Review of linear transformations in finite-dim'l inner product spaces

A good reference (among many) is

Linear Algebra by Friedberg, Insel, Spence
3^{rd} ed. Prentice Hall 1997

The relevant chapter, Chapter 6, can be downloaded in pdf from the course website.

Defn Let V be a vector space over $\mathbb{C}$. An inner product in V is a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ with the following properties:

* $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
* $\langle cu, v \rangle = c \langle u, v \rangle$
* $\overline{\langle u, v \rangle} = \langle v, u \rangle$
* $\langle u, u \rangle > 0$ if $u \neq 0$

The fllg properties are implied by the above:

* $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$
* $\langle u, cv \rangle = \bar{c} \langle u, v \rangle$
* $\langle 0, u \rangle = 0$

The map $V \to \mathbb{R} : v \longmapsto \langle v, v \rangle^{1/2}$ is a norm on V.

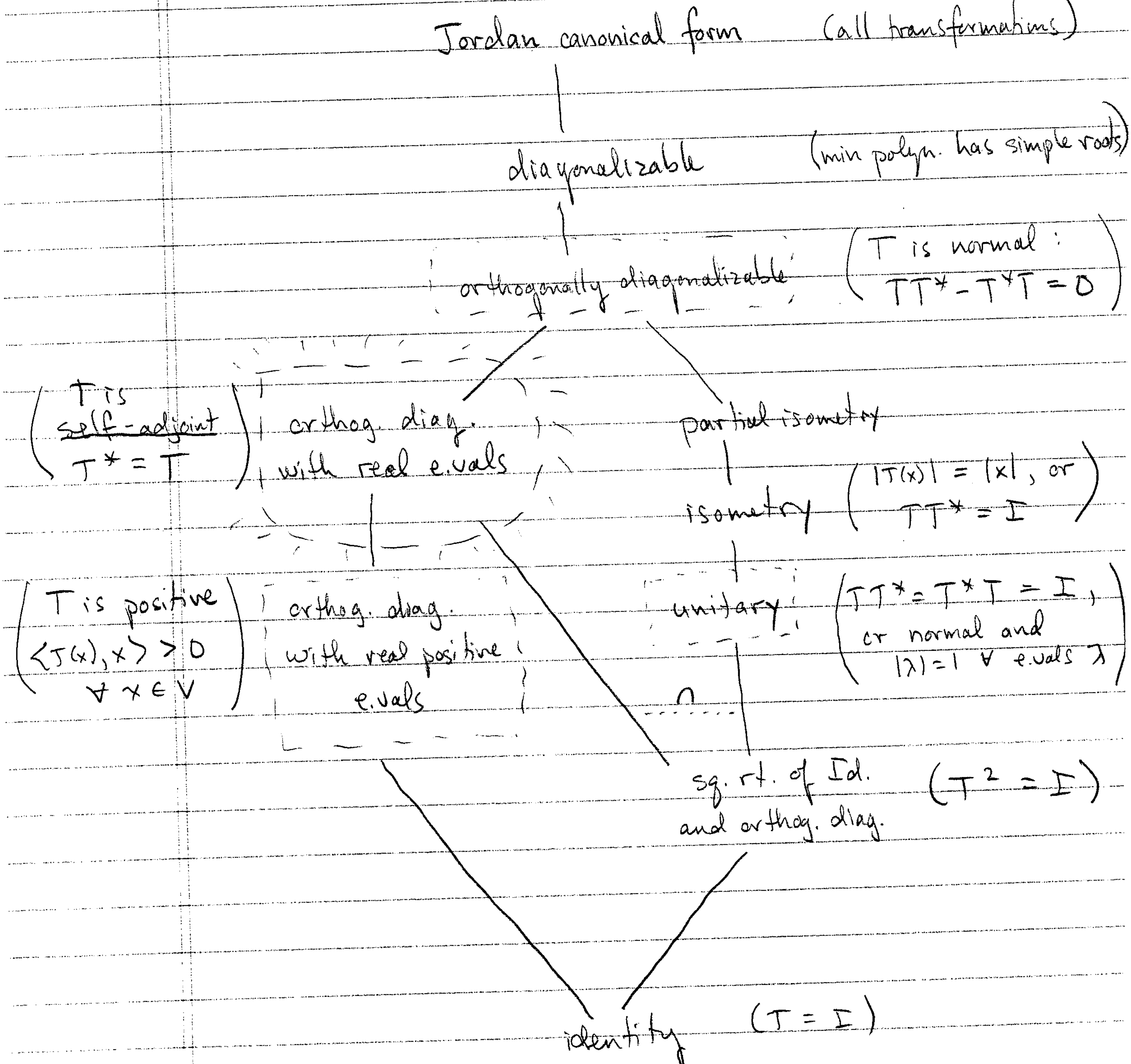
[Recall the properties of a norm.]

