Solutions Problem Set 3

1. a. For $f \in L^2(\mathbb{C}, d\mu)$, we have

$$\|Tf\|^2 = \sum_{i=1}^{\infty} |\lambda_i f(\lambda_i)|^2 \le \max_{i=1}^{\infty} |\lambda_i|^2 \sum_{i=1}^{\infty} |f(\lambda_i)|^2$$

$$= \max_{i=1}^{\infty} |\lambda_i|^2 \, \|f\|^2,$$

which proves that T is bounded, since $\max_{i=1}^{\infty} |\lambda_i|^2 < \infty$.

b. To prove that $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty} \cup \{0\}$, one proves that the latter set is the essential range of the function $\lambda \mapsto \lambda$ with respect to the measure μ and that the result of problem set 2, (c), holds for general measures.
Moreover, part (e) of that problem can be generalized to arbitrary measures, and we conclude that $\lambda \in \sigma_p(T)$ if and only if $\lambda = \lambda_i$ for some i.

c. This is also analogous to problem set 2. But let's do it directly: T is bounded, so $\mathcal{D}(T) = L^2(\mathbb{C}, d\mu)$, $\therefore T$ is closed, and T^* exists with $\mathcal{D}(T^*) = \mathcal{D}(T) = L^2(\mathbb{C}, d\mu)$. Now,

$$\langle Tf, h \rangle = \sum_{i=1}^{\infty} \lambda_i f(\lambda_i) \overline{h(\lambda_i)} = \sum_{i=1}^{\infty} f(\lambda_i) \overline{\overline{\lambda_i}\, h(\lambda_i)}$$

$$= \langle f, T^* h \rangle,$$

and we see that

$$(*) \qquad (T^* h)(\lambda_i) = \overline{\lambda_i}\, h(\lambda_i).$$

d. From $(*)$, we see that $T^* = T$ if and only if $\lambda_i = \overline{\lambda_i}$ for each i, that is, if $\lambda_i \in \mathbb{R}$ for each i.

2. $S: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$

$$(Sf)(x) = \begin{cases} 0, & x \le 1 \\ f(x-1), & 1 < x \end{cases}$$

a. To see that S is an isometry, let $f \in L^2(\mathbb{R}_+)$ be given:

$$\|Sf\|^2 = \int_0^\infty |Sf|^2 = \int_1^\infty |f(x-1)|^2 dx = \int_0^\infty |f(y)|^2 dy = \|f\|^2 .$$

To see that S is not unitary, we observe that each function with support in $(0,1)$ is not in the range of S. This implies that S is not invertible.

b. Define the operator $L: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ by

$$(Lf)(x) = f(x+1), \quad x > 0 .$$

We demonstrate that $L = S^*$. For $f, g \in L^2(\mathbb{R}_+)$,

$$\langle Sf, g \rangle = \int_0^\infty (Sf)(x)\overline{g(x)}\,dx = \int_1^\infty f(x-1)\overline{g(x)}\,dx$$

$$= \int_0^\infty f(y)\overline{g(y+1)}\,dy = \int_0^\infty f(y)\overline{(Lg)(y)}\,dy = \langle f, Lg \rangle .$$

c. (i) We prove first that each $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue of L. Given $f \in L^2$, the condition $(\lambda I - L)f = 0$, or $\lambda f(x) - f(x-1) = 0$ a.e., is equivalent to the statement

$$(*) \qquad f(x+n) = \lambda^n f(x) \quad \forall\, n = 0 : \infty$$

In fact, for each measurable function f satisfying $(*)$, we have

$$\int_0^\infty |f(x)|^2\,dx = \sum_{n=0}^\infty \int_n^{n+1} |f(x)|^2\,dx = \sum_{n=0}^\infty \int_0^1 |f(x+n)|^2\,dx$$
$$= \sum_{n=0}^\infty |\lambda^{2n}| \int_0^1 |f(x)|^2\,dx = \left(\int_0^1 |f(x)|^2\,dx\right) \sum_{n=0}^\infty |\lambda|^{2n}.$$

Thus, if $|\lambda|<1$ and $\chi_{(0,1)} f \in L^2$, then $f \in L^2$ also, and f is determined by $\chi_{(0,1)} f$ and the recursion $(*)$. So $\mathcal{N}ull(\lambda I - L)$ can be identified with $L^2(0,1)$, and we see that $\lambda \in \sigma_p(L)$.

