Solutions Problem Set 1

Suppose $T : \mathcal{D}(T) \to \mathcal{H}$ is closed, $\lambda I - T$ is injective, and $\operatorname{Ran}(\lambda I - T)$ is dense in $\mathcal{H}$.

First, since $\lambda I - T$ is injective, it is invertible (and linear) on its range:

$$(\lambda I - T)^{-1} : \operatorname{Ran}(\lambda I - T) \longrightarrow \mathcal{D}(T).$$

Second, we prove that $\lambda I - T$ is closed:

Suppose $(y_n, x_n) \in \Gamma((\lambda I - T)^{-1})$ and $(y_n, x_n) \to (y, x)$ in $\mathcal{H} \oplus \mathcal{H}$.

Then $(x_n, y_n) \in \Gamma(\lambda I - T)$ and $x_n \to x$ and $y_n \to y$ in $\mathcal{H}$.

We have $x_n \in \mathcal{D}(\lambda I - T) = \mathcal{D}(T)$ and $Tx_n = \lambda x_n - y_n \longrightarrow \lambda x - y$ $(n \to \infty)$

Thus $(x, \lambda x - y) \in \Gamma(T)$ by the closedness of T, that is, $Tx = \lambda x - y$, or $y = (\lambda I - T)x$, and we obtain $x = (\lambda I - T)^{-1} y$ and hence $(y, x) \in \Gamma((\lambda I - T)^{-1})$.

(a) Suppose $\operatorname{Ran}(\lambda I - T) = \mathcal{H}$. Since $(\lambda I - T)^{-1}$ is closed, the closed-graph theorem tells us that $(\lambda I - T)^{-1}$ is bounded.

(b) Suppose $(\lambda I - T)^{-1}$ is bounded, and let $y_n \to y$ in $\mathcal{H}$, with $y_n \in \operatorname{Ran}(\lambda I - T)$. Since $(\lambda I - T)^{-1}$ is continuous, $(\lambda I - T)^{-1} y_n$ converges, say to $x \in \mathcal{H}$. By the closure of $(\lambda I - T)^{-1}$, $(y, x) \in \Gamma((\lambda I - T)^{-1})$, so $y \in \operatorname{Ran}(\lambda I - T)$.

We conclude that, if $\operatorname{Ran}(\lambda I - T) \neq \mathcal{H}$, then $(\lambda I - T)^{-1}$ is unbounded. This is tantamount to the existence of a sequence $\{w_n\}$ from $\operatorname{Ran}(\lambda I - T)$ such that $w_n \to 0$ and $\|(\lambda I - T)^{-1} w_n\| = 1$.

Setting $v_n = (\lambda I - T)^{-1} w_n$, we obtain

$$(\lambda I - T) v_n \to 0 \quad \text{and} \quad \|v_n\| = 1.$$

Solutions Problem Set 2

Let g be a measurable fcn on $\mathbb{R}$, and define the operator T_g in $L^2(\mathbb{R})$ by

$$\begin{cases} \mathcal{D}(T_g) = \{ f \in L^2(\mathbb{R}) : gf \in L^2(\mathbb{R}) \} \\ T_g f = gf, \quad f \in \mathcal{D}(T_g). \end{cases}$$

a. Suppose $g \in L^\infty(\mathbb{R})$ and $f \in \mathcal{D}(T_g)$. Then $\|T_g f\|_2 =$

(*) $$\|gf\|_2 = \left[\int |g|^2 |f|^2\right]^{1/2} \le \|g\|_\infty \left[\int |f|^2\right]^{1/2} \le \|g\|_\infty \|f\|_2,$$

and we see that $\|T_g\| \le \|g\|_\infty$. To prove $\|T_g\| \ge \|g\|_\infty$, let $\varepsilon > 0$ be given. By defn. of $\|g\|_\infty$, the set $E_\varepsilon = \{x \in \mathbb{R} : |g(x)| > \|g\|_\infty - \varepsilon\}$ has positive measure. Let F_ε be a subset of E_ε with finite positive measure, and observe that $\chi_{F_\varepsilon} \in L^2(\mathbb{R})$. By (*), $\chi_{F_\varepsilon} \in \mathcal{D}(T_g)$, and we obtain

$$\|T_g \chi_{F_\varepsilon}\|_2 = \left[\int_{F_\varepsilon} |g|^2\right]^{1/2} \ge (\|g\|_\infty - \varepsilon)\left[\int_{\mathbb{R}} \chi_{F_\varepsilon}^2\right]^{1/2} = (\|g\|_\infty - \varepsilon)\|\chi_{F_\varepsilon}\|_2$$

This shows that $\|T_g\| \ge (\|g\|_\infty - \varepsilon)$, and since ε was chosen arbitrarily, we obtain $\|T_g\| \ge \|g\|_\infty$.

