1) > 1$ for $\epsilon_1 < \epsilon_0$. As $\alpha = 1$ corresponds to the unaugmented periodic slab structure $\tilde{\Omega}$ with material constant ϵ_1 surrounded by a medium with constant ϵ_0, we conclude that no guided modes exist in $\tilde{\Omega}$ for $\epsilon_1 < \epsilon_0$.

Indeed, if $\epsilon_1 > \epsilon_0$, we have seen that nonrobust modes exist in symmetric structures if κ_1 or κ_2 vanishes. In fact, this restriction is not necessary: in [6], nonrobust modes are constructed for arbitrary Bloch wavevectors for two-dimensional slabs.

4.1 Eigenvalues for a flat slab

We compute explicitly the eigenvalues α_j and corresponding eigenfunctions ψ_j when $\epsilon_1 = \epsilon_0$ and $\mu_1 = \mu_0$ (equation 27). In this situation, the eigenfunctions satisfy the strong form of the Helmholtz equation in $\mathcal{R}$ (with $\psi = \psi_j$ and $\alpha = \alpha_j$):

$$\begin{array}{l}(\nabla + i\boldsymbol{\kappa})^2\psi + \alpha\epsilon_0\mu_0\omega^2\psi = 0 \text{ in } \mathcal{R},\\ \psi \in X \text{ and } \partial_n\psi|_\Gamma = B\psi,\\ \psi \text{ has periodic boundary conditions in } x \text{ and } y.\end{array} \tag{28}$$

Since $\varepsilon(x, y, z)$ is constant in x and y and $\mathcal{R}$ is bounded by planes parallel to the three coordinate planes, the method of separation of variables is applicable. The separable solutions have the simple form

$$\psi(x, y, z) = \left(A_{mn}e^{\eta_{mn}z} + B_{mn}e^{-\eta_{mn}z}\right) e^{i(mx+ny)}, \quad m, n \in \mathbb{Z}, \tag{29}$$

in which

$$\eta_{mn}^2 = (m + \kappa_1)^2 + (n + \kappa_2)^2 - \alpha\epsilon_0\mu_0\omega^2 \tag{30}$$

and $\operatorname{Im}\eta_{mn} > 0$ if $\eta_{mn}^2 < 0$ and $\operatorname{Re}\eta_{mn} > 0$ if $\eta_{mn}^2 > 0$. If $\eta_{mn} = 0$, then

$$\psi(x, y, z) = (A_{mn} + B_{mn}z)\, e^{i(mx+ny)}. \tag{31}$$

Each solution of the Helmholtz equation with periodic boundary conditions is a series superposition of separable solutions:

$$\psi(x, y, z) = \sum_{m,n=-\infty}^{\infty} \phi_{mn}(z)e^{i(mx+ny)},$$

in which ϕ_{mn} is of the form shown in (29) or (31). Moreover, the conditions that $\psi \in X$ and $\partial_n\psi|_\Gamma = B\psi$ impose *independent* conditions on the Fourier harmonics indexed by m and n on the boundary Γ:

$$\begin{array}{l}(\psi|_\Gamma)\widehat{\ }_{mn} = 0 \text{ for } (m, n) \in \tilde{\mathcal{P}},\\ (\partial_z\psi|_{\Gamma_1})\widehat{\ }_{mn} = \nu_{mn}\, (\psi|_{\Gamma_1})\widehat{\ }_{mn} \text{ and } (\partial_z\psi|_{\Gamma_2})\widehat{\ }_{mn} = -\nu_{mn}\, (\psi|_{\Gamma_2})\widehat{\ }_{mn} \text{ for } (m, n) \notin \tilde{\mathcal{P}}.\end{array} \tag{32}$$

Because of this, if ψ satisfies the Helmholtz equation as well as the boundary conditions (28), then each separable component (29) of ψ in its series representation also satisfies both. Therefore, each solution of (28) is composed of separable solutions.

To find the values of α that admit such solutions and the solutions themselves, we impose the condition (32) on the separable solution (29) or (31) for each pair (m, n). For each fixed $(m, n) \in \tilde{\mathcal{P}}$, the condition (32) is possible only for values of α for which

$\eta_{mn}^2 < 0$, which give oscillatory solutions in the interval from z_1 to z_2. In addition, in order for (32) to hold, η_{mn} must be of the form $\eta_{mn} = i\left(\frac{j\pi}{z_2-z_1}\right)$ for some j (independent of (m,n)), and we thus arrive at a sequence of eigenvalues $\alpha = \alpha_{mnj}$ satisfying

$$\alpha_{mnj}\epsilon_0\mu_0\omega^2 = \left(\frac{j\pi}{z_2 - z_1}\right)^2 + (m+\kappa_1)^2 + (n+\kappa_2)^2, \quad j = 1, 2, 3, \dots .$$

It is straightforward to deduce from the condition (32) that the constants A_{mn} and B_{mn} in (29) have the same modulus, so that, by multiplying the solution by a unitary number $e^{i\theta}$, we may take $\phi_{mnj}(z)$ to be a shifted sine function inside the region $\mathcal{R}$. These solutions do not represent guided modes because they do not satisfy the second part of Condition 2.5 requiring the normal derivative of ϕ_{mnj} to vanish, and therefore their extensions to all of $\mathcal{S}$ do not decay as $|z| \to \infty$.

For $(m,n) \notin \tilde{\mathcal{P}}$, the condition (32) amounts to matching a solution that is decaying as $z \to -\infty$ for $z < z_1$ to one that is decaying as $z \to \infty$ for $z > z_2$ through a solution in the interval from z_1 to z_2. This is possible only if the solution in this interval is oscillatory, and this is only achievable when $\eta_{mn}^2 < 0$. By enforcing the decay of the solution as $|z| \to \infty$, we obtain a sequence of eigenvalues $\alpha = \alpha_{mnj}$ with $\alpha_{mnj} \to \infty$ as $j \to \infty$ satisfying

$$\tan\zeta(z_2 - z_1) = \frac{2\nu_{mn}\zeta}{\zeta^2 - \nu_{mn}^2}, \quad \zeta = \left(\alpha_{mnj}\epsilon_0\mu_0\omega^2 - (m+\kappa_1)^2 - (n+\kappa_2)^2\right)^{1/2}.$$

Again, by multiplying the solution by a unitary number, we may take $\phi_{mnj}(z)$ to be a shifted sine function inside the region $\mathcal{R}$. These solutions satisfy Condition 2.5, even when $\tilde{\mathcal{P}} \neq \emptyset$, as they involve only one Fourier harmonic.

The union of the sequences $\{\alpha_{mnj}\}$, arranged in increasing order, gives the sequence $\{\alpha_j\}$ that we seek.

As ϵ_1 is perturbed away from ϵ_0 the structure attains a genuine periodicity, and separable solutions are no longer valid. Typically all Fourier harmonics are represented in the eigenfunctions so that the guided modes disappear in a regime admitting linear or propagating harmonics. As we have seen, however, antisymmetric nonrobust modes persist, for example, in symmetric structures for which there is only one propagating harmonic, the rest being evanescent.