2008.01.14

(2)

The *adjoint* T^* of a linear transformation T is *defined through* $\langle Tx, y\rangle = \langle x, T^*y\rangle \;\forall x, y \in V$

[Show that this is well-defined.]



Math 7390-2
2008.01.16

Let V be a complex inner-product space.
$\dim V = n < \infty$.

<u>The Spectral Theorem</u> for finite-dim'l inner-product spaces.
[see Friedberg, et. al., for example]
A linear transformation $T: V \to V$ is normal if and only if it is orthogonally diagonalizable.

Let us briefly look at the main points in a proof.

First observe that, if $\mathcal{B} = \{e_i\}_{i=1}^{n}$ is a basis for V and $[T]_{\mathcal{B}}$ denotes the matrix for T with respect to $\mathcal{B}$, then if $\mathcal{B}$ is orthonormal, the adjoint T^* of T is represented by the conjugate of the transpose of $[T]_{\mathcal{B}}$ [The conj. transpose of a matrix A is denoted by A^*.]:

$$[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^* .$$

1. If T is orthogonally diagonalizable, there is an orthonormal basis $\mathcal{B}$ such that $A := [T]_{\mathcal{B}}$ is diagonal. Therefore $A^* = [T^*]_{\mathcal{B}}$ is diagonal and therefore commutes with A. Thus T commutes with T^*, so T is normal
2. Let T be normal. First we use Thm. 6.14 in [FIS] to obtain an orthonormal basis for V such that $[T]_{\mathcal{B}} =: A$ is upper triangular. By computing successive diagonal elements of $AA^* - A^*A = 0$, one shows that the off-diagonal elements must be zero.

2008. 01. 16

Our immediate objective is to present the spectral theorem for finite dimensional inner-product spaces in such a way that it can be easily generalized into the statement of the spectral theorem for (normal operators in) infinite-dimensional Hilbert spaces.

We begin with considering the meaning of the statement that T is orthogonally diagonalizable. This means that

- T has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ (as does any operator!)
  The spectrum of T $= \sigma(T) = \{\lambda_1, \ldots, \lambda_k\}$ (closed)
  The resolvent set of T $= \rho(T) = \mathbb{C} \setminus \sigma(T)$ (open)
- The corresponding eigenspaces $W_1, \ldots, W_k$ are mutually orthogonal and

(*) $$V = \bigoplus_{i=1}^{k} W_i \quad \text{(as a direct sum of inner prod. spaces)}$$

  Let P_i be the (orthogonal) projection onto W_i ($\operatorname{Ran} P_i = W_i$)
  The statement (*) is equivalent to the statement
  $$\sum_{i=1}^{k} P_i = I,$$
  Notice that $\operatorname{Ran} P_j = W_j$ and $\operatorname{Null} P_j = \bigoplus_{\substack{i=1 \\ i \neq j}}^{k} P_i =: V \ominus P_j$
- $T = \sum_{i=1}^{k} \lambda_i P_i$.

Note: The word "projection" P in an inner-product space refers to an orthogonal projection, that is $P^2 = P$ and $\operatorname{Ran} P \perp \operatorname{Null} P$.

<u>The Spectral Theorem</u> (finite dim.), reformulated.
Let V be a finite-dim'l inner-product space and $T: V \to V$ a linear transformation. Then the following two statements are equivalent.

