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Spectral measure and the equivalence of a normal operator in $V$ ($\dim V < \infty$) to a multiplication operator in an $L^2$ space.

Suppose first that the eigenvalues of the normal operator $T$ in $V$ are simple, and let us fix an orthonormal basis of $V$ consisting of eigenvectors of $T$:

$$\mathcal{B} = \{ e_\lambda : \lambda \in \sigma(T) \}$$

Define a measure $\mu$ on the Borel sets $\mathbb{B}$ of $\mathbb{C}$ (a Borel measure) by

$$\mu(\omega) = |\omega \cap \sigma(T)| = \begin{array}{l}\#\text{ of eigenvalues of } T\\ \text{that are contained in } \omega\end{array} \quad (\omega \in \mathbb{B})$$

$$= \sum_{\lambda \in \sigma(T)} \delta(z-\lambda) \quad \text{(where } z \text{ is the complex variable)}$$

Notice that the integral of a function with respect to $\mu$ is simply the sum of the values of the function at the points in $\sigma(T)$:

$$\int f \, d\mu = \sum_{\lambda \in \sigma(T)} f(\lambda) .$$

Let $\mathcal{M}$ denote the Borel-measurable functions on $\mathbb{C}$ and observe that all functions in $\mathcal{M}$ are square integrable, that is

$$\int |f|^2 d\mu = \sum_{\lambda \in \sigma(T)} |f(\lambda)|^2 < \infty$$

because $\sigma(T)$ is finite.

As usual, the Hilbert space $L^2(\mathbb{C},\mu)$ is defined by the space of square-integrable functions factored by the subspace of functions that are equal to zero almost everywhere (in this case functions vanishing on $\sigma(T)$), which we denote by $\mathcal{N}$

$$L^2(\mathbb{C},\mu) = \mathcal{M}/\mathcal{N},$$

(f and g are equivalent if they coincide on $\sigma(T)$)

together with the inner product

$$\langle f, g\rangle = \int f\,\bar{g}\,d\mu = \sum_{\lambda\in\sigma(T)} f(\lambda)\overline{g(\lambda)}.$$

At this point, it is evident that there is a unitary operator $\mathcal{U}: L^2(\mathbb{C},\mu) \longrightarrow V$ given by

$$\mathcal{U}(g) = \sum_{\lambda\in\sigma(T)} g(\lambda)\, e_\lambda$$

$$\mathcal{U}^{-1}(v) = \sum_{\lambda\in\sigma(T)} \langle v, e_\lambda\rangle\, \chi_{\{\lambda\}}$$

[ $\chi_E$ is the char. fcn of $E$. Notation is sloppy in the sense that $g$ is a representative of an element of $L^2(\mathbb{C},\mu)$. ]

(equality of fcns refers to equivalence in the sense of the defn of $L^2$)

The point of introducing $\mathcal{U}$ is that it establishes a conjugacy between $T$ and a standard multiplication operator $\tilde{T}$ in $L^2(\mathbb{C},\mu)$:

$$(\tilde{T}g)(z) := (\mathcal{U}^{-1}T\,\mathcal{U}g)(z) = \sum_{\lambda\in\sigma(T)} \lambda\, g(\lambda)\,\chi_{\{\lambda\}} = z\,g(z)$$

$*$ Now we see that a function of $T$, say $f(T): V \to V$ (defined by $f(T) = \varphi(f)$ from p. 7) goes over to multiplication by $f(z)$ on $L^2(\mathbb{C},\mu)$.

This is the "functional calculus" for $T$.

Spectral measure and $L^2$-representation of $T$, cont.

Now let the dimension of the eigenspaces for $T$ be arbitrary, $\dim W_\lambda = n_\lambda$ for $\lambda \in \sigma(T)$, and $\sum_{\lambda \in \sigma(T)} n_\lambda = n$.

Let $m = \max_{\lambda \in \sigma(T)} n_\lambda$.

