Review of linear transformations in finite-dimen\'sion inner product spaces

A good reference (among many) is

**Linear Algebra** by Friedberg, Insel, Spence


The relevant chapter, Chapter 6, can be downloaded in pdf from the course website.

**Defn** Let \( V \) be a vector space over \( \mathbb{C} \). An inner product in \( V \) is a function \( \langle \cdot , \cdot \rangle : V \times V \to \mathbb{C} \) with the following properties:

\[
\begin{align*}
\star \quad \langle u + v, w \rangle &= \langle u, w \rangle + \langle v, w \rangle \\
\star \quad \langle c u, v \rangle &= c \langle u, v \rangle \\
\star \quad \langle u, v \rangle &= \langle v, u \rangle \\
\star \quad \langle u, u \rangle &> 0 \text{ if } u \neq 0
\end{align*}
\]

The other properties are implied by the above:

\[
\begin{align*}
\star \quad \langle u, v + w \rangle &= \langle u, v \rangle + \langle u, w \rangle \\
\star \quad \langle u, c v \rangle &= c \langle u, v \rangle \\
\star \quad \langle 0, u \rangle &= 0
\end{align*}
\]

The map \( V \to \mathbb{R} : v \mapsto \|v\|^2 \) is a norm on \( V \).

[Recall the properties of a norm.]
The adjoint $T^*$ of a linear transformation $T$ is defined through $\langle Tx, y \rangle = \langle x, T^*y \rangle \forall x,y \in V$

Show that this is well-defined.

Jordan canonical form (all transformations)

- Diagonalizable (min poly. has simple roots)

- Orthogonally diagonalizable ($T$ is normal: $TT^* - T^*T = 0$)

- $T$ is self-adjoint ($T^* = T$) orthonormal with real eigenvalues

- $T$ is positive ($\langle T(x), x \rangle > 0 \forall x \in V$) orthonormal with real positive eigenvalues

- Isometry ($\langle T(x), x \rangle = \langle x, x \rangle$; $T$ is a norm-preserving mapping)

- Unitary ($TT^* = T^*T = I$; or normal and $|\lambda| = 1 \forall \lambda$ eigenvalues)

- $T^2 = I$; $T$ is an orthogonal projection

- $T = I$; $T$ is the identity matrix

Example: $T = \text{Id}$. $T^2 = I$.

and orthonormal.
Let \( V \) be a complex inner-product space, 
\[
\dim V = n < \infty.
\]

The **Spectral Theorem** for finite-dim'l inner-product spaces. 
[See Friedberg, et al., for example]

A linear transformation \( T : V \to V \) is normal if and only if it is orthogonally diagonalizable.

Let us briefly look at the main points in a proof.

First observe that, if \( \mathcal{B} = \{e_1, e_2, \ldots, e_n \} \) is a basis for \( V \) and 
\([T]_{\mathcal{B}}\) denotes the matrix for \( T \) with respect to \( \mathcal{B} \), 
then if \( \mathcal{B} \) is orthonormal, the adjoint \( T^* \) of \( T \) 
is represented by the conjugate of the transpose of 
\([T]_{\mathcal{B}}\). [The conj. transpose of a matrix \( A \) is denoted by \( A^* \).]

\[
[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^*.
\]

1. If \( T \) is orthogonally diagonalizable, there is an orthonormal basis \( \mathcal{B} \) such that \( A := [T]_{\mathcal{B}} \) is diagonal. Therefore 
\( A^* = [T^*]_{\mathcal{B}} \) is diagonal and therefore commutes with \( A \). Thus \( T \) commutes with \( T^* \), so \( T \) is normal.

2. Let \( T \) be normal. First we use Thm. 6.14 in [FIS] 
to obtain an orthonormal basis for \( V \) such that \( [T]_{\mathcal{B}} = A \) is upper triangular. By computing successive diagonal 
elements of \( A A^* - A^* A = 0 \), one shows that 
the off-diagonal elements must be zero.
Our immediate objective is to present the spectral theorem for finite dimensional inner-product spaces in such a way that it can be easily generalized into the statement of the spectral theorem for (normal operators in) infinite-dimensional Hilbert spaces.