1. T is normal
2. There exists a finite set $\sigma(T) \subset \mathbb{C}$ and projections $\{P_\lambda : \lambda \in \sigma(T)\}$ such that
   - $P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda$ [$\mathrm{Ran}\, P_\lambda \perp \mathrm{Ran}\, P_\mu$ for $\lambda \neq \mu$]
   - $\sum_{\lambda \in \sigma(T)} P_\lambda = I$ [$V = \bigoplus_{\lambda \in \sigma(T)} \underbrace{\mathrm{Ran}\, P_\lambda}_{W_\lambda}$]
   - $\sum_{\lambda \in \sigma(T)} \lambda P_\lambda = T$ ["T is diagonal"]

Notice that the statement $T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda$ is saying that

T is a "multiplication operator" in the sense that, with respect to the decomposition $V = \bigoplus_{\lambda \in \sigma(T)} W_\lambda$, <u>$T$ just multiplies the components by the eigenvalues – it does not couple the components</u> :

$$T\left(v_{\lambda_1} + v_{\lambda_2} + \cdots + v_{\lambda_k}\right) = \lambda_1 v_{\lambda_1} + \lambda_2 v_{\lambda_2} + \cdots + \lambda_k v_{\lambda_k}$$

By choosing an orthornormal basis for each W_i and taking the union of all these bases, we obtain an orthonormal basis for V with respect to which T is diagonal, or a "multiplication operator"

By choosing an orthonormal basis for each W_i and taking the union of all those bases, we obtain an orthonormal basis $\mathcal{B}$ for V with respect to which T is diagonal, or a "multiplication operator", when elements of V are represented by n-tuples with respect to $\mathcal{B}$:

$$[T]_{\mathcal{B}} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda_1 a_1 \\ \vdots \\ \lambda_n a_n \end{bmatrix}, \text{ or } [T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

where the λ_i are not necessarily distinct.

Another way to say this is that the map

$$U : \mathbb{C}^n \to V \text{ such that } (0, 0, \ldots, \underset{i^{th}\text{ place}}{1}, \ldots 0) = e_i$$

is <u>unitary</u> and $U^{-1} T U$ is a diagonal matrix Λ:

<u>The Spectral Theorem</u> (fin. dim.), reformulated.

Let V be a fin. dim. inner-prod. space and $T: V \to V$ linear. Then T is normal if and only if there exists a unitary operator $U : \mathbb{C}^n \to V$ and a diagonal matrix Λ such that

$$T = U \Lambda U^{-1}$$

Some observations

- $(T-\lambda)^* = T^* - \bar{\lambda}$
- $\sigma(T^*) = [\sigma(T)]^*$
- If T is normal, then $T-\lambda$ and $T^*-\bar{\lambda}$ have the same nullspaces (the eigenspaces of T and T^* coincide).
- If T is normal, then "$\sigma(T) \subset \mathbb{R}$" is equivalent to $T=T^*$, that is, those normal operators that are self-adjoint are exactly those with real spectrum.

Note: All of these statements can be proven quite directly from the definitions of "adjoint" and "normal", without resorting to matrix representations. One of the key observations is that

Thm If W_1 and W_2 are subspaces of V such that $W_1 = \{v \in V : T(v) \in W_2\}$ (W_1 is the preimage of W_2 under T), then $W_2^\perp = \{v \in V : T^*(v) \in W_1^\perp\}$ ($W_2^\perp$ is the preimage of $W_1^\perp$ under T^*)

Furthermore, $T(W_1) = W_2$ and $T^*(W_2^\perp) = W_1^\perp$.

Actually, you only need the weaker statement

Cor $\operatorname{Ran}(T^*) = [\operatorname{Null}(T)]^\perp$.

Since $T^{**} = T$, we have also $\operatorname{Ran} T = [\operatorname{Null}(T^*)]^\perp$.
This is a concise statement of the "Fredholm alternative" for operators in fin. dim.l inner-prod. spaces.