Let us represent the "spectrum of $T$, counted according to multiplicity" as the disjoint union of subsets $\sigma_j(T)$ of $\sigma(T)$, $j = 1, \ldots, m$, (in a unique way):

$$\sigma(T) = \sigma_1(T) \supset \ldots \supset \sigma_m(T)$$

For each, $j = 1, \ldots, m$, let $\mathcal{B}_j$ be an orthornormal sets of eigenvalues of $T$ for the eigenvalues in $\sigma_j(T)$, with $\bigcup_{j=1}^{m} \mathcal{B}_j$ being an orthonormal basis for $V$ (diagonalizing $T$). $T$ is invariant on each $\operatorname{span} \mathcal{B}_j$, and we obtain a decomposition

$$T = \sum_{j=1}^{m} T_j \,, \qquad T_j : \operatorname{span} \mathcal{B}_j \to \operatorname{span} \mathcal{B}_j$$

of $T$ into multiplicity-one normal operators $T_j$, each of which has an $L^2$-representation. The measures $\mu_j$ are defined by

$$\mu_j(\omega) = |\omega \cap \sigma_j(T)| \,, \quad \mu_{j+1} \leq \mu_j \,,$$

and unitary operators $U_j : L^2(\mathbb{C}, \mu_j) \to \operatorname{span} \mathcal{B}_j$ by

$$U_j(g) = \sum_{\lambda \in \sigma_j(T)} g(\lambda)\, e^j_\lambda \,,$$

where $e^j_\lambda$ is the eigenvector in $\mathcal{B}_j$ for eigenvalue $\lambda$.

Thus each $T_j$ is, as before, conjugate by $U_j$ to a multiplication operator on $L^2(\mathbb{C}, \mu_j)$

$$(\tilde{T}_j g)(z) := (U_j^{-1} T_j U_j g)(z) = z g(z).$$

We have arrived at a representation theorem for normal operators in fin. dim inner-prod. spaces that will generalize easily to the inf.-dim'l case.

The Spectral Thm (again) in fin. dim.

Let $V$ be an inner-product space (over $\mathbb{C}$) with $\dim V < \infty$. Then $T$ is normal if and only if there are measures $\mu_j$ supported on finite sets $\sigma_j(T)$ such that $T$ is unitarily equivalent the multiplication operator on

$$L^2(\mathbb{C}, \mu_1) \oplus \dots \oplus L^2(\mathbb{C}, \mu_m)$$

defined by

$$f_1(\lambda) \oplus \dots \oplus f_m(\lambda) \longmapsto \lambda f_1(\lambda) \oplus \dots \oplus \lambda f_m(\lambda).$$

The sets $\sigma_j(T)$ can be taken to be nested: $\sigma_1(T) \supset \dots \supset \sigma_m(T)$.

<u>Resolution of the identity and projection-valued measures for <u>self-adjoint</u> operators ($T$ is normal and $\sigma(T) \subset \mathbb{R}$).</u>

[We will use the defn. of a <u>resolution of the identity</u> as in Akhiezer/Glazman, §61, p.16 (except I'll use right continuity instead of left continuity).]

In the finite case (still), let all previous notation associated with the self-adjoint operator $T$ continue to hold.

Define a <u>projection-valued function</u> of $\mathbb{R}$, $\lambda \mapsto E_\lambda$, where, for each $\lambda \in \mathbb{R}$, the projection $E_\lambda$ is given by

$$E_\lambda = \sum_{\substack{\mu \leq \lambda \\ \mu \in \sigma(T)}} P_\mu$$

<u>Properties of $E_\lambda$</u>

- increasing ($\langle E_\lambda v, v\rangle : \mathbb{R} \to \mathbb{R}$ is increasing for each $v \in V$)
- $E_{-\infty} = 0$, $E_\infty = I$
- $E_\lambda = E_{\lambda+0}$ ($\langle E_\lambda v, v\rangle$ is right-continuous for each $v \in V$)
- $E_\lambda E_\mu = E_\nu$, where $\nu = \min\{\lambda, \mu\}$ (This is equivalent to the embedding of images: $\operatorname{Ran} E_\lambda \subset \operatorname{Ran} P_\mu$ for $\lambda < \mu$)

We obtain integral expressions for the fllg statements:

$$\sum_{\lambda \in \sigma(T)} P_\lambda = I \quad \Longleftrightarrow \quad v = \int_{-\infty}^{\infty} d(P_\lambda v)$$

$$\sum_{\lambda \in \sigma(T)} \lambda P_\lambda = T \quad \Longleftrightarrow \quad Tv = \int_{-\infty}^{\infty} \lambda \, d(P_\lambda v)$$

$$\sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda = f(T) \quad \Longleftrightarrow \quad f(T)v = \int_{-\infty}^{\infty} f(\lambda) \, d(P_\lambda v)$$