We begin with considering the meaning of the statement that $T$ is orthogonally diagonalizable. This means that

- $T$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ (as does any operator!)
- The spectrum of $T = \sigma(T) = \{\lambda_1, \ldots, \lambda_k\}$ (closed)
- The resolvent set of $T = \rho(T) = \mathbb{C} \setminus \sigma(T)$ (open)
- The corresponding eigenspaces $W_1, \ldots, W_k$ are mutually orthogonal and

\[ V = \bigoplus_{i=1}^{k} W_i \quad \text{(at a direct sum of inner product spaces)} \]

Let $P_i$ be the (orthogonal) projection onto $W_i$ ($\text{Ran} P_i = W_i$).

The statement (*) is equivalent to the statement

\[ \sum_{i=1}^{k} P_i = I, \]

Notice that $\text{Ran} P_i = W_i$ and $\text{Null} P_i = \bigoplus_{i \neq j} W_j = V \ominus W_i$.

\[ T = \sum_{i=1}^{k} \lambda_i P_i. \]

Note: The word "projection" $P$ in an inner-product space refers to an orthogonal projection, that is, $P^2 = P$ and $\text{Ran} P \perp \text{Null} P$. 
The Spectral Theorem (finite dim.), reformulated.

Let $V$ be a finite-dimensional inner-product space and $T : V \to V$ a linear transformation. Then the following two statements are equivalent.

1. $T$ is normal.

2. There exists a finite set $\sigma(T) \subset \mathbb{C}$ and projections $\{ P_\lambda : \lambda \in \sigma(T) \}$ such that

   \[ P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda \quad \text{[Range } P_\lambda \perp \text{Range } P_\mu \text{ for } \lambda \neq \mu] \]

   \[ \sum_{\lambda \in \sigma(T)} P_\lambda = I \quad \text{[} V = \bigoplus_{\lambda \in \sigma(T)} \text{Range } P_\lambda \text{]} \]

   \[ \sum_{\lambda \in \sigma(T)} \lambda P_\lambda = T \quad \text{["} T \text{ is diagonal"}] \]

Notice that the statement $T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda$ is saying that $T$ is a "multiplication operator" in the sense that, with respect to the decomposition $V = \bigoplus_{\lambda \in \sigma(T)} W_\lambda$, $T$ just multiplies the components by the eigenvalues — it does not couple the components:

\[ T(V_1 + V_2 + \cdots + V_\kappa) = \lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_\kappa V_\kappa \]

By choosing an orthonormal basis for each $W_\lambda$ and taking the union of all these bases, we obtain an orthonormal basis for $V$ with respect to which $T$ is diagonal, or a "multiplication operator."
By choosing an orthonormal basis for each $W_i$ and taking the union of all these bases, we obtain an orthonormal basis $B$ for $V$ with respect to which $T$ is diagonal, or a "multiplication operator" when elements of $V$ are represented by $n$-tuples with respect to $B$:

$$[T]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \text{or} \quad [T]_B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where the $\lambda_i$ are not necessarily distinct.

Another way to say this is that the map $U : \mathbb{C}^n \to V$ such that $(0, 0, \ldots, 1, \ldots, 0) = e_i$ is unitary and $U^* T U$ is a diagonal matrix $\Lambda$.

The Spectral Theorem (fin. dim.), reformulated.

Let $V$ be a fin. dim. inner-prod. space and $T : V \to V$ linear. Then $T$ is normal if and only if there exists a unitary operator $U : \mathbb{C}^n \to V$ and a diagonal matrix $\Lambda$ such that

$$T = U \Lambda U^*.$$
Some observations

- $(T^*)^* = T^* - I$
- $\sigma(T^*) = [\sigma(T)]^*$
- If $T$ is normal, then $T - I$ and $T^* - I$ have the same nullspaces (the eigenspaces of $T$ and $T^*$ coincide).
- If $T$ is normal, then "$\sigma(T) \subseteq \mathbb{R}$" is equivalent to $T = T^*$.

That is, those normal operators that are self-adjoint are exactly those with real spectrum.

Note: All of these statements can be proven quite directly from the definitions of "adjoint" and "normal", without resorting to matrix representations. One of the key observations is that

If $W_1$ and $W_2$ are subspaces of $V$ such that $W_1 = \{ v \in V : T(v) \in W_2 \}$ ($W_1$ is the preimage of $W_2$ under $T$), then $W_2^+ = \{ v \in V : T^*(v) \in W_1^+ \}$ ($W_2^+$ is the preimage of $W_1^+$ under $T^*$).

Furthermore, $T(W_1) = W_2$ and $T^*(W_2^+)$.