Observations, continued

Let T be normal and $\sigma(T)$ and P_λ for $\lambda \in \sigma(T)$ be defined as in the spectral theorem. Observe that

- $\|v\|^2 = \sum_{\lambda \in \sigma(T)} \|P_\lambda v\|^2$
- $\|Tv\|^2 = \sum_{\lambda \in \sigma(T)} |\lambda|^2 \|P_\lambda v\|^2 \leq \left(\max_{\lambda \in \sigma(T)} |\lambda|^2\right) \sum_{\lambda \in \sigma(T)} \|P_\lambda v\|^2$

$$= \left(\max_{\lambda \in \sigma(T)} |\lambda|^2\right) \|v\|^2$$

with equality if and only if $v \in W_{\lambda^+}$ (λ^+ = e.val of max modulus)

- $\Rightarrow \|T\| = \sup_{\|v\|=1} \|T(v)\| = \max_{\lambda \in \sigma(T)} |\lambda|$.

Polynomial functions of T (normal)

Let $\mathcal{P}$ denote the space of polynomials over $\mathbb{C}$ in one indeterminate: $\mathcal{P} = \mathbb{C}[x]$

$$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda \Rightarrow T^n = \sum_{\lambda \in \sigma(T)} \lambda^n P_\lambda$$

$\Rightarrow$ for $p \in \mathcal{P}$, $\boxed{p(T) = \sum_{\lambda \in \sigma(T)} p(\lambda) P_\lambda}$

Consequences:

- $p(T)$ is normal with adjoint $\bar{p}(T^*)$, where $\bar{p}$ is obtained from p by conjugating the coefficients.
- $\sigma(p(T)) = p[\sigma(T)] := \{p(\lambda) : \lambda \in \sigma(T)\}$. This is the "spectral mapping theorem".

(**) • $\|p(T)\| = \max_{\mu \in \sigma(p(T))} |\mu| = \max_{\lambda \in \sigma(T)} |p(\lambda)| = \|p\|_{L^\infty(\sigma(T))}$

Continuous functions of a normal operator $T: V \to V$, $\dim V < \infty$

This presentation will seem too technical for the simple case of finite dimension, but remember that it is for the purpose of setting the framework for the infinite-dim'l. case.

$C(\sigma(T)) = \mathcal{F}(\sigma(T)) = \mathbb{C}^{\sigma(T)}$ = set of all complex-valued functions on $\sigma(T)$ with the "uniform" norm $\|f\| = \max_{\lambda \in \sigma(T)} |\lambda|$

(continuous fcns) (all fcns)

These coincide because $\sigma(T)$ is finite.

We now define a map $\varphi : C(\sigma(T)) \longrightarrow \mathcal{L}(V)$ that is an <u>isometric</u> <u>homomorphism</u>

$\mathcal{L}(V)$ is the space of linear transformations from V to V with the <u>operator norm</u>.

- Given the spectral theorem, we can do this directly by

$$\varphi : f \longmapsto \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda$$

- A more abstract framework arises from this:

$$\mathcal{P} = \mathcal{P}_k \oplus \mathrm{Null}(\Phi)$$

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathcal{L}(V) & & C(\sigma(T)) \arrow[ll, dashed, "\varphi"'] \end{tikzcd}$$

$\|S\| = \max_{\|v\|=1} |S(v)|$ $\qquad$ $\|f\| = \max_{\lambda \in \sigma(T)} |f(\lambda)|$

Ψ is defined by restricting a polynomial to $\sigma(T)$:

$[\Psi(p)](\lambda) = p(\lambda)$ for $\lambda \in \sigma(T)$.

Φ is defined by applying a polynomial to T:

$\Phi(p) = p(T)$.

By $(**)$, we see that $\|\Phi(p)\| = \|\Psi(p)\|$, so $\mathrm{Null}\,\Phi = \mathrm{Null}\,\Psi$, and since Ψ is surjective, the homomorphism φ exists:

$$\varphi(f) = \Phi(p), \text{ where } p \text{ is chosen in the preimage } \Psi^{-1}[\{f\}]$$

We denote $\varphi(f)$ by $f(T)$.

You may show that φ is a <u>$*$-homomorphism of algebras</u>. This means that it preserves all of the algebraic structure and, in addition, takes the conjugates to adjoints:

$$\varphi : C(\sigma(T)) \longrightarrow \mathcal{L}(V)$$

$$\begin{aligned}
\varphi(af + bg) &= a\varphi(f) + b\varphi(g) \\
\varphi(fg) &= \varphi(f)\varphi(g) \\
\varphi(1) &= I \\
\varphi(\bar{f}) &= (\varphi(f))^*
\end{aligned}$$