4.2 Nonexistence of guided modes

We use the foregoing analysis to prove a theorem stating that guided modes do not exist in certain structures in which the interior product of the material coefficients $\mu_1\epsilon_1$ is greater than the exterior product $\mu_0\epsilon_0$.

Theorem 4.1 (Nonexistence of guided modes) *Let $0 < \mu_1 \leq \mu_0$ and $0 < \epsilon_1 \leq \epsilon_0$, and let the frequency ω and wavevector $\boldsymbol{\kappa} = \langle \kappa_1, \kappa_2, 0 \rangle$ be given with $\boldsymbol{\kappa}$ in the first Brillouin zone: $-\frac{1}{2} \leq \kappa_1, \kappa_2 < \frac{1}{2}$. Suppose that the slab structure $\tilde{\Omega}$ (Fig. 2 with $\alpha = 1$) lies between two planes $\{z = z_1\}$ and $\{z = z_2\}$ satisfying*

$$(z_2 - z_1)(\epsilon_0 \mu_0 \omega^2 - \kappa_1^2 - \kappa_2^2)^{1/2} \leq \pi \tag{33}$$

in the case that $\epsilon_0 \mu_0 \omega^2 - \kappa_1^2 - \kappa_2^2 \geq 0$, that is, $\tilde{\mathcal{P}} \neq \emptyset$ (otherwise, there is no restriction). Then the slab with material properties

$$\varepsilon(r) = \begin{cases} \epsilon_0, & r \notin \tilde{\Omega} \\ \epsilon_1, & r \in \tilde{\Omega} \end{cases} \qquad \text{and} \qquad \mu(r) = \begin{cases} \mu_0, & r \notin \tilde{\Omega} \\ \mu_1, & r \in \tilde{\Omega} \end{cases}$$

admits no guided modes at the given frequency and wavevector.

Proof. We begin showing that, for $\epsilon_1 = \epsilon_0, \mu_1 = \mu_0$, the slab admits no guided modes. This is the case of a flat slab analyzed in subsection 4.1. Recall the definition of ν_{mn}^2:

$$\nu_{mn}^2 = -\epsilon_0 \mu_0 \omega^2 + (m + \kappa_1)^2 + (n + \kappa_2)^2,$$

and define, for each $\alpha > 0$, as before,

$$\eta_{mn}^2(\alpha) = -\alpha \epsilon_0 \mu_0 \omega^2 + (m + \kappa_1)^2 + (n + \kappa_2)^2.$$

In subsection 4.1, we have seen that the eigenvalues α_j (for $\epsilon_1 = \epsilon_0, \mu_1 = \mu_0$) correspond to eigenfunctions containing a single Fourier harmonic, and we wish to show that all of these eigenvalues are greater than or equal to 1.

For those pairs (m, n) for which $\nu_{mn}^2 > 0$, corresponding to the evanescent Fourier harmonics ($(m, n) \notin \tilde{\mathcal{P}}$), we have seen in subsection 4.1 that the matching conditions at $z = z_1$ and $z = z_2$ require that $\eta_{mn}^2(\alpha) < 0$. From the definitions of ν_{mn}^2 and $\eta_{mn}^2(\alpha)$, we conclude that $\alpha > 1$, so that all the eigenvalues corresponding to the evanescent harmonics are at least greater than 1.

For $(m, n) \in \tilde{\mathcal{P}}$, we still require that $\eta_{mn}^2(\alpha) < 0$. Since $\boldsymbol{\kappa}$ is taken to lie in the first symmetric Brillouin zone, that is, $-1/2 \leq \kappa_1 < 1/2$ and $-1/2 \leq \kappa_2 < 1/2$, we have (recall that $\mathrm{Im}\,(\nu_{mn}(\alpha)) > 0$ (page 7))

$$-i\nu_{mn} \leq -i\nu_{00} \quad \text{for all} \quad (m, n) \in \tilde{\mathcal{P}}.$$

According to the discussion of the preceding subsection, to satisfy the boundary conditions at $z = z_1$ and $z = z_2$, α must be chosen such that

$$-i\eta_{mn}(\alpha) = \left(\frac{j\pi}{z_2 - z_1} \right), \quad j \text{ a positive integer.}$$

From condition (33), we obtain

$$-i\eta_{mn}(\alpha) = \frac{j\pi}{z_2 - z_1} \geq j\,(\epsilon_0\mu_0\omega^2 - \kappa_1^2 - \kappa_2^2)^{1/2} \geq -i\nu_{00} \geq -i\nu_{mn},$$

from which it follows that $\alpha \geq 1$, so that all the eigenvalues corresponding to the propagating harmonics are at least 1. As we have mentioned in the previous subsection, these eigenvalues do not correspond to guided modes.

Since the eigenvalues α_j are strictly decreasing in ϵ_1, as well as in μ_1 (Lemma 3.2), we have $\alpha_j > 1$ if both $\epsilon_1 \leq \epsilon_0$ and $\mu_1 \leq \mu_0$, which proves the theorem. □

5 Numerical computations

We compute guided modes for the Helmholtz equation. These are scalar functions u satisfying Condition 2.2, for which $\alpha = 1$. We focus on the case of one propagating Fourier harmonic, and we consider a two-dimensional reduction, in which the slab is constant in the y-direction and $\kappa_2 = 0$. In this case, only the $(m, 0)$ Fourier harmonics enter the fields. Our method begins with a geometry Ω that is symmetric about the yz-plane (in the two-dimensional reduction to the x and z variables, this implies symmetry about the z-axis) and given values of $\boldsymbol{\kappa}$, ϵ_0, μ_0, μ_1, and ω. The code then computes the values of ϵ_1 which give rise to a solution of the first part of Condition 2.6, in other words, it computes one of the values $E_j(\alpha)$. The second part of the condition is automatically satisfied because only one harmonic is propagating, namely that with $(m, n) = (0, 0)$. The corresponding antisymmetric nonrobust guided mode u is also computed.

We use a finite element solver in the finite rectangular region $\mathcal{R}$ for the eigenvalue problem, with ϵ_1 as the eigenvalue, for the Helmholtz equation in two variables, x and z:

$$(\nabla + i\kappa)\cdot(\nabla + i\kappa)u + \epsilon_0\mu_0\omega^2 u = 0 \text{ in } \mathcal{R}\setminus\Omega, \tag{34}$$

$$(\nabla + i\kappa)\cdot(\nabla + i\kappa)u = -\epsilon_1\mu_0\omega^2 u \text{ in } \Omega, \tag{35}$$

$$\mu_1\left(\partial_n u_+ + (i\boldsymbol{\kappa}\cdot n)u\right) = \mu_0\left(\partial_n u_- + (i\boldsymbol{\kappa}\cdot n)u\right) \text{ on } \Sigma, \tag{36}$$

$$u(-\pi, z) = u(\pi, z) \text{ and } \partial_n u(-\pi, z) = -\partial_n u(\pi, z), \tag{37}$$

$$\partial_n u + \nu_0 u = 0 \text{ on the edges } z = z_1 \text{ and } z = z_2. \tag{38}$$

Note that since we assumed that only one harmonic propagates, $\nu_0 = i\sqrt{\epsilon_0\mu_0\omega^2 - |\kappa|^2}$, and condition (38) expresses that there are no incoming harmonics impinging the rectangular zone $\mathcal{R}$. In fact, condition (38) is a first approximation of the Dirichlet to Neumann operator B. If only one harmonic is allowed to propagate and if the rectangular region $\mathcal{R}$ is chosen to be wide enough, it is reasonable to believe that this first approximation leads to an exponentially small error. Numerical methods hinging on this boundary approximation idea have been used in the literature, for example by Kriegsmann and Volkov [11, 12], albeit in the case of regular transmission problems instead of eigenvalue problems.

