Spectrum and spectral gaps in 1D periodic media

We consider the "Helmholtz equation" on the real line:

$$\psi'' + \omega^2 \varepsilon(x) \psi = 0 . \tag{1}$$

This is a reduction of the Maxwell equations in a medium in which $\mu = 1$ an ε depends only on one spatial coordinate. The frequency is ω.

We take ε to be a periodic function of x. Perhaps the simplest example to calculate is a singular one in which ε is a constant plus a periodically-placed delta-function.

$$\varepsilon(x) = 1 + \beta \sum_{n=-\infty}^{\infty} \delta(x-n)$$

and $\beta \geq 0$ is a nonnegative strength parameter.

The spectrum (we are disregarding the proper functional-analytic framework for now) consists of those ω for which there exists a bounded solution of (1).

Let's first see what the delta-functions in ε do to the solutions ψ. If ψ is a continuous function, then the singular part of equation (1) gives

$$\phi(x+0) = \lim_{\xi \to x^+} \phi(\xi)$$

$$\psi'(n+0) - \psi'(n-0) + \omega^2 \beta \psi(n) = 0 , \quad n \in \mathbb{Z} , \tag{2}$$

in other words, the jump in the derivative of ψ at integer values of x is equal to $\omega^2\beta$ times the value of ψ.

Now, if we have a solution ψ to (1), we can compare its "Cauchy data" (ψ, ψ') at $n+0$ to its Cauchy data at $n+1+0$. Since $\varepsilon(x)$ is periodic, $\psi(x-n)$ is also a solution to (*), and we might as well take $n=0$.
The solution in the interval $(0,1)$ with

$$\begin{cases} \psi(0+0) = a \\ \psi'(0+0) = b \end{cases}$$

is $\psi(x) = a\cos\omega x + \frac{b}{\omega}\sin\omega x$. From this, we obtain

$$\begin{cases} \psi(1-0) = a\cos\omega + \frac{b}{\omega}\sin\omega \\ \psi'(1-0) = -a\omega\sin\omega + b\cos\omega \end{cases}.$$

Now, using the jump condition (2) and continuity of ψ, we obtain

$$\begin{cases} \psi(1+0) = a\cos\omega + \frac{b}{\omega}\sin\omega \\ \psi'(1+0) = (-\omega a - \beta\omega b)\sin\omega + (b - \beta\omega^2 a)\cos\omega \end{cases}$$

We obtain the relation

$$(3) \qquad \begin{bmatrix} \psi(n+1+0) \\ \psi'(n+1+0) \end{bmatrix} = \begin{bmatrix} \cos\omega & \frac{1}{\omega}\sin\omega \\ -\omega\sin\omega - \beta\omega^2\cos\omega & \cos\omega - \beta\omega\sin\omega \end{bmatrix} \begin{bmatrix} \psi(n+0) \\ \psi'(n+0) \end{bmatrix}.$$

To understand the behavior of solutions ψ, it suffices to know the eigenvalues of the "transfer matrix" T in (3).

The characteristic polynomial of T is

$$\lambda^2 - 2\tau\lambda + 1, \quad \text{where} \quad \tau = \cos\omega - \frac{\beta\omega}{2}\sin\omega,$$

which has roots

$$\lambda^{\pm} = \cos\omega - \frac{\omega\beta}{2}\sin\omega \pm \left[\left(\frac{\omega^2\beta^2}{4} - 1\right)\sin^2\omega - \omega\beta\sin\omega\cos\omega\right]^{1/2}$$

Since $\det(T) = 1$, we have $\lambda^+\lambda^- = 1$. This means that λ^+ and λ^- are either conjugate complex numbers with modulus 1 (case 1) or both are real (case 2). Cases 1 and 2 overlap when $\lambda^+ = \lambda^- = 1$ or $\lambda^+ = \lambda^- = -1$.

In the case $|\lambda^{\pm}| < 1$ and $|\lambda^{\mp}| > 1$, it is evident that there are no bounded solutions, for each solution is a linear combination of an exponentially growing solution and an exponentially decaying solution. Thus we try to find those values of ω for which $|\lambda^{\pm}| = 1$. The condition for $|\lambda^{\pm}| = 1$ is

(4) $$\underbrace{\sin\omega\left[\left(\frac{\omega^2\beta^2}{4} - 1\right)\sin\omega - \omega\beta\cos\omega\right]}_{D} \leq 0 .$$

As this function is symmetric in ω, we may restrict attention to $\omega > 0$. We also take $\beta > 0$. Of course, $\beta = 0$ corresponds to the problem

$$\psi'' + \omega^2\psi = 0 \qquad (\beta = 0),$$

for which the spectrum is $\mathbb{R}$.

