Applications of Spectral Theory in the Material Sciences

Math 7390-2, Spring 2008
Louisiana State University
Monday, Wednesday, and Friday, from 9:40 to 10:30
Room 218 of Prescott Hall

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Office hours: Monday, Wednesday, and Friday, 10:40–12:00; or by appointment
Course web site: `http://www.math.lsu.edu/~shipman/class/08A-7390`

Course Synopsis:

The spectrum of an operator is one of the most fundamental objects that arises in problems of physics. It describes, for example, fundamental modes of vibration or fundamental shapes that are preserved through time in the dynamics of material structures. The aim of the course is a development of the role of Hilbert space in mathematical physics and especially the spectral decomposition of operators in Hilbert space, most of which are unbounded.

The syllabus is rather ambitious. I hope to delve substantially into the first four applications listed below and at least touch on the fifth. These applications represent only a tiny but important selection of problems in mathematical physics, and they will serve to build a foundation in the methods of Hilbert space and spectral analysis.

We will begin with the theory of self-adjoint operators in Hilbert space and learn the spectral theorem. We will not learn the proof of the spectral theorem, as this would require a course in itself, but rather learn what is means, how it is manifest in examples, and how it is useful in mathematical physics. We will treat several theorems in functional analysis in this manner, taking them for granted but then building rigorously upon them.

- Review of Hilbert space and some results of functional analysis.
- Review of the spectral decomposition for transformations in finite-dimensional inner-product spaces. We will pursue an exposition that leads to a generalization of the spectral decomposition for normal operators in infinite-dimensional Hilbert space, known as the "spectral theorem".
- The spectral theorem says that every normal operator in a Hilbert space is unitarily equivalent to a multiplication operator on an L^2 space. We will discuss the main steps in a proof of the spectral theorem, referring to the literature for the details.
- Stone's Theorem. This theorem associates with each self-adjoint operator A in a Hilbert space a unique norm-preserving dynamics through the exponential function e^{-iAt}, which is the unitary *semigroup* associated to A. This semigroup gives unique solutions to the Cauchy problem $\dot{x} = -iAx$.

- Special classes of operators and their spectrum. The most important for us will be *compact operators* and *multiplication operators*.
- Embedded eigenvalues and resonance.
- The Fourier transform, discrete and continuous, finite and infinite. Spectral representation of the Laplacian in all these cases.

Applications.

- Quantum mechanics. The Schrödinger operator on the line with a well potential. The Schrödinger operator in 3-space with a $1/r$ potential well as a model of the hydrogen atom. The helium atom and associated concepts of embedded eigenvalues and resonance.
- The Laplacian and more general elliptic operators on bounded domains, which applies to such problems as diffusion (as heat), electric conduction, linear waves and vibration, electrostatics, and incompressible fluid flow. The mathematical ideas include the *weak formulation* of PDEs and the form domain of operators, the resolvent of operators, and the spectrum of compact operators.
- The Laplacian on the interval $[0, 1]$. The beautiful correspondence between self-adjoint extensions of the symmetric Laplacian to the vibrating string (or diffusion in a rod) by means of boundary values is probably the best illustration of the drastic effect that the subtle notion of the domain of an operator has on the associated dynamics, or semigroup.
- Photonic crystals. We will trudge through the review article of Peter Kuchment (2001). This is a difficult article and will require students to read portions of the literature referenced therein and present expositions thereof to the class.
- Effective conductivity of micro-composite materials. The paper of Papanicolaou and Golden (1983) deals with the bulk conductivity properties of micro-structured composites in the framework of the spectral theory.

Literature:

I will place a bibliography of relevant works on the web site for the course. I will also make available, through links from the bibliography, PDF files of research articles and excerpts of relevant parts of longer works.

Assignments:

I will periodically assign problems that are designed to illuminate the theory we are discussing and that are motivated by applications. When we come to the subject of photonic crystals, each student will choose a topic that arises in the review article of P. Kuchment and present it to the class.

Evaluation:

Evaluation of performance in the course is based on performance on the assignments and presentations.