We discretize (34-38) by finite elements on a meshing of $\mathcal{R}$, and then solve the discretized problem as an eigenvalue problem in ϵ_1. A function u satisfying (34-38) satisfies

$$0 = \nu_0 \int_\Gamma |u|^2 - \int_{\mathcal{R}} |(\nabla + i\kappa)u|^2 + \epsilon_0\mu_0\omega^2 \int_{\mathcal{R}\setminus\Omega} |u|^2 + \epsilon_1\mu_0\omega^2 \int_\Omega |u|^2.$$

Thus, loosely speaking, if the imaginary part of ϵ_1 is very small, u and $\partial_n u$ are very small on Γ which is made up by the two narrow edges of the rectangle $\mathcal{R}$. If the rectangle $\mathcal{R}$ is long enough, this simulates the exponential decay expected from a Bloch solution to the Helmholtz equation that is a guided mode.

We first use our numerical method to reproduce a computation of eigenvalues for bound states that appeared in [2]. The geometry under consideration is that of a dielectric made up of one large circle of radius 3, and 8 small circles of radius 1 (the circles are cross-sections of rods that extend infinitely in the y-direction). Their centers lie on the line $x = 0$, and two consecutive centers are 2π units of length apart. Fixing $\mu_0 = \mu_1 = 1$, $\epsilon_0 = 1$, $\epsilon_1 = 12$, it was found in [2] that guided modes (referred to as "bound states" in that paper) exist for the pairs ($\kappa_1 = 0.0, \omega = 0.4017$), ($\kappa_1 = 0.14, \omega = 0.3863$), ($\kappa_1 = 0.22, \omega = 0.3707$), and ($\kappa_1 = 0.44, \omega = 0.3306$). Thus with our present numerical method, we fix $\mu_0 = \mu_1 = 1$, $\epsilon_0 = 1$, and (κ_1, ω) at one of these pairs, and compute values for ϵ_1 for which there appears to be a guided mode. The computations lead to $\epsilon_1 \approx 12.0$, as expected. The corresponding eigenfunction is plotted using a gray scale coloring in Figures 5.

The first of these guided modes, at ($\kappa_1 = 0.0, \omega = 0.4017$), is antisymmetric about the yz-plane. It is nonrobust because it exists in the (κ_1, ω)-regime of one propagating Fourier harmonic, which is suppressed by the symmetry of the structure and the vanishing of κ_1. The last of them, at ($\kappa_1 = 0.44, \omega = 0.3306$), is a robust guided mode, for it exists in the (κ_1, ω)-regime in which all Fourier harmonics are evanescent. A dispersion relation for these is shown in [2].

The other two nonrobust modes are not discussed in the analysis in this paper, for they are in a (κ_1, ω)-regime of one propagating harmonic but the wave number κ_1 in the x-direction is not zero. However, at these two pairs, the coefficient for the one propagating Fourier harmonic happens to be zero, that is, the second part of Condition 2.6 happens to be satisfied. These values of κ_1 and ω occur at points of a complex dispersion curve calculated in [2] (Fig. 7.2, part 7) at which the imaginary part of the frequency appears to vanish. The existence of nonrobust modes at nonzero wavenumbers is proved in the final section of [6].

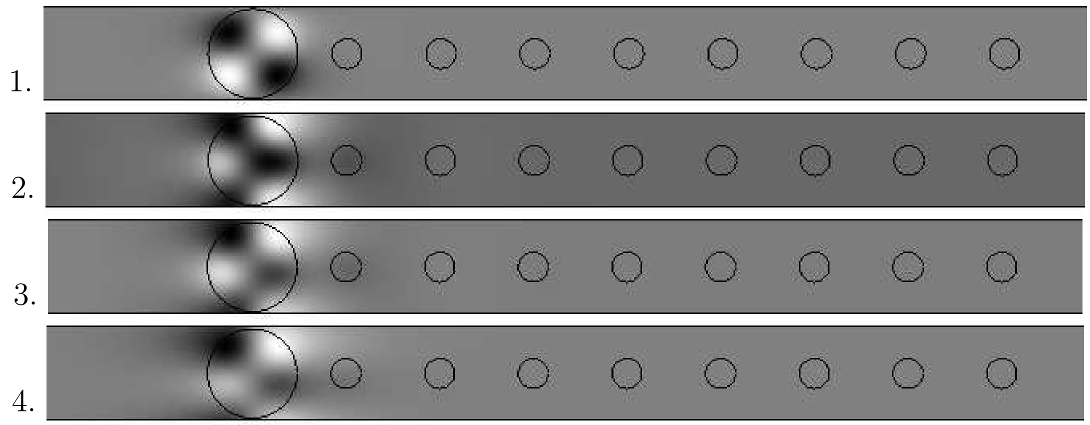


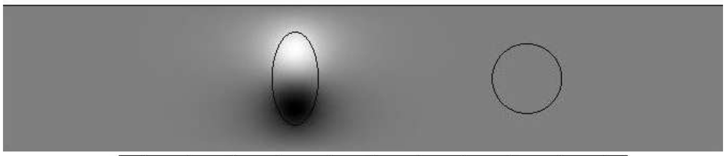
Figure 3: Four guided modes in a two-dimensional structure investigated in [2] (Fig. 7.2). One period is shown; the structure continues periodically in the vertical direction on the page. 1. A nonrobust antisymmetric guided mode at Bloch wavevector zero, $(\kappa_1, \omega) = (0.0, 0.4017)$. 2 and 3. Nonrobust guided modes at nonzero wavenumbers in the direction perpendicular to the line of symmetry, $(\kappa_1, \omega) = (0.14, 0.3863), (0.22, 0.3707)$. 4. A robust guided mode, $(\kappa_1, \omega) = (0.44, 0.3306)$.

We also show computations involving geometries that are not exclusively circular. We choose to place two dielectrics, one shaped as an ellipse of focal lengths 1 and 2 and centered at $(-5, \pi)$, the other shaped as a circle of a radius 1 and centered at $(5, \pi)$. Their boundaries appear in figures 5, 5, and 6. We pick the values $z_1 = -50, z_2 = 50$, to bound the rectangle $\mathcal{R}$. We first assume that $\omega = 1, \epsilon_0 = 1, \mu_0 = \mu_1 = 1, \kappa = (0, 0)$, ensuring that one harmonic only propagates. Thus, we know that nonrobust guided modes do exist at certain values of ϵ_1.

