Spectrum and spectral gaps in 1D periodic media

We consider the "Helmholtz equation" on the real line:

$$\psi'' + \omega^2 \epsilon(x) \psi = 0.\tag{1}$$

This is a reduction of the Maxwell equations in a medium in which \(\mu=1\) and \(\epsilon\) depends only on one spatial coordinate. The frequency is \(\omega\).

We take \(\epsilon\) to be a periodic function of \(x\). Perhaps the simplest example to calculate is a singular one in which \(\epsilon\) is a constant plus a periodically-placed delta-function:

$$\epsilon(x) = 1 + \rho \sum_{n=-\infty}^{\infty} \delta(x-n)$$

and \(\rho \geq 0\) is a nonnegative strength parameter.

The spectrum (we are disregarding the proper functional-analytic framework for now) consists of these \(\omega\) for which there exists a bounded solution of (1).

Let's first see what the delta-functions in \(\epsilon\) do to the solutions \(\psi\). If \(\psi\) is a continuous function, then the singular point of equation (1) gives

$$\phi'(n+0) - \phi'(n-0) + \omega^2 \rho \phi(n) = 0, \quad n \in \mathbb{Z},$$

(2)
in other words, the jump in the derivative of $\psi$ at integer values of $x$ is equal to $\omega^2 \beta$ times the value of $\psi$.

Now, if we have a solution $\psi$ to (1), we can compare its “Cauchy data” $(\psi, \psi')$ at $n+0$ to its Cauchy data at $n+1+0$. Since $e^x$ is periodic, $\psi(x-n)$ is also a solution to (x), and we might as well take $n=0$.

The solution in the interval $(0, 1)$ with

\[
\begin{cases}
\psi(0+0) = a \\
\psi'(0+0) = b
\end{cases}
\]

is $\psi(x) = a \cos \omega x + \frac{b}{\omega} \sin \omega x$. From this, we obtain

\[
\begin{cases}
\psi(1+0) = a \cos \omega + \frac{b}{\omega} \sin \omega \\
\psi'(1+0) = -a \omega \sin \omega + b \cos \omega
\end{cases}
\]

Now, using the jump condition (2) and continuity of $\psi$, we obtain

\[
\begin{cases}
\psi(1+0) = a \cos \omega + \frac{b}{\omega} \sin \omega \\
\psi'(1+0) = (-\omega a - \beta w) \sin \omega + (b - \beta \omega^2 a) \cos \omega
\end{cases}
\]

We obtain the relation

\[
\begin{bmatrix}
\psi(n+1+0) \\
\psi'(n+1+0)
\end{bmatrix} =
\begin{bmatrix}
\cos \omega & \frac{1}{\omega} \sin \omega \\
-\omega \sin \omega - \beta w^2 \cos \omega & \omega \cos \omega - \beta \omega \sin \omega
\end{bmatrix}
\begin{bmatrix}
\psi(n+0) \\
\psi'(n+0)
\end{bmatrix}.
\]

To understand the behavior of solutions $\psi$, it suffices to know the eigenvalues of the "transfer matrix" $\mathbf{T}$ in (3).
The characteristic polynomial of $T$ is

$$
\lambda^2 - 2\tau \lambda + 1, \quad \text{where } \tau = \cos \omega - \frac{\beta \omega}{2} \sin \omega,
$$

which has roots

$$
\lambda^\pm = \cos \omega - \frac{\omega \beta}{2} \sin \omega \pm \left[ \frac{\omega^2 \beta^2}{4} - 1 \right] \sin^2 \omega - \omega \beta \sin \omega \cos \omega \right]^{1/2}.
$$

Since $\det(T) = 1$, we have $\lambda^+ \lambda^- = 1$. This means that $\lambda^+$ and $\lambda^-$ are either conjugate complex numbers with modulus 1 (case 1) or both are real (case 2).

Cases 1 and 2 overlap when $\lambda^+ = \lambda^- = 1$ or $\lambda^+=\lambda^-=-1$.

In the case $|\lambda^+| < 1$ and $|\lambda^-| > 1$, it is evident that there are no bounded solutions, for each solution is a linear combination of an exponentially growing solution and an exponentially decaying solution. Thus we try to find those values of $\omega$ for which $|\lambda^\pm| = 1$. The condition for $|\lambda^\pm| = 1$ is

$$
(4) \quad \sin \omega \left[ \frac{(\omega^2 \beta^2)}{4} - 1 \right] \sin \omega - \omega \beta \cos \omega \right] \leq 0.
$$

As this function is symmetric in $\omega$, we may restrict attention to $\omega > 0$. We also take $\beta > 0$. Of course, $\beta = 0$ corresponds to the problem

$$
\psi'' + \omega^2 \psi = 0 \quad (\beta = 0),
$$

for which the spectrum is $\mathbb{R}$. 
One observes that the roots of the LHS of (4) have simple and consist of the numbers \((n-1)\pi\) and \(\omega_n\), where

\[(n-1)\pi < \omega_n < n\pi\]

Graphically, the second factor of \(D\) has roots equal to the values of \(\omega\) at which

\[
\left(\frac{\omega^2}{4} - \frac{1}{\omega^2}\right) = \cot \omega
\]


\[D < 0: \text{spectral bands}, \quad D > 0: \text{spectral gaps}\]

Properties:

1. \(\omega_n \to (n-1)\pi\) as \(n \to \infty\) for fixed \(\beta\)

   \[\Rightarrow\] The lengths of the spectral bands shrink to zero as they tend to \(\infty\).

2. \(\omega_n \to n\pi\) as \(\beta \to 0\) for fixed \(n\)

   \[\Rightarrow\] In a finite part of the \(\omega\)-axis, the spectral gaps tend to zero as \(\beta \to 0\), that is, the strength of \(D\) tends to zero.