Spectral measure and the equivalence of a normal operator in $V$ (dim $V < \infty$) to a multiplication operator in an $L^2$ space.

Suppose first that the eigenvalues of the normal operator $T$ in $V$ are simple, and let us fix an orthonormal basis of $V$ consisting of eigenvectors of $T$:

$$\mathcal{B} = \{e_\lambda : \lambda \in \sigma(T)\}$$

Define a measure $\mu$ on the Borel sets $B$ of $\mathbb{C}$ (a Borel measure) by

$$\mu(B) = |\omega \cap \sigma(T)| = \# \text{ of eigenvalues of } T \text{ that are contained in } \omega \quad \text{(we } B)$$

$$= \sum_{\lambda \in \sigma(T)} \delta(\zeta - \lambda) \quad \text{(where } \zeta \text{ is the complex variable)}$$

Notice that the integral of a function with respect to $\mu$ is simply the sum of the values of the function at the points in $\sigma(T)$:

$$\int f \, d\mu = \sum_{\lambda \in \sigma(T)} f(\lambda)$$

Let $M$ denote the Borel measurable functions on $\mathbb{C}$ and observe that all functions in $M$ are square integrable, that is

$$\int |f|^2 \, d\mu = \sum_{\lambda \in \sigma(T)} |f(\lambda)|^2 < \infty$$

because $\sigma(T)$ is finite.
As usual, the Hilbert space $L^2(\mathbb{R}, \mu)$ is defined by
the space of square-integrable functions factored by
the subspace of functions that are equal to zero almost
everywhere (in this case functions vanishing on $\sigma(T)$), which we denote by $\mathcal{N}$

$$L^2(\mathbb{R}, \mu) = \mathcal{M}/\mathcal{N},$$

if they coincide on $\sigma(T)$

Together with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(\tau) \overline{g(\tau)} d\mu = \sum_{\lambda \in \sigma(T)} f(\lambda) \overline{g(\lambda)}.$$

At this point, it is evident that there is a unitary operator
$$U : L^2(\mathbb{R}, \mu) \to V$$
given by

$$U(g) = \sum_{\lambda \in \sigma(T)} g(\lambda) e_{\lambda}$$

where $\{e_{\lambda}\}$ is an orthonormal basis for $V$

$$U^{-1}(v) = \sum_{\lambda \in \sigma(T)} \langle v, e_{\lambda} \rangle e_{\lambda}$$

Notation is sloppy in the sense that $g$ is a representative
of an element of $L^2(\mathbb{R}, \mu)$.

The point of introducing $U$ is that it establishes a conjugacy
between $T$ and a standard multiplication operator $\tilde{T}$ in $L^2(\mathbb{R}, \mu)$:

$$\tilde{T}g(z) := (U^{-1} T U g)(z) = \sum_{\lambda \in \sigma(T)} \lambda g(\lambda) e_{\lambda}$$

Now we see that a function of $T$, say $f(T) : V \to V$
(defined by $f(T)v = f(v)$ from $\mathcal{M}$) goes over to multiplication
by $f(z)$ on $L^2(\mathbb{R}, \mu)$.

This is the "functional calculus" for $T$. 
Spectral measure and $L^2$-representation of $T$, cont.

Now let the dimension of the eigenspaces for $T$ be arbitrary, $\dim \mathcal{W}_\lambda = n_\lambda$ for $\lambda \in \sigma(T)$, and $\sum_{\lambda \in \sigma(T)} n_\lambda = N$.

Let $m = \max n_\lambda$.