The first guided mode that we find corresponds to $\epsilon_1 \approx 9.762$, and is plotted in figure 5. That values were obtained with a mesh containing 4592 elements. Numerical convergence was verified by either quadrupling the number of mesh elements or changing $z_1 = -50, z_2 = 50$ into $z_1 = -60, z_2 = 60$. These refinements did not change the first four digits of the numerical value for ϵ_1. The numerical method employed finds complex eigenvalues, and sorts them in increasing real part order. Some of those eigenvalues do not have a small imaginary part: we ignore them, as they are unrelated to the solutions we are trying to compute. We verify decay of the solution as we move away from the dielectrics. This is illustrated in the graph in Figure 5, which shows the absolute value of the solution along the line $z = \pi/2$. We also show the graphs of the second and third guided modes, still for the same values of $\omega, \epsilon_0, \mu_0, \mu_1, \kappa_1$. They appear in Figure 5.

Finally, we compute a robust guided mode with $\kappa_1 \neq 0$. To guarantee existence, we choose values for the material parameters such that no harmonic can propagate. More

1.



2.

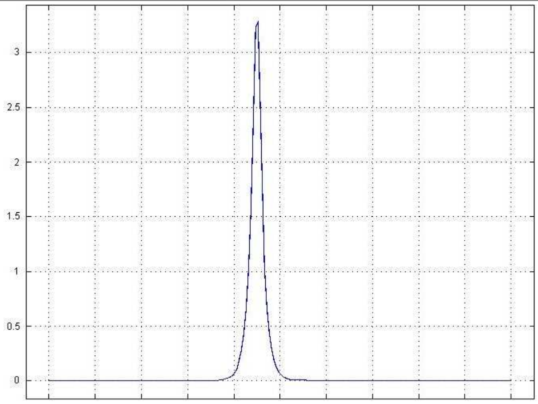


Figure 4: 1. Eigenfunction for the first real eigenvalue $\epsilon_1 \approx 9.762$ for the parameters $\omega = 1, \epsilon_0 = 1, \mu_0 = \mu_1 = 1, \kappa = (0, 0)$. 2. Cross section of the solution along the line $z = \pi/2$; the magnitude of the solution is plotted.

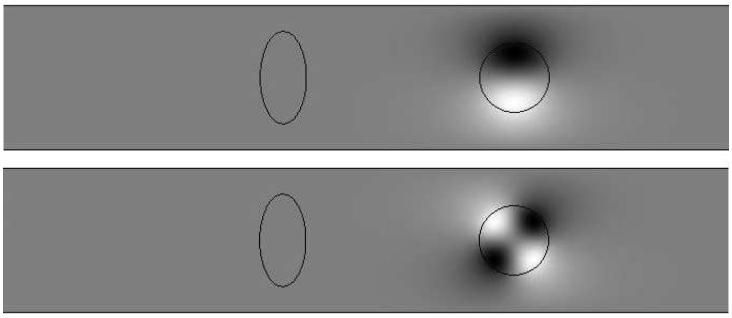


Figure 5: Eigenfunctions for the second and third real eigenvalues $\epsilon_1 \approx 11.00$ and $\epsilon_1 \approx 25.66$ for the parameters $\omega = 1, \epsilon_0 = 1, \mu_0 = \mu_1 = 1, \kappa_1 = 0$. These are nonrobust guided modes.

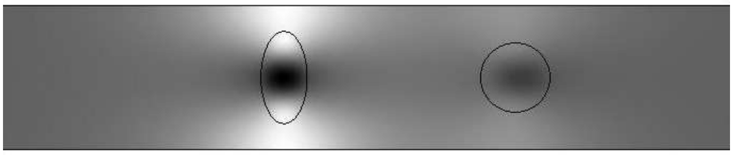


Figure 6: Eigenfunction for the third real eigenvalue $\epsilon_1 \approx 11.42$ for the parameters $\omega = 0.3, \epsilon_0 = 1, \mu_0 = 1, \mu_1 = 3, \kappa_1 = 0.4$. This is a robust guided mode.

precisely, we select, $\omega = 0.3, \epsilon_0 = 1, \mu_0 = 1, \mu_1 = 3, \kappa_1 = 0.4$. The third eigenfunction for that case is plotted in Figure 6.

6 Appendix: Proofs of Theorems

Proof of Theorem 2.7. (Equivalence of strong and weak conditions.) Many of the arguments are standard in the literature on elliptic equations; details of the relevant theory can be found in ([9], Chapter 8), for example. We confine discussion to the basic elements of the proof and those aspects that are unique to this problem.

i. That the strong formulation satisfies the weak is a matter of application of the divergence theorem (integration by parts). The relevant identity is

$$\nabla\cdot \left[\left(\frac{1}{\mu}(\nabla + i\boldsymbol{\kappa})u\right)\bar{v}\right] = \left[(\nabla + i\boldsymbol{\kappa})\cdot\left(\frac{1}{\mu}(\nabla + i\boldsymbol{\kappa})u\right)\right]\bar{v} + \frac{1}{\mu}(\nabla + i\boldsymbol{\kappa})u\cdot(\nabla - i\boldsymbol{\kappa})\bar{v}. \quad (39)$$

Applying this identity for a function u that satisfies the strong Condition 2.2, the left-hand side of part (i) of the weak Condition 2.4 becomes

$$-\int_{\mathcal{R}}\left[\frac{1}{\mu}\left((\nabla + i\boldsymbol{\kappa})^2 u\right) + \varepsilon\omega^2 u\right]\bar{v} + \frac{1}{\mu_0}\int_{\Gamma}(Bu + \partial_n u_-)\,T\bar{v} - \int_{\partial\mathcal{R}\setminus\Gamma}\frac{1}{\mu}\partial_n u T\bar{v} + \int_{\Sigma}\left[\frac{1}{\alpha\mu_1}(\partial_n u_- + (i\boldsymbol{\kappa}\cdot n)u) - \frac{1}{\alpha\mu_0}(\partial_n u_+ + (i\boldsymbol{\kappa}\cdot n)u)\right]T\bar{v}. \quad (40)$$

The integral over $\mathcal{R}$ vanishes by properties (i–iii), the integral over Γ by property (v), that over $\partial\mathcal{R}\setminus\Gamma$ by properties (vii–$viii$), and the integral over Σ by property (vi).

ii. Let u satisfy Condition 2.4. The functions v of class C^∞ with compact support in $\mathcal{R} \setminus \Sigma$ are contained in $H^1(\mathcal{R})$, and this is sufficient to establish that u satisfies the Helmholtz equation in $\mathcal{R} \setminus \Sigma$ (*i-iii*) and that $u \in H^2(\mathcal{R})$ ([9], §8.3). Thus u has values on $\partial\mathcal{R}$ (including the interior side of Γ) and Σ that are of class $H^{3/2}$ and normal derivatives of class $H^{1/2}$. Integration by parts, using the Helmholtz equation away from these boundaries, establishes properties (*iv–viii*). The extension of u to all of $\mathcal{S}$ is achieved by the formula (22).