Now suppose $g \notin L^\infty(\mathbb{R})$. Let $M \in \mathbb{R}$ be given arbitrarily, and define

$$E_M = \{x \in \mathbb{R} : |g| > M\}.$$

Since $g \notin L^\infty(\mathbb{R})$, $\mu(E_M) > 0$. Let F_M be a subset of E_M of finite positive measure, and set $G_{M'} = F_M \setminus E_{M'}$

Since $F_M = \bigcup G_{M'}$, with $G_{M'} \subset G_{M''}$ if $M' < M''$, we have

$$\lim_{M' \to \infty} \mu(G_{M'}) = \mu(F_M), \quad 0 < \mu(F_M) < \infty.$$

Therefore $\exists\, M'$ such that $0 < \mu(G_{M'}) < \infty$.
By definition of $G_{M'}$, we have $|g(x)| < \infty$ for $x \in G_{M'}$ and $\|\chi_{G_{M'}}\|_2 < \infty$. Thus $g\chi_{G_{M'}} \in L^2$ so $\chi_{G_{M'}} \in \mathcal{D}(T_g)$, and we obtain

$$\|T_g \chi_{G_{M'}}\|_2 = \left[\int_{G_{M'}} |g|^2\right]^{1/2} \geq M \left[\int \chi_{G_{M'}}^2\right]^{1/2} = M \|\chi_{G_{M'}}\|_2,$$

which proves that $\|T_g\| \geq M$. Since M was chosen arbitrarily, we see that T_g is unbounded.

b. To prove that $\mathcal{D}(T_g)$ is dense in $L^2(\mathbb{R})$, let $f \in L^2$ be given. For each ε, there exists numbers L and δ such that, if $\mu(E) < \delta$, then

$$(**) \qquad \int_{(\mathbb{R} \setminus (-L,L)) \cup E} |f|^2 < \varepsilon.$$

$\left[\begin{array}{l} \text{and } G_M = (-L,L) \setminus F_M \\ \quad = (-L,L) \cap E_M \\ \text{so } F_M = (-L,L) \setminus G_M \end{array}\right]$

Let E_M be defined as in part (a), and put $F_M = (-L,L) \setminus E_M$. Since $\mu(G_M) < \infty$, $G_M \supset G_{M'}$ for $M < M'$, and $\bigcap G_M = \phi$, we have $\lim_{M \to \infty} \mu(G_M) = 0$. Let M be such that $\mu(G_M) < \delta$, and define

$$f_\varepsilon = f \chi_{F_M} \in L^2.$$

By definition of F_M and E_M, we have $f \in \mathcal{D}(T_g)$ because $|g(x)| \leq M$ for $x \in F_M$, and by $(**)$,

$$\|f - f_\varepsilon\|_2^2 = \int_{(\mathbb{R} \setminus (-L,L)) \cup G_M} |f|^2 < \varepsilon,$$

and this proves that $f_\varepsilon \to f$ in $L^2(\mathbb{R})$ as $\varepsilon \to 0$.

If T_g is bounded, $\|g\|_\infty < \infty$ by part (a), and we obtain, for each $f \in L^2$,

$$\int |gf|^2 \leq \|g\|_\infty^2 \int |f|^2 < \infty,$$

so that $f \in \mathcal{D}(T_g)$, so $\mathcal{D}(T_g) = L^2(\mathbb{R})$

If T_g is unbounded, $\|g\|_\infty = \infty$ by part (a), and we can infer quite readily that there exists a sequence $\{M_n\}_{n=1}^\infty$ with $M_n \geq n$ such that $\{x \in \mathbb{R} : M_n \leq |g(x)| < M_{n+1}\} > 0$. Let E_n be measurable sets such that

$$E_n \subset \{x \in \mathbb{R} : M_n \leq |g(x)| < M_{n+1}\} \quad \text{and}$$

$$0 < \mu(E_n) < \infty .$$

The sets E_n are disjoint by construction. Define a function f by

$$f(x) = \frac{1}{n(\mu(E_n))^{1/2}} \quad \text{for} \quad x \in E_n, \quad n = 1:\infty$$