Let us represent the "spectrum of $T$, counted according to multiplicity" as the disjoint union of subsets $\sigma_j(T)$ of $\sigma(T)$, $j = 1, \ldots, m$, (in a unique way):$
\sigma(T) = \sigma_1(T) \cup \ldots \cup \sigma_m(T)
$

For each $j = 1, \ldots, m$, let $B_j$ be an orthonormal sets of eigenvalues of $T$ for the eigenvalues in $\sigma_j(T)$, with $\mathbb{U}B_j$ being an orthonormal basis for $V$ (diagonalizing $T$). $T$ is invariant on each span $B_j$, and we obtain a decomposition

$$
T = \sum_{j=1}^m T_j : \text{span } B_j \to \text{span } B_j
$$

of $T$ into multiplicity-one normal operators $T_j$, each of which has an $L^2$-representation. The measures $\mu_j$ are defined by

$$
\mu_j(\omega) = |\omega \cap \sigma_j(T)|, \quad \mu_{j+1} = \mu_j,
$$

and unitary operators $U_j : L^2(\mu_j) \to \text{span } B_j$ by

$$
U_j(g) = \sum_{\lambda \in \sigma_j(T)} g(\lambda) e_\lambda^j,
$$

where $e_\lambda^j$ is the eigenvector in $B_j$ for eigenvalue $\lambda$. 

Thus each \( T_j \) is, as before, conjugate by \( U_j \) to a multiplication operator on \( L^2(\mathbb{C}, \mu_j) \)

\[
(\tilde{T}_j g)(z) = (U_j^{-1} T_j U_j g)(z) = z g(z).
\]

We have arrived at a representation theorem for normal operators in fin. dim. inner-prod. spaces that will generalize easily to the inf.-dim. case.

The Spectral Theorem (again) in fin. dim.
Let \( V \) be an inner-product space (over \( \mathbb{C} \)) with \( \dim V < \infty \). Then \( T \) is normal if and only if there are measures \( \mu_j \) supported on finite sets \( \sigma_j(T) \) such that \( T \) is unitarily equivalent to the multiplication operator on

\[
L^2(\mathbb{C}, \mu_1) \oplus \cdots \oplus L^2(\mathbb{C}, \mu_m)
\]

defined by

\[
f_1(z) \oplus \cdots \oplus f_m(z) \mapsto \lambda f_1(z) \oplus \cdots \oplus \lambda f_m(z).
\]

The sets \( \sigma_j(T) \) can be taken to be nested: \( \sigma_1(T) \supset \cdots \supset \sigma_m(T) \).
Resolution of the identity and projection-valued measures for self-adjoint operators (T is normal and \( \sigma(T) \subseteq \mathbb{R} \)).

We will use the defn. of a resolution of the identity as in Akhiezer/Glazman, Elel., p.116 (except I'll use right continuity instead of left continuity.)

In the finite case (still), let all previous notation associated with the self-adjoint operator T continue to hold.

Define a projection-valued function of \( \mathbb{R} \), \( T \mapsto E_x \), where, for each \( x \in \mathbb{R} \), the projection \( E_x \) is given by

\[
E_x = \sum_{\mu \in \sigma(T)} P_{\mu}
\]

Properties of \( E_x \):
- increasing \((\langle E_x v, v \rangle : \mathbb{R} \rightarrow \mathbb{R} \) is increasing for each \( v \in V \))
- \( E_x = 0 \) if \( x < -1 \)
- \( E_x = E_{x+} \) \((\langle E_x v, v \rangle \) is right-continuoues for each \( v \in V \))
- \( E_x E_y = E_y \), where \( y = \min \{ \beta, \gamma \} \)

We obtain integral expressions for the silly statements:

\[
\sum_{\mu \in \sigma(T)} P_{\mu} = I \quad \Leftrightarrow \quad v = \int_{-\infty}^{\infty} d(P_{\mu} v)
\]

\[
\sum_{\mu \in \sigma(T)} \lambda P_{\mu} = T \quad \Leftrightarrow \quad Tv = \int_{-\infty}^{\infty} \lambda d(P_{\mu} v)
\]

\[
\sum_{\mu \in \sigma(T)} f(\lambda) P_{\mu} = f(T) \quad \Leftrightarrow \quad f(T) v = \int_{-\infty}^{\infty} f(\lambda) d(P_{\mu} v)
\]

\( \mathbb{R} \)