$\omega > 0$

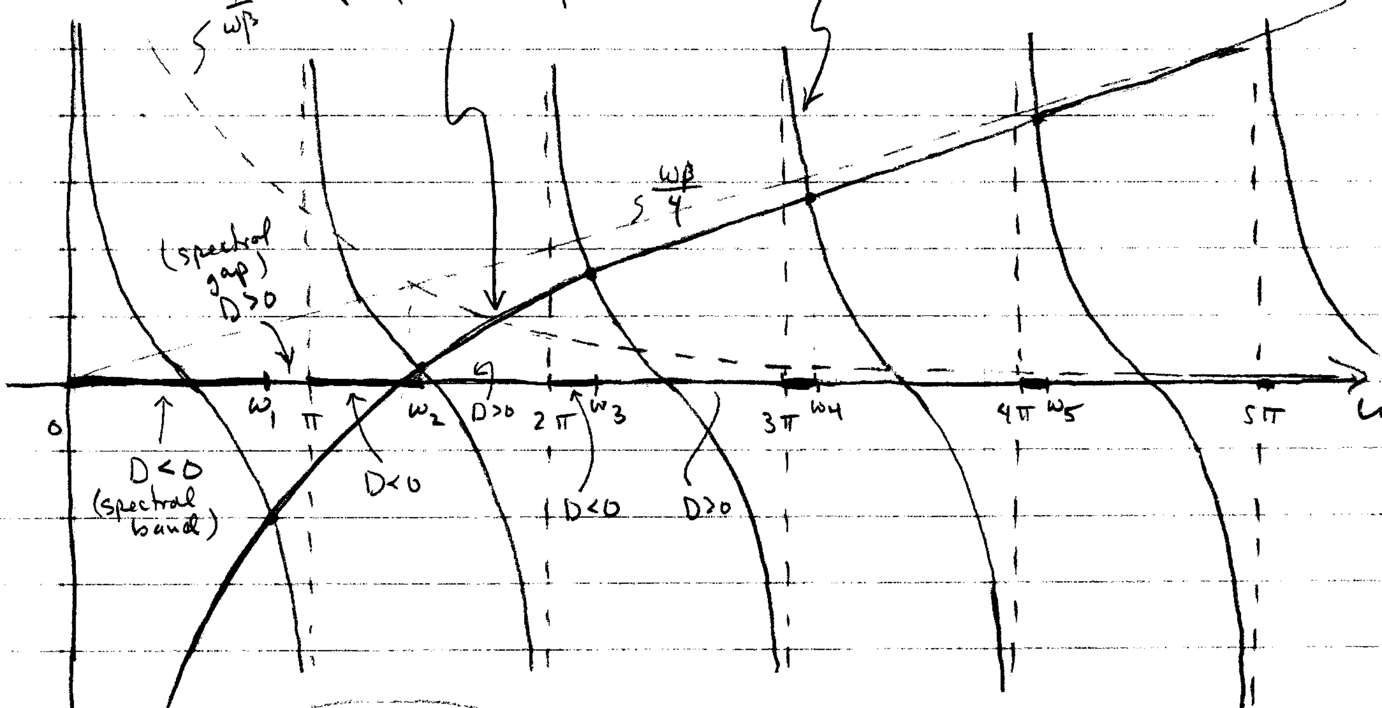
$\beta > 0$

One observes that the roots of the LHS of (4) called D are simple and consist of the numbers $n\pi$ and ω_n, where

$$(n-1)\pi < \omega_n < n\pi .$$

Graphically, the second factor of D has roots equal to the values of ω at which

$$\left(\frac{\omega\beta}{4} - \frac{1}{\omega\beta}\right) = \cot\omega$$



D < 0 : spectral bands, $[(n-1)\pi, \omega_n]$

D > 0 : spectral gaps, $(\omega_n, n\pi)$

Properties:

(1) $\omega_n \to (n-1)\pi$ as $n \to \infty$ for fixed β

$\Rightarrow$ ~~The lengths of the spectral bands shrink to zero as they tend to ∞.~~

(2) $\omega_n \to n\pi$ as $\beta \to 0$ for fixed n

$\Rightarrow$ In a finite part of the ω-axis, the lengths of the spectral gaps tend to zero as $\beta \to 0$, that is, the strength of ε tends to zero.