$$f(x) = 0 \quad \text{for} \quad x \notin \bigcup_{n=1}^{\infty} E_n .$$

We compute $f \in L^2$ and $gf \notin L^2$:

$$\int |f|^2 = \sum_{n=1}^{\infty} \int_{E_n} \frac{1}{n^2} \frac{1}{\mu(E_n)} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$\int |T_g f|^2 = \sum_{n=1}^{\infty} \int_{E_n} |g(x)|^2 \frac{1}{n^2} \frac{1}{\mu(E_n)} \geq \sum_{n=1}^{\infty} 1 = \infty .$$

c. Let $\lambda \in ER(g)$ be given. $\forall\ \varepsilon > 0$, set $E_\varepsilon = \{x \in \mathbb{R} : |g(x) - \lambda| < \varepsilon\}$. By defn. of ER, $\mu(E_n) > 0$. Set $f_\varepsilon = \chi_{E_\varepsilon}$ and observe

$$\|(\lambda I - T_g) f_\varepsilon\|_2^2 = \int_{E_\varepsilon} |\lambda - g(x)|^2 < \varepsilon^2 \int_{E_\varepsilon} 1 = \varepsilon^2 \|f_\varepsilon\|_2^2 .$$

This proves that $\lambda I - T_g$ does not have a bounded inverse, and we conclude that $\lambda \in \sigma(T_g)$.

Now let $\lambda \notin ER(g)$. $\exists\ \varepsilon > 0$ s.th. $E = \{x : |g(x) - \lambda| < \varepsilon\}$ has measure 0. Let $k \in L^2(\mathbb{R})$ be given, and define f by

(***) $$f(x) = \frac{1}{\lambda - g(x)} k(x) \quad \text{for } x \in \mathbb{R} \setminus E .$$

Since $|(g(x) - \lambda)^{-1}| \le 1/\varepsilon$ a.e., $f \in L^2$, and

(†) $$(\lambda - g(x)) f(x) = k(x) \in L^2,$$

So $\operatorname{Ran}(\lambda I - T_g) = L^2(\mathbb{R})$. Now suppose that $(\lambda I - T_g) h = 0$ in L^2 for some $h \in L^2$. Then, since $\lambda - g(x) \neq 0$ a.e., we have $h(x) = 0$ a.e., so $h = 0$ in L^2, and we see that $\lambda I - T_g$ is injective. In addition, we see that the inverse $(\lambda I - T_g)^{-1} : k \mapsto f$ is given by (***), which is bounded. Therefore, $\lambda \in \rho(T_g)$.

'Tis mere child's play to prove that $ER(g)$ is closed.

d. For $\lambda \in \rho(T_g)$, we have seen in part (c) that $(\lambda I - T_g)^{-1}$ is given by multiplication by $(\lambda - g(x))^{-1}$ (see (†)):

$$[(\lambda I - T_g)^{-1} f](x) = (\lambda - g(x))^{-1} f(x).$$

Define $\varepsilon_0 = \sup\{\varepsilon : \mu\{x : |\lambda - g(x)| < \varepsilon\} = 0\} = \sup\{\varepsilon : \mu(g^{-1}(B_\lambda(\varepsilon))) = 0\}$.
Since $\lambda \notin ER(g)$, we have $\varepsilon_0 > 0$. We also have that

$$\|(\lambda - g(x))^{-1}\|_\infty = \inf\{M : \mu\{x : |\lambda - g(x)|^{-1} > M\} = 0\}$$

$$= \left[\sup\{\varepsilon : \mu\{x : |\lambda - g(x)|^{-1} > \tfrac{1}{\varepsilon}\} = 0\}\right]^{-1}$$

$$= \left[\sup\{\varepsilon : \mu\{x : |\lambda - g(x)| < \varepsilon\} = 0\}\right]^{-1} = \varepsilon_0^{-1}.$$

We now prove that $\varepsilon_0 = \operatorname{dist}(\lambda, ER(g))$.
First, let $\lambda' \in B_\lambda(\varepsilon_0)$ be given, and let $\varepsilon < \varepsilon_0$ be such that $\lambda' \in B_\lambda(\varepsilon)$. By defn. of ε_0, we have $\mu(g^{-1}(B_\lambda(\varepsilon))) = 0$. Since $B_\lambda(\varepsilon)$ is open, $\exists\, \varepsilon' > 0$ such that $B_{\lambda'}(\varepsilon') \subset B_\lambda(\varepsilon)$. As $\mu(g^{-1}(B_{\lambda'}(\varepsilon'))) = 0$, $\lambda' \notin ER(g)$.