The form of the extension (22) of u to $\mathcal{S}$ outside of $\mathcal{R}$ shows that $a^-_{mn} = b^+_{mn} = 0$ for all (m, n). The relation (15) expressing conservation of energy, which is obtained by integration by parts with $v = u$, shows that $a^+_{mn} = b^-_{mn} = 0$ for $(m, n) \in \mathcal{P}$. The additional requirement in Condition 2.4 for $\nu^2_{mn} = 0$ establishes $a^+_{mn} = b^-_{mn} = 0$ for $(m, n) \in \tilde{\mathcal{P}}$. Therefore each harmonic with $(m, n) \in \tilde{\mathcal{P}}$ in the expansion (9) has vanishing value and normal derivative on Γ, implying that $u \in X$ and part (*ii*) of the second form of the weak Condition 2.5 are satisfied.

iii. The functions v of class C^∞ with compact support in $\mathcal{R}\setminus\Sigma$ are contained in X, and again we obtain that u satisfies the Helmholtz equation in $\mathcal{R}\setminus\Sigma$ and $u \in H^2(\mathcal{R})$. It suffices to prove that, for each $(m, n) \in \mathcal{P}$, the weak form in Condition 2.5 holds for v such that $(v|_{\Gamma_1}\widehat{)}_{mn} = 1$, $(v|_{\Gamma_1}\widehat{)}_{m'n'} = 0$ for $(m', n') \neq (m, n)$, and $(v|_{\Gamma_2}\widehat{)}_{m'n'} = 0$ for all (m', n') (and similarly with Γ_1 and Γ_2 interchanged). Applying integration by parts for such v together with the Helmholtz equation for u yields for the left-hand side of the equation in the weak Condition 2.4,

$$\text{left-hand side} = \int_{\Gamma_1} (\partial_n u + Bu)\bar{v} = \big((\partial_n u|_{\Gamma_1}\widehat{)}_{mn} - \nu_{mn}(u|_{\Gamma_1}\widehat{)}_{mn}\big) = 0,$$

in which $(\partial_n u|_{\Gamma_1}\widehat{)}_{mn} = 0$ by property (*ii*) of Condition 2.5 and $(u|_{\Gamma_1}\widehat{)}_{mn} = 0$ because $u \in X$.

iv. To prove part (*i*) of Condition 2.5, it suffices to prove the equality for $v \in X^{\text{sym}}$, which follows from the observation that the integrands are antisymmetric over the regions of integration. To prove part (*ii*), we observe that the Fourier coefficients with m even are zero because u is antisymmetric in the x-variable.

□

Recall the sesquilinear forms in $H^1_{\text{per}}(\mathcal{R})$:

$$A(u, v) = \int_{\mathcal{R}} \frac{1}{\mu} (\nabla + i\boldsymbol{\kappa})\, u \cdot (\nabla - i\boldsymbol{\kappa})\, \bar{v} + \frac{1}{\mu_0} \int_{\Gamma} (BTu)(T\bar{v}), \tag{41}$$

$$\ell(u, v) = \int_{\mathcal{R}\setminus\Omega} \epsilon_0 \omega^2 u\bar{v} + \int_{\Omega} \epsilon_1 \omega^2 u\bar{v}. \tag{42}$$

and that A depends on $\boldsymbol{\kappa}$, ω, ϵ_0, μ_0, and μ_1 (the dependence on ω and ϵ_0 is through B—see equations (19) and (10)) and ℓ depends on ω, ϵ_0, and ϵ_1; neither form depends on α.

Lemma 6.1 (Estimates) *There exist positive numbers C and δ such that, for all $u, v \in H^1(\mathcal{R})$,*

i. $\min\{\epsilon_0, \epsilon_1\}\|u\|_{L^2}^2 \le \ell(u,u) \le \max\{\epsilon_0, \epsilon_1\}\|u\|_{L^2}^2$ *(equivalence of $\|\cdot\|_{L^2}$ and $\ell(\cdot,\cdot)$),*

ii. $|A(u,v)| \le C\|u\|_{H^1}\|v\|_{H^1}$ *(boundedness of A),*

iii. $\delta\|u\|_{H^1}^2 \le A(u,u)$ *(coercivity of A).*

These constants depend on the parameters $\boldsymbol{\kappa}$, ω, ϵ_0, μ_0, and μ_1.

Proof.

i. Part (*i*) is straightforward to verify.

ii. Because the trace operator $T : H^1(\mathcal{R}) \to H^{1/2}(\Gamma)$ and the operator $B : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are bounded, there is a constant C_1 such that

$$\left|\int_\Gamma (BTu)(T\bar{v})\right| \le C_1\|u\|_{H^1}\|v\|_{H^1}.$$

This, together with the estimate

$$\begin{aligned}\min\{\mu_0,\mu_1\}\left|\int_{\mathcal{R}} \frac{1}{\mu}\,(\nabla + i\boldsymbol{\kappa})\,u\cdot(\nabla - i\boldsymbol{\kappa})\,\bar{v}\right| &\le \left|\int_{\mathcal{R}} (\nabla + i\boldsymbol{\kappa})\,u\cdot(\nabla - i\boldsymbol{\kappa})\,\bar{v}\right| \\ &\le \|\,(\nabla + i\boldsymbol{\kappa})\,u\|_{L^2}\|\,(\nabla + i\boldsymbol{\kappa})\,v\|_{L^2} \\ &\le (\|\nabla u\|_{L^2} + |\boldsymbol{\kappa}|\|u\|_{L^2})\,(\|\nabla u\|_{L^2} + |\boldsymbol{\kappa}|\|u\|_{L^2}) \le C_2\|u\|_{H^1}\|u\|_{H^1},\end{aligned}$$

prove the estimate.

iii. Suppose, to the contrary, that there exists a sequence $\{u_n\}_{n=0}^\infty$ from X such that

$$\|u_n\|_{H^1} = 1 \quad \text{and} \quad A(u_n,u_n) \to 0 \text{ as } n \to \infty. \tag{43}$$

Because of the compact embedding of X into $L^2(\mathcal{R})$, we simply assume that there is a function $u \in L^2(\mathcal{R})$ such that

$$\|u - u_n\|_{L^2(\mathcal{R})} \to 0,$$

and from the definition of A, we see that

$$\| (\nabla + i\boldsymbol{\kappa})\, u_n \|_{L^2(\mathcal{R})} \to 0,$$

whence

$$\|\nabla u_n + i\boldsymbol{\kappa} u\|_{L^2(\mathcal{R})} \to 0.$$

It follows that $u_n \to u$ in the H^1-norm so that $u \in X$ and

$$(\nabla + i\boldsymbol{\kappa})\, u \;=\; 0,$$

from which we obtain

$$u = C_3 e^{i(\kappa_1 x + \kappa_2 y)} \tag{44}$$

for some constant C. From the convergence of u_n to u in X, the boundedness of T and B, and the definition of A, we have

$$\int_\Gamma (BTu) T\bar{u} = \lim_{n\to\infty} \int_\Gamma (BTu_n) T\bar{u}_n = 0.$$

Since B is a positive operator on $H^{1/2}(\Gamma)$, we obtain $Tu = 0$, and the form (44) therefore gives $u = 0$. This is in contradiction to the supposition that $\|u_n\|_{H^1} = 1$, and (43) is therefore untenable. $\square$

Because of the equivalence of the norms $\ell(f,f)$ and $\|f\|_{L^2(\mathcal{R})}$ (property (*ii*) of Lemma 6.1), we may define by $L^2(\mathcal{R},\ell)$ to be the linear space of functions in $L^2(\mathcal{R})$ endowed with the inner product $\ell(f,g)$.