(ii) Next we show that, for each $\lambda \in \mathbb{C}$, λ is not an eigenvalue of S. Let $f \in L^2$ be such that $(\lambda I - S)f = 0$. If $\lambda = 0$, then $f(x-1) = 0$ a.e. $(1,\infty)$, that is $f(x) = 0$ a.e. $(\mathbb{R}_+)$, so $f = 0$. Otherwise, set $x_0 = \sup\{x \in \mathbb{R} : \mu\{y \in (0,x) : f(y) \neq 0\} = 0\}$. If $x_0 \neq \infty$, then $\mu\{y \in (x_0,1) : f(y) \neq 0\} > 0$. By defn. of S, $\mu\{y \in (x_0,1) : (Sf)(y) \neq 0\} = 0$, and this implies (for $\lambda \neq 0$), that $\mu\{y \in (x_0,1) : \lambda f(y) - (Sf)(y)\} > 0$, so that $(\lambda I - S)f \neq 0$ in L^2. We conclude that $x_0 = \infty$, so that $f = 0$ in L^2.

prove $\lambda \in \sigma_{res}(S)$ for $|\lambda|<1$.

(iii) For $|\lambda| < 1$, it follows from (i) that $Ran(\lambda I - S)$ is not dense in $L^2(\mathbb{R}_+)$. Indeed, for $f \in \mathcal{N}ull(\bar\lambda I - L)$ and all $g \in L^2(\mathbb{R}_+)$, we have

$$\langle (\lambda I - S)g, f\rangle = \langle g, (\bar\lambda I - L)f\rangle = 0,$$

which shows that the nullspace of $\bar\lambda I - L$, which is nonempty, is orthogonal to the range of $\lambda I - S$. Thus $Ran(\lambda I - S)$ is not dense in $L^2(\mathbb{R}_+)$. By (ii), λ is not an eigenvalue of S. We conclude that $\lambda \in \sigma_{res}(S)$, i.e., λ is in the residual spectrum of S.

(iv) Now we prove that, if $|\lambda| > 0$, then $\lambda \in \rho(S)$.
Since $\rho(S^*) = \overline{\rho(S)}$, we will also obtain that $|\lambda|>0 \Rightarrow \lambda \in \rho(S^*)$.
Let $\lambda \in \mathbb{C}$ with $|\lambda| > 0$ be given, and let $g \in L^2(\mathbb{R}_+)$ be given. Suppose that f is a measurable function on $\mathbb{R}_+$, which we extend by zero to $\mathbb{R}$, such that

$$\lambda f(x) - f(x-1) = g(x), \quad x \in \mathbb{R}_+$$

(which is equivalent to $(\lambda I - S)f = g$ if $f \in L^2$). From this, we obtain $\lambda f(x+n) - f(x+(n-1)) = g(x+n)$, and finally

(†) $$f(x) = \sum_{i=0}^{n} \frac{1}{\lambda^{i+1}} g(x-i), \quad n < x \le n+1, \ n = 0:\infty,$$

Let $x_0 \in (0,1]$ be given. We have $\forall\ n = 0:\infty$,

$$|f(x_0+n)|^2 = \left(\sum_{i=0}^{n} \frac{1}{\lambda^{i+1}} g(x_0+n-i)\right)\left(\sum_{j=0}^{n} \frac{1}{\bar\lambda^{j+1}} \overline{g(x_0+n-j)}\right)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{\lambda^{i+1}\bar\lambda^{j+1}} g(x_0+n-i)\overline{g(x_0+n-j)},$$

where we have replaced n by ∞ in the sums by extending g by 0 on $\mathbb{R}$. Next we compute, for $N \in \mathbb{N}$,

$$\sum_{n=0}^{N} |f(x_0+n)|^2 = \sum_{n=0}^{N}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{\lambda^{i+1}\bar\lambda^{j+1}} g(x_0+n-i)\overline{g(x_0+n-j)}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{\lambda^{i+1}\bar\lambda^{j+1}} \sum_{n=0}^{\infty} g(x_0+n-i)\overline{g(x_0+n-j)}$$