This proves that $\operatorname{dist}(\lambda, ER(g)) \geq \varepsilon_0$

On the other hand, let $\varepsilon > \varepsilon_0$ be given. By defn. of ε_0, we have $\mu(g^{-1}(B_\lambda(\varepsilon) \setminus B_\lambda(\varepsilon_0))) > 0$. It is straightforward to obtain, for each $n = 1:\infty$, a number $\lambda_n \in B_\lambda(\varepsilon) \setminus B_\lambda(\varepsilon_0)$ s.th. $\mu(g^{-1}(B_{\lambda_n}(1/n))) > 0$.
Let $\hat\lambda$ be an accumulation point of $\{\lambda_n\}$, which is in $\overline{B_\lambda(\varepsilon)}$, and let $\varepsilon' > 0$ be given. One can choose n s.th. $|\lambda_n - \hat\lambda| < \varepsilon'/2$ and s.th. $1/n < \varepsilon'/2$, so that $B_{\lambda_n}(1/n) \subset B_{\hat\lambda}(\varepsilon')$, to conclude that $\mu(g^{-1}(B_{\hat\lambda}(\varepsilon'))) > 0$. Since ε' was chosen arbitrarily, we have, by defn of $ER(g)$, that $\hat\lambda \in ER(g)$.
This proves that $\operatorname{dist}(\lambda, ER(g)) \leq \varepsilon_0$.
We conclude finally that $\operatorname{dist}(\lambda, ER(g)) = \varepsilon_0$.

e. Given $\lambda \in \mathbb{C}$, define $E = \{x : \lambda - g(x) = 0\}$.
Suppose that $(\lambda I - T_g) f = 0$ in L^2, that is, $(\lambda - g(x)) f(x) = 0$ a.e.
This is equivalent to $f(x) = 0$ a.e. $(\mathbb{R} \setminus E)$. We conclude that the eigenspace of λ for T_g is identified with $L^2(E)$. Thus $\lambda \in \sigma_p(T_g)$ iff and only if $\mu(E) > 0$.

f. Suppose that λ is not an eigenvalue of T_g. By (e), we have $(\lambda - g(x)) \neq 0$ a.e. Put $r(x) = (\lambda - g(x))^{-1} \in L^\infty(\mathbb{R})$, and consider the multiplication operator T_r defined by

$$\mathcal{D}(T_r) = \{k \in L^2(\mathbb{R}) : T_r k \in L^2(\mathbb{R})\},$$

$$(T_r k)(x) = r(x) k(x) = (\lambda - g(x))^{-1} k(x).$$

We have seen that $\mathcal{D}(T_r)$ is dense in $L^2(\mathbb{R})$. But $\forall\, k \in \mathcal{D}(T_r)$, $(\lambda I - T_g)(T_r k) = k$, which shows that $\mathcal{D}(T_r) \subset \operatorname{Ran}(\lambda I - T_g)$, and therefore T_g has no residual spectrum.

g. Suppose $h \in \mathcal{D}(T_g^*)$. This means that the functional

$$f \longmapsto \int (T_g f) \bar{h} \quad \text{is bounded on } \mathcal{D}(T_g)$$

On one hand, we have $\int (T_g f) \bar{h} = \int f(x) \overline{g(x) h(x)}\, dx$, and on the other hand, by the Riesz Lemma, $\exists\, k \in L^2$ s.th. $\int (T_g f) \bar{h} = \int f(x) \overline{k(x)}\, dx \quad \forall\, f \in \mathcal{D}(T_g)$. Thus we have

$$\int f(x) \left(\overline{g(x) h(x)} - \overline{k(x)} \right) dx = 0 \quad \forall\, f \in \mathcal{D}(T_g)$$

This implies (see analysis books) that $\overline{g(x)} h(x) = k(x)$ a.e., or $\bar{g} h = k \in L^2$. Thus $h \in \mathcal{D}(T_{\bar{g}})$, and by defn. of T_g^*,

$$T_g^* h = k = T_{\bar{g}} h$$

Now suppose $h \in \mathcal{D}(T_{\bar{g}})$. Then $f \mapsto \int f\, \overline{T_{\bar{g}} h}$ is bounded on $\mathcal{D}(T_g)$. Again, we have

$$\int f\, \overline{T_{\bar{g}} h} = \int g(x) f(x) \overline{h(x)}\, dx = \int (T_g f) \bar{h},$$

$f \mapsto \int (T_g f)\bar{h}$ is bounded, and we obtain $h \in \mathcal{D}(T_g^*)$.