Proof of Theorem 3.1. By the Lax-Milgram theorem, there exists a linear operator $K : L^2(\mathcal{R},\ell) \to X$ such that, for each $f \in L^2(\mathcal{R})$,

$$A(Kf, v) = \ell(f, v) \quad \text{for all } v \in X,$$

and by the sesquilinearity of A and ℓ, we also have

$$A(v, Kf) = \ell(v, f) \quad \text{for all } v \in X.$$

K is self-adjoint in the inner product $\ell(\cdot,\cdot)$ because

$$\ell(Kf, g) = A(Kf, Kg) = \ell(f, Kg) \quad \text{for all} \quad f,\, g \in L^2(\mathcal{R}).$$

In fact, K is positive because

$$\ell(Kf, f) = A(Kf, Kf) > 0 \quad \text{for all} \quad f \in L^2(\mathcal{R}).$$

K is injective because, if $Kf = 0$, then $\ell(f, v) = 0$ for all $v \in X$, and since X contains the functions of class C^∞ with compact support in $\mathcal{R}$, $f = 0$ almost everywhere, so that $f = 0$ in $L^2(\mathcal{R})$. The estimate

$$\delta \|Kf\|_{H^1}^2 \leq A(Kf, Kf) = \ell(f, Kf) \leq C\|f\|_{L^2}\|Kf\|_{L^2} \leq C\|f\|_{L^2}\|Kf\|_{H^1}$$

gives us

$$\delta \|Kf\|_{H^1} \leq C\|f\|_{L^2},$$

which shows that K is compact as an operator on $L^2(\mathcal{R})$. As the $L^2(\mathcal{R})$-norm with respect to Lebesgue measure on $\mathcal{R}$ and the norm in $L^2(\mathcal{R}, \ell)$ are equivalent, K is compact as an operator on $L^2(\mathcal{R}, \ell)$. The spectrum of K therefore consists of a nonincreasing sequence of eigenvalues $\{\lambda_j\}_{j=0}^\infty$ converging to zero, in which eigenvalues are repeated according to multiplicity, and there is a corresponding sequence of eigenfunctions $\{\psi_j\}_{j=0}^\infty$ that form an orthonormal Hilbert-space basis for $L^2(\mathcal{R}, \ell)$.

By definition of A, and K, we see that, for $\alpha \in \mathbb{R}$,

$$A(u, \cdot) = \alpha \ell(u, \cdot) \iff Ku = \alpha^{-1} u, \tag{45}$$

in which the dot in the second argument of the forms indicates action on X. The sequence of eigenvalues α_j we seek is therefore

$$\{\alpha_j = \lambda_j^{-1}\}_{j=0}^\infty.$$

By properties (*i, iii*) of Lemma 6.1,

$$\ell(u, u) \leq C_4 A(u, u),$$

which shows that $\alpha_0 > 0$.

We now show that the eigenvalues and eigenfunctions arise from minimization of the Rayleigh quotient. Define

$$\beta_0 = \inf_{u \in X, u \neq 0} J(u).$$

We prove that there exists a nonzero function $\phi_0 \in X$ such that

$$\beta_0 = J(\phi_0) > 0.$$

Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence, that is, $u_n \neq 0$ for each n and $\lim_{n\to\infty} J(u_n) = \beta_0$. By homogeneity of $J(u)$, we may assume that $\ell(u_n, u_n) = 1$ for each n, and since the sequence $\{J(u_n)\}$ is bounded, $\{A(u_n, u_n)\}$ is also bounded. Part (*iii*) of the Lemma shows that $\{\|u_n\|_{H^1}\}$ is bounded.

By the compact embedding of $H^1_{\text{per}}(\mathcal{R})$ into $L^2(\mathcal{R})$, there is a subsequence that is strongly convergent in $L^2(\mathcal{R})$; we simply assume therefore that $\{u_n\}$ is strongly convergent in $L^2(\mathcal{R})$, say to a function ϕ_0. By the second inequality in part (i) of Lemma 6.1, $\ell(\phi_0, \phi_0) = 1$. We now prove that $\|u_n - u_m\|_{H^1} \to 0$ as $m, n \to \infty$. The parallelogram law holds for A:

$$A(u_m - u_n, u_m - u_n) = A(u_m, u_m) + A(u_n, u_n) - A(u_m + u_n, u_m + u_n). \tag{46}$$

Because $\ell(u_n, u_n) = 1$, $A(u_n, u_n) = J(u_n)$, and the sum of the first two terms on the right-hand side of (46) converges to $4\beta_0$. By definition of β_0 and because of the second inequality in (i) of the lemma,

$$A(u_m + u_n, u_m + u_n) \geq \beta_0 \ell(u_m + u_n, u_m + u_n) \to \beta_0 \ell(2u, 2u) = 4\beta_0 \text{ as } m, n \to \infty.$$

We thus obtain $A(u_m - u_n, u_m - u_n) \to 0$ as $m, n \to \infty$, and from (iii) of the lemma, $\|u_n - u_m\|_{H^1} \to 0$. Therefore, $u_n \to \phi_0 \in X$ in the H^1 norm. Part (ii) shows that $A(u_n, u_n) \to A(\phi_0, \phi_0)$ as $n \to \infty$, and therefore

$$J(\phi_0) = \lim_{n\to\infty} J(u_n) = \beta_0.$$

To define β_1 and ϕ_1, we set Y_1 to be the orthogonal complement of $\text{span}\{\phi_0\}$ in X with respect to the sesquilinear form $A(u, v)$,

$$Y_1 = \{v \in X : A(\phi_0, v) = 0\},$$

and define

$$\beta_1 = \inf_{u \in Y_1, u \neq 0} J(u).$$

The proof of the existence of a minimizer ϕ_1 in Y_1 is essentially the same as the proof of the existence of ϕ_0. Continuing in this way, we obtain a sequence $\{Y_j\}$ of subspaces of X, numbers β_j, and functions ϕ_j such that

$$Y_j = \{v \in X : A(\phi_k, v) = 0 \text{ for } k = 0, \ldots, j-1\}$$

and

$$\beta_j = \inf_{u \in Y_j, u \neq 0} J(u) = J(\phi_j), \quad \phi_j \in Y_j.$$

Taking the first variation of the relation $A(u, u) = J(u)\ell(u, u)$ at $u = \phi_j$ and using the fact that J is minimized by ϕ_j in Y_j and that $A(\phi_k, \phi_j) = \ell(\phi_k, \phi_j) = 0$ for $k = 0, \ldots, j-1$, we obtain

$$A(\phi_j, v) = \beta_j \ell(\phi_j, v) \quad \text{for all } v \in X. \tag{47}$$