$$\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{|\lambda|^{i+1}|\lambda|^{j+1}} \sum_{n=0}^{\infty} \left| g(x_0+n-i)\overline{g(x_0+n-j)} \right|$$

$$\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{|\lambda|^{i+1}|\lambda|^{j+1}} \left[\sum_{n=0}^{\infty} |g(x_0+n-i)|^2\right)^{1/2} \left[\sum_{n=0}^{\infty} |g(x_0+n-j)|^2\right)^{1/2}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{|\lambda|^{i+1}|\lambda|^{j+1}} \sum_{n=0}^{\infty} |g(x_0+n)|^2$$

$$= \left[\sum_{i=0}^{\infty}\frac{1}{|\lambda|^{i+1}}\right]^2 \sum_{n=0}^{\infty} |g(x_0+n)|^2$$

$$= \left(\frac{1}{|\lambda|-1}\right)^2 \sum_{n=0}^{\infty} |g(x_0+n)|^2.$$

From this, we obtain

$$\|f\|^2 = \int_0^\infty |f(x)|^2\,dx = \int_0^1 \sum_{n=0}^{\infty} |f(x+n)|^2\,dx$$

$$\leq \left(\frac{1}{|\lambda|-1}\right)^2 \int_0^1 \sum_{n=0}^{\infty} |g(x+n)|^2\,dx = \left(\frac{1}{|\lambda|-1}\right)^2 \int_0^\infty |g(x)|^2\,dx$$

$$= \left(\frac{1}{|\lambda|-1}\right)^2 \|g\|^2.$$

This proves that $f \in L^2$. It is simple, using $(\dagger)$, to prove that $(\lambda I - S)f = g$.
We conclude that, given $g \in L^2$, the formula $(\dagger)$ inverts $(\lambda I - S)$ in a bounded fashion, so $\lambda \in \rho(S)$.

(v) It remains to investigate $\lambda \in \mathbb{C}$, with $|\lambda| = 1$.
Since $\sigma(S)$ is closed and $|\lambda| < 1 \Rightarrow \lambda \in \sigma(S)$ and $|\lambda| > 1 \Rightarrow \lambda \notin \sigma(S)$, we see that $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
Also, $\sigma(S^*) = \overline{\sigma(S)} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ be given. We have shown in (ii) that $\lambda \notin \sigma_p(S)$. We show now that $\lambda \notin \sigma_p(S^*)$.
Let f be a measurable function on $\mathbb{R}_+$ such that $f(x+1) = \lambda f(x)$ a.e. This is equivalent to $(\lambda I - L)f = 0$ if $f \in L^2$.

It follows that $f(x+n) = \lambda^n f(x) \quad \forall \; x \in \mathbb{R}_+$.

Now we compute

$$\int_0^\infty |f(x)|^2 dx = \sum_{n=0}^\infty \int_n^{n+1} |f(x)|^2 dx$$

$$= \sum_{n=0}^\infty \int_0^1 |f(x+n)|^2 dx = \sum_{n=0}^\infty |\lambda|^{2n} \int_0^1 |f(x)|^2 dx$$

$$= \sum_{n=0}^\infty \int_0^1 |f(x)|^2 dx = \infty. \quad (\text{since } |\lambda| = 1)$$

Thus $f \notin L^2$, and we conclude that $\lambda \notin \sigma_p(L) = \sigma_p(S^*)$.

Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ be given. We have shown $\lambda \notin \sigma_p(S)$ (resp. $\lambda \notin \sigma_p(S^*)$). Since $\bar{\lambda} \notin \sigma_p(S^*)$ (resp. $\bar{\lambda} \notin \sigma_p(S)$), we see that $\lambda \notin \sigma_{res}(S)$ (resp. $\lambda \notin \sigma_{res}(S^*)$).

We conclude that $\lambda \in \sigma_c(S)$ and $\lambda \in \sigma_c(S^*)$.

d. We have already demonstrated and explicit expression for the resolvent of S in (†), for $|\lambda| > 1$.

$$\left[(\lambda I - S)^{-1} g\right](x) = \sum_{i=0}^{n} \frac{1}{\lambda^{i+1}} g(x-i), \quad n < x \le n+1, \quad n = 0:\infty.$$