Actually, you only need the weaker statement

$,\text{Ran}(T^*) = [\text{Null}(T)]^+.$

Since $T^{**} = T$, we have also $\text{Ran}T = [\text{Null}(T^*)]^+$.

This is a concise statement of the "Fredholm alternative" for operators in finite-dimensional inner-product spaces.
Observations, continued

Let $T$ be normal and $\sigma(T)$ and $P_\lambda$ for $\lambda \in \sigma(T)$ be defined as in the spectral theorem. Observe that

- $\|v\|^2 = \sum_{\lambda \in \sigma(T)} \|P_\lambda v\|^2$
- $\|Tv\|^2 = \sum_{\lambda \in \sigma(T)} |\lambda|^2 \|P_\lambda v\|^2 \leq (\max_{\lambda \in \sigma(T)} |\lambda|^2) \sum_{\lambda \in \sigma(T)} \|P_\lambda v\|^2$
  
  \[= (\max_{\lambda \in \sigma(T)} |\lambda|^2) \|v\|^2\]

with equality if and only if $v \in W_{\lambda^+}$ ($\lambda^+$ is the right $\sigma$-module)

- $\Rightarrow \|T\| = \sup_{\|v\| = 1} \|Tv\| = \max_{\lambda \in \sigma(T)} |\lambda|$

Polynomial functions of $T$ (normal)

Let $\mathcal{P}$ denote the space of polynomials and $\lambda$ in one indeterminate $p(\lambda) \in \mathbb{C}[\lambda]$

$T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda \implies T^n = \sum_{\lambda \in \sigma(T)} \lambda^n P_\lambda$

$\Rightarrow$ for $p \in \mathcal{P}$, $p(T) = \sum_{\lambda \in \sigma(T)} p(\lambda) P_\lambda$

Consequences:

- $p(T)$ is normal with adjoint $p(T^*)$, where $T^*$ is defined from $T$ by conjugating the coefficients.
- $\sigma(p(T)) = p(\sigma(T)) = \{ p(\lambda) : \lambda \in \sigma(T) \}$

This is the "Spectral Mapping Theorem".

\[\|p(T)\| = \max_{\mu \in \sigma(p(T))} |\mu| = \max_{\lambda \in \sigma(T)} |p(\lambda)| = \|p\|_{\infty}(\sigma(T)}\]
Continuous functions of a normal operator $T: V \rightarrow V$, dim $V < \infty$

This presentation will seem too technical for the simple case of finite dimension, but remember that it is for the purpose of setting the framework for the infinite-dimensional case.

$$C(\sigma(T)) = \mathcal{F}(\sigma(T)) = C^0(\sigma(T))$$

The set coincides because $\sigma(T)$ is finite.

We now define a map $\Phi: C(\sigma(T)) \rightarrow \mathbb{L}(V)$ that is an isometric homomorphism.

- Given the spectral theorem, we can do this direktly by
  $$\Phi: f \mapsto \sum_{\lambda \in \sigma(T)} f(\lambda) P_{\lambda}$$

- A more abstract framework arises from this:
  $$\Phi = \Phi_k \oplus \text{Null}(\Phi)$$

  $\Phi_k$ is defined by restricting a polynomial to $\sigma(T)$:
  $$[\Phi_k(p)](\lambda) = p(\lambda) \text{ for } \lambda \in \sigma(T)$$

  $\Phi$ is defined by applying a polynomial to $T$:
  $$\Phi(p) = p(T)$$

By (**) we see that $\|\Phi(p)\| = \|\Phi_k(p)\|$; so $\text{Null } \Phi = \text{Null } \Phi_k$, and since $\Phi$ is surjective, the homomorphism $\Phi$ exists:

$$\Phi(f) = \Phi_k(p)$$

where $p$ is chosen in the preimage $E$. [SFE3]

We denote $\Phi(f)$ by $f(T)$. 

You may show that \( \psi \) is a \( \ast \)-homomorphism of algebras. This means that it preserves all of the algebraic structure and, in addition, takes the conjugates to adjoints:

\[
\psi : \mathcal{L}^{-}(\mathfrak{h}(\tau)) \rightarrow \mathcal{L}(\mathfrak{v})
\]

\[
\begin{align*}
\psi(af + bg) &= a\psi(f) + b\psi(g) \\
\psi(fg) &= \psi(f)\psi(g) \\
\psi(1) &= 1 \\
\psi(\overline{f}) &= (\psi(f))^\ast
\end{align*}
\]