By definition, ϕ_{j+1} minimizes the same functional as ϕ_j, but over a smaller set, and therefore the sequence $\{\beta_j\}$ is nondecreasing:

$$0 < \beta_0 \leq \beta_1 \leq \cdots \leq \beta_j \leq \ldots .$$

Because of (45) and (47), we have

$$\{\beta_j\}_{j=0}^{\infty} \subseteq \{\alpha_j\}_{j=0}^{\infty} \quad \text{and} \quad \mathrm{span}\{\phi_j : j = 0, \dots, \infty\} \subseteq \mathrm{span}\{\psi_j : j = 0, \dots, \infty\}.$$

To show for, $j = 0, \dots, \infty$, that $\alpha_j = \beta_j$, that ψ_j can be taken to be equal to ϕ_j, and that $X_j = Y_j$, we prove that any eigenvalue α with eigenfunction $0 \neq \psi \in X$, in the sense that

$$A(\psi, v) = \alpha \ell(\psi, v) \quad \text{for all } v \in X$$

is necessarily one of the β_j and that ψ is in the span of $\{\phi_j : \beta_j = \alpha\}$. If, to the contrary, $\alpha \neq \beta_j$ for all n, then $A(\phi_j, \psi) = 0$ for all n because

$$A(\psi, \phi_j) = \alpha \ell(\psi, \phi_j) \quad \text{and} \quad A(\phi_j, \psi) = \beta_j \ell(\phi_j, \psi),$$

whence we obtain, from conjugating the first relation and keeping in mind that $\alpha \geq \alpha_0 > 0$ and $\beta_j \geq \alpha_0 > 0$,

$$(\alpha^{-1} - \beta_j^{-1}) A(\phi_j, \psi) = 0.$$

Since $\alpha \neq \beta_j$, we obtain $A(\phi_j, \psi) = 0$, as desired. This implies that $\psi \in Y_{j+1}$ so that

$$\alpha = \frac{A(\psi, \psi)}{\ell(\psi, \psi)} \geq \inf_{u \in Y_j, u \neq 0} J(u) = \beta_j \quad \text{for all } j,$$

which is impossible because $\beta_j \to \infty$. Therefore we may let k be

$$k = \max\{j : \beta_j = \alpha\}.$$

We still have $A(\psi, \phi_j) = 0$ for all j with $\beta_j \neq \alpha$. If we also have $A(\psi, \phi_j) = 0$ for all j with $\beta_j = \alpha$, then

$$\alpha = J(\psi) \geq \inf_{u \in Y_{k+1}, u \neq 0} J(u) = \beta_{k+1} > \beta_k. \quad \text{(contradiction)}$$

We now see that ψ, which was taken to be an *arbitrary* nonzero element of the eigenspace for α, is such that $A(\psi, \phi_j)$ for some j with $\beta_j = \alpha$. This implies that the eigenspace for α is in fact equal to $\mathrm{span}\{\phi_j : \beta_j = \alpha\}$.

The last part of the theorem on the symmetric and antisymmetric eigenfunctions is proved analogously by replacing X by X^{sym} and X^{ant} and using the fact that these two spaces are orthogonal with respect to the sesquilinear form $A(\cdot, \cdot)$. There are no essential changes in the proof.

□

The form ℓ depends on the parameter ϵ_1; we make this dependence explicit by introducing the variable ϵ:

$$\ell_\epsilon(u,v) = \int_{\mathcal{R}\setminus\Omega} \epsilon_0 \omega^2 u\bar{v} + \int_\Omega \epsilon\omega^2 u\bar{v}, \tag{48}$$

$$J_\epsilon(u) = \frac{A(u,u)}{\ell_\epsilon(u,u)} = \frac{\int_{\mathcal{R}} \frac{1}{\mu} |\,(\nabla + i\boldsymbol{\kappa})\,u|^2 \;+\; \frac{1}{\mu_0}\int_\Gamma (BTu)(T\bar{u})}{\epsilon_0\omega^2 \int_{\mathcal{R}\setminus\Omega} |u|^2 \;+\; \epsilon\omega^2 \int_\Omega |u|^2}. \tag{49}$$

The eigenvalues and eigenfunctions also depend on ϵ, and we denote them by $\alpha_j(\epsilon)$ and $\psi_j(\epsilon)$. Normalizing the eigenfunctions so that $\ell_\epsilon(\psi_j(\epsilon), \psi_j(\epsilon)) = 1$, we have

$$\ell_\epsilon(\psi_j(\epsilon), \psi_k(\epsilon)) = \delta_{jk}, \qquad A(\psi_j(\epsilon), \psi_k(\epsilon)) = 0 \text{ for } j \neq k.$$

The compact operator $K = K_\epsilon$ also depends on ϵ:

$$A(K_\epsilon f, v) = \ell_\epsilon(f, v) \quad \text{for all } v \in X, \tag{50}$$

and the eigenvalues of K_ϵ are $\{\alpha_j(\epsilon)^{-1}\}_{j=0}^\infty$ with corresponding eigenvectors $\{\psi_j(\epsilon)\}_{j=0}^\infty$.

Lemma 6.2 *K_ϵ is continuous in ϵ with respect to the operator norm on K_ϵ.*

Proof. Let $\epsilon_1 > 0$ be given. For an arbitrary variation $\Delta\epsilon > 0$ with $0 < |\Delta\epsilon| < \epsilon_1$, set

$$\Delta K = K_{\epsilon_1 + \Delta\epsilon} - K_{\epsilon_1} \quad \text{and} \quad \Delta\ell = \ell_{\epsilon_1 + \Delta\epsilon} - \ell_{\epsilon_1}.$$

Applying the defining relation (50) for K_ϵ to $K_{\epsilon_1+\Delta\epsilon}$ and K_{ϵ_1}, with $v = \Delta K f$, and subtracting yields the relation

$$A(\Delta K f, \Delta K f) = \Delta\ell(f, \Delta K f). \tag{51}$$

We have the following lower estimate for the left-hand side of (51):

$$\delta \|\Delta K f\|_{L^2}^2 \leq \delta \|\Delta K f\|_{H^1}^2 \leq A(\Delta K f, \Delta K f),$$

and upper estimate for the right-hand side:

$$|\Delta\ell(f, \Delta K f)| = |\Delta\epsilon| \left| \int_\Omega f \Delta K \bar{f} \right| \leq |\Delta\epsilon| \, \|f\|_{L^2} \, \|\Delta K f\|_{L^2}.$$

Putting these inequalities together, we obtain

$$\|\Delta K f\|_{L^2} \leq \frac{|\Delta\epsilon|}{\delta} \|f\|_{L^2},$$

so that $\|\Delta K\| \le |\Delta\epsilon|/\delta$. $\square$

We now prove the lemma of Section 3 that states that the eigenvalues α are continuous strictly decreasing functions of ϵ_1, and $\alpha_j \to 0$ as $\epsilon_1 \to \infty$. A similar result was stated for the μ_1 dependency, but we omit the proof in that case, as it is similar.