To prove that T_g^* is a closed operator, let $\{h_n\}$ be a sequence from $\mathcal{D}(T_g^*)$ such that $h_n \to h \in L^2$ and $T_g^* h_n \to k \in L^2$. Then we have $\langle T_g f, h_n\rangle \to \langle T_g f, h\rangle$ and $\langle f, T_g^* h_n\rangle \to \langle f, k\rangle$ as $n \to \infty$. As $\langle T_g f, h_n\rangle = \langle f, T_g^* h_n\rangle$, we obtain $\langle T_g f, h\rangle = \langle f, k\rangle$ and conclude that $h \in \mathcal{D}(T_g^*)$ and $T_g^* h = k$.

Now we observe that $g \in L^\infty(\mathbb{R})$ in this problem was chosen arbitrarily from $L^\infty(\mathbb{R})$, and since $T_{\bar{g}} = T_g^*$ is closed, we conclude that T_g is closed.

h. Such a sequence arises from the functions f_ε constructed in (c). We simply take $h_n = f_{1/n}/\|f_{1/n}\|$ and see that $\|(\lambda I - T_g) f_n\| \le \frac{1}{n}$.

i. First observe that $T_g = \mathcal{D}(T_{\bar{g}})$.

By definition, $\mathcal{D}(T_{\bar{g}} T_g) = \{ f \in \mathcal{D}(T_g) : gf \in \mathcal{D}(T_{\bar{g}})\}$
$= \{ f \in L^2(\mathbb{R}) : \bar{g} g f \in L^2(\mathbb{R})\} = \mathcal{D}(T_{|g|^2})$.

Similarly, $\mathcal{D}(T_g T_{\bar{g}}) = \mathcal{D}(T_{|g|^2})$.

Now, for $f \in \mathcal{D}(T_{|g|^2})$, we have

$$T_{\bar{g}} T_g f = T_{\bar{g}}(gf) = |g|^2 f = T_{|g|^2} f.$$
$$T_g T_{\bar{g}} f = T_g(\bar{g} f) =$$

Therefore $T_g T_g^* = T_g T_{\bar g} = T_{|g|^2} = T_{\bar g} T_g = T_g^* T_g$; in other words, T_g is normal.

j. To see that P_E is a projection, we write
$$P_E P_E f = P_E(\chi_E f) = \chi_E^2 f = \chi_E f,$$
since $\chi_E^2 = \chi_E$. To see that P_E is orthogonal,

prove this → it suffices to prove that P_E is self-adjoint. This is true by part (g), in which we proved that $P_E^* = P_{\bar E} = P_E$.

The nullspace of P_E is the eigenspace for $\lambda = 0$, which is the natural image of $L^2(\mathbb{R} \setminus E)$ in $L^2(\mathbb{R})$. The Range of P_E is the image of $L^2(E)$ in $L^2(\mathbb{R})$, as one can observe easily in a variety of ways.

By arguments similar to those in (i), we find that
$$P_E T_g = T_{\chi_E} T_g = T_{\chi_E g} = T_{g \chi_E} = T_g T_{\chi_E} = T_g P_E.$$

Easier: just observe $T_g^* = T_{\bar g} = T_{g^{-1}} = (T_g)^{-1}$.

k. Let g be such that $|g(x)| = 1$ for all x. Then for all $f \in \mathcal{D}(T_g) = L^2(\mathbb{R})$,
$$\|T_g f\|_{L^2}^2 = \|gf\|_{L^2}^2 = \int |gf|^2 = \int |f|^2 = \|f\|_{L^2}^2.$$
Since $T_g^{-1} = T_{g^{-1}}$, we conclude that T_g is unitary.

Let g be an arbitrary function in $L^\infty(\mathbb{R})$, and let h be such that $|h(y)| = 1$ for all $y \in \mathbb{R}$. Then we have $|h(g(x))| = 1$ for all $x \in \mathbb{R}$, and thus $h(T_g) = T_{h \circ g}$ is unitary.

l. Let $g(x) \in i\mathbb{R}$ for all $x \in \mathbb{R}$, so that $\bar g = -g$. Then $T_g^* = T_{\bar g} = T_{-g} = -T_g$, that is, T_g is anti-self-adjoint.