Proof of Lemma 3.2. By Lemma 6.2, K_ϵ is continuous in ϵ with respect to the operator norm on K_ϵ and its spectrum is the set $\{\alpha_j(\epsilon)^{-1}\}_{j=0}^{\infty}$. Because the eigenvalues of compact operators are continuous functions of the operator in the operator norm (Kato [10], Ch. IV, §3.5), we conclude that the functions $\alpha_j(\epsilon)$ are continuous functions of ϵ.

To prove that $\alpha_j(\epsilon)$ is strictly decreasing in ϵ, let ϵ_1 and ϵ_2 be given with $0 < \epsilon_1 < \epsilon_2$, and let an integer $N \ge 0$ be given. Define

$$V_N = \text{span}\{\psi_j(\epsilon_1) : 0 \le j \le N\},$$

($V_0 = \{0\}$) in which the eigenvectors $\psi_j(\epsilon_1)$ are orthonormal with respect to $\ell_{\epsilon_1}(\cdot,\cdot)$ and orthogonal with respect to $A(\cdot,\cdot)$:

$$\ell_{\epsilon_1}(\psi_j(\epsilon_1), \psi_k(\epsilon_1)) = \delta_{jk}, \qquad A(\psi_j(\epsilon_1), \psi_k(\epsilon_1)) = 0 \text{ for } j \ne k.$$

For each $\psi \in V_N$ with $\ell_{\epsilon_1}(\psi,\psi) = 1$, there are numbers a_j, for $0 \le j \le N$ such that

$$\psi = \sum_{j=0}^{N} a_j \psi_j(\epsilon_1), \qquad \sum_{j=0}^{N} |a_j|^2 = 1,$$

and we obtain $A(\psi,\psi) = \sum_{j=0}^{N} |a_j|^2 A(\psi_j(\epsilon_1), \psi_j(\epsilon_1))$ so that

$$A(\psi,\psi) = J_{\epsilon_1}(\psi) \;=\; \sum_{j=0}^{N} |a_j|^2 J_{\epsilon_1}(\psi_j(\epsilon_1)) \;=\; \sum_{j=0}^{N} |a_j|^2 \alpha_j(\epsilon_1) \;\le\; \alpha_N(\epsilon_1). \tag{52}$$

From the definition of J_ϵ, it is evident that $J_{\epsilon_2}(\phi) \le J_{\epsilon_1}(\phi)$ for each $\phi \in X$; however, we need to show strict inequality for $\phi \in V_N$, which requires showing that, for each nonzero $\phi \in V_N$, it is not true that ϕ is equal to zero almost everywhere on Ω. To this end, let

$$\phi = \sum_{j=0}^{N} b_j \psi_j(\epsilon_1) = 0 \quad \text{a.e. in} \quad \Omega.$$

Put $\beta_j = \alpha_j(\epsilon_1)\epsilon_1\mu_1\omega^2$. As the $\psi_j(\epsilon_1)$ are smooth in Ω, for each $k = 0, \ldots, N$, we have

$$0 = \prod_{\beta_j \ne \beta_k} \left((\nabla + i\boldsymbol{\kappa})^2 + \beta_j \right) \phi = \prod_{\beta_j \ne \beta_k} (-\beta_k + \beta_j) \sum_{\beta_j = \beta_k} b_j \psi_j(\epsilon_1) \quad \text{in} \quad \Omega.$$

Since $\prod_{\beta_j \neq \beta_k}(-\beta_k+\beta_j) \neq 0$, we obtain $\sum_{\beta_k=\beta_h} b_k \psi_k(\epsilon_1) = 0$ in Ω. However, $\sum_{\beta_j=\beta_k} b_j \psi_j(\epsilon_1)$ satisfies Condition 2.2, and therefore $\sum_{\beta_j=\beta_k} b_j \psi_j(\epsilon_1)$ is zero in $\mathcal{R}$, and therefore the zero element of X, in which the $\psi_j(\epsilon_1)$ are linearly independent, and we infer that $b_j = 0$ for j such that $\beta_j = \beta_k$. As k was chosen arbitrarily from $\{0, \ldots, N\}$, we obtain $b_j = 0$ for $0 \leq j \leq N$. We conclude that $\psi = \sum_{j=0}^{N} a_j \psi_j(\epsilon_1)$ from above is not zero in $L^2(\Omega)$. It follows now from the definitions of J_ϵ and ℓ_ϵ and from (52) that $J_{\epsilon_2}(\psi) < J_{\epsilon_1}(\psi) \leq \alpha_N(\epsilon_1)$, and by the homogeneity of J_{ϵ_2}, we obtain

$$J_{\epsilon_2}(\phi) < J_{\epsilon_1}(\phi) \leq \alpha_N(\epsilon_1) \quad \text{for all} \quad \phi \in V_N. \tag{53}$$

Define, for each $\epsilon > 0$,

$$X_N(\epsilon) = \{v \in X : A(\psi_j(\epsilon), v) = 0 \text{ for } j = 0, \ldots, N-1\}.$$

The dimension of $V_N \cap X_N(\epsilon_2)$ is at least 1; let ϕ be a nonzero vector in this intersection. We obtain

$$\alpha_N(\epsilon_2) = \inf_{u \in X_N(\epsilon_2), u \neq 0} J_{\epsilon_2}(u) \leq J_{\epsilon_2}(\phi) < \alpha_N(\epsilon_1),$$

and we have proved that $\alpha_N(\epsilon)$ is a decreasing function of ϵ.

We now prove that $\alpha_N(\epsilon)$ tends to zero as ϵ tends to infinity. The set

$$S = \{\psi \in V_N : \ell_{\epsilon_1}(\psi, \psi) = 1\}.$$

S is compact in $L^2(\mathcal{R}, \ell_{\epsilon_1})$ and therefore also in $L^2(\mathcal{R})$. Since $\int_\Omega |\psi|^2$ is continuous in $L^2(\mathcal{R})$ and $\int_\Omega |\psi|^2 \neq 0$ for all $\psi \in S$ there is a number M such that

$$0 < M < \int_\Omega |\psi|^2 \quad \text{for all} \quad \psi \in S.$$

Therefore

$$\ell_\epsilon(\psi, \psi) \geq \epsilon\omega^2 \int_\Omega |\psi|^2 > \epsilon\omega^2 M \quad \text{for all} \quad \psi \in S,$$

and, using (52) for $\epsilon > \epsilon_1$,

$$J_\epsilon(\psi) = \frac{A(\psi, \psi)}{\ell_\epsilon(\psi, \psi)} < \frac{\alpha_N(\epsilon_1)}{\epsilon\omega^2 M} \quad \text{for all} \quad \psi \in S.$$

The dimension of $V_N \cap X_N(\epsilon)$ is at least 1. Let ϕ be a nonzero vector in this intersection, which we may take to be in S. We then obtain

$$\alpha_N(\epsilon) = \inf_{u \in X_N(\epsilon), u \neq 0} J_\epsilon(u) \leq J_\epsilon(\phi) < \frac{\alpha_N(\epsilon_1)}{\epsilon\omega^2 M}. \tag{54}$$

□

The estimate (54) shows that the eigenvalues decay at least proportionally to $1/\epsilon$.

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