2008.01.23

Unbounded operators in Hilbert space

In general, an unbounded linear operator T in a Hilbert space $\mathcal{H}$ is defined only on a sub-vector-space of $\mathcal{H}$, called its domain and denoted by $\mathcal{D}(T)$. We will always assume that $\mathcal{D}(T)$ is dense in $\mathcal{H}$, in which case we say T is densely defined in $\mathcal{H}$.

The adjoint T^* of T is defined on the vector space $\mathcal{D}(T^*)$ of all elements $z \in \mathcal{H}$ such that the functional

$$\mathcal{D}(T) \longrightarrow \mathbb{C} :: x \longmapsto \langle Tx, z \rangle$$

is bounded. In this case, the Riesz lemma (see Reed/Simon, vol 1, for example) guarantees that there is an element w of $\mathcal{H}$ that represents this functional, that is,

$$\langle Tx, z \rangle = \langle x, w \rangle \quad \text{for all } x \in \mathcal{D}(T).$$

By defining $T^* z = w$, we have a linear operator defined on $\mathcal{D}(T^*)$ such that

$$\langle Tx, z \rangle = \langle x, T^* z \rangle \quad \forall\, x \in \mathcal{D}(T),\ z \in \mathcal{D}(T^*).$$

An operator T is said to be self-adjoint if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $Tx = T^* x$ for all $x \in \mathcal{D}(T)$. In other words,

$$\langle Tx, z \rangle = \langle x, Tz \rangle \quad \forall\, x, z \in \mathcal{D}(T) = \mathcal{D}(T^*).$$

<u>The Hellinger-Toeplitz theorem</u> (see Reed/Simon Vol 1, p. 84)
Let T be an everywhere-defined ($\mathcal{D}(T) = \mathcal{H}$) self-adjoint linear operator in a Hilbert space $\mathcal{H}$. Then T is bounded.

<u>Proof</u> By the Closed-Graph Theorem, it is sufficient to prove that the graph of T is closed, that is, whenever $x_n \to x$ ($n \to \infty$) and $Tx_n \to y$ ($n \to \infty$), we have $Tx = y$. To see this, note that, for each $z \in \mathcal{H}$,

$$\langle Tx, z \rangle = \langle x, Tz \rangle = \lim_{n\to\infty} \langle x_n, Tz \rangle = \lim_{n\to\infty} \langle Tx_n, z \rangle = \langle y, z \rangle,$$

which implies $Tx = y$.

This theorem says that any <u>unbounded</u> self-adjoint operator T in a Hilbert space $\mathcal{H}$ cannot be defined on all of $\mathcal{H}$. Instead, it will be defined on a <u>dense</u> <u>proper</u> <u>subspace</u> (meaning sub-vector-space) of $\mathcal{H}$, the domain $\mathcal{D}(T) \subset \mathcal{H}$.

There is much much more that can be said about unbounded operators — the concepts of <u>closed</u> operators, the <u>closure</u> of an operator, those that are or are not <u>closeable</u>, the <u>core</u> for the domain of an operator, <u>essentially self-adjoint</u> operators, etc. Reed/Simon has a good presentation of this material.

However, for now, we have introduced enough mathematics to present the spectral theorem and proceed with examples.

Before stating the spectral theorem, let us discuss the spectrum of an operator (bounded or unbounded) in Hilbert space. Notice that the spectrum does not need to be mentioned in the statement of the theorem, but it is there, encoded in the measure (its support) and the multiplication operator, just as we saw in the finite-dimensional case.

Let T be an operator in $\mathcal{H}$ with domain $\mathcal{D}(T)$, $T : \mathcal{D}(T) \to \mathcal{H}$. In considerations of spectrum it is typically assumed that T is a closed operator.

Defn. $T : \mathcal{D}(T) \to \mathcal{H}$ is closed if the graph $\Gamma(T)$ of T is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. Recall the definition of $\Gamma(T)$:

$$\Gamma(T) = \{ (x, Tx) : x \in \mathcal{D}(T) \}$$

We will not pursue the issue of closedness at this point. For now we will simply note its significance in the following fact:

Fact The closure of $\Gamma(T)$ in $\mathcal{H} \oplus \mathcal{H}$ is the graph of an operator if and only if $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$.
[Reed/Simon Theorem VIII.1 p. 252]

spectrum

Defn The resolvent set $\rho(T)$ of an operator T is the set of $\lambda \in \mathbb{C}$ such that the operator

$$(\lambda I - T) : \mathcal{D}(T) \to \mathcal{H}$$

and $(\lambda I - T)^{-1}$ is called the resolvent of T at λ.

is bijective and has a bounded inverse. The spectrum of T is defined to be $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

<u>Fact</u> $\sigma(T)$ is a closed subset of $\mathbb{C}$ [Reed/Simon Theorem VIII.2 p254]

We will assume now that <u>T is closed</u>.

Several cases arise for the operators $\lambda I - T$:

1. If $\lambda I - T$ is not injective, then λ is an eigenvalue of T, that is, there exists $v \in \mathcal{D}(T)$ such that

$$(\lambda I - T)v = 0, \quad \|v\| = 1.$$

In this case, we say that λ is in the <u>point spectrum</u> of T.

2. If $\lambda I - T$ is injective but $\mathrm{Ran}(\lambda I - T)$ is not dense in $\mathcal{H}$, then λ is an element of the <u>residual spectrum</u> of T.

EXERCISE 1 to turn in :

If $\lambda I - T$ is injective and $\mathrm{Ran}(\lambda I - T)$ is dense in $\mathcal{H}$, then one of the following hold :

a. $\mathrm{Ran}(\lambda I - T) = \mathcal{H}$ and $(\lambda I - T)^{-1}$ is bounded.

b. $\mathrm{Ran}(\lambda I - T) \neq \mathcal{H}$ and there is a sequence $\{v_n\}_{n=1}^{\infty}$ from $\mathcal{D}(T)$ such that

$$(\lambda I - T)v_n \longrightarrow 0 \text{ as } n \to \infty \text{ and } \|v_n\| = 1.$$

In this exercise, remember that T is closed, and recall the closed-graph theorem.

The Spectral Theorem for self-adjoint operators

Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then there exists a family $\{\mu_\alpha\}_{\alpha\in\mathcal{A}}$ ($\mathcal{A}$ is some index set) of Borel measures on $\mathbb{R}$ and a unitary operator $\mathcal{U}$

$$\mathcal{U} : \bigoplus_{\alpha\in\mathcal{A}} L^2(\mathbb{R},\mu_\alpha) \longrightarrow \mathcal{H}$$

such that, if we define $\tilde{T} = \mathcal{U}^{-1} T\, \mathcal{U}$ on $\mathcal{D}(\tilde{T}) = \mathcal{U}^{-1}[\mathcal{D}(T)]$, then

- $\mathcal{D}(\tilde{T}) = \left\{ \{f_\alpha\}_{\alpha\in\mathcal{A}} \in \bigoplus_{\alpha\in\mathcal{A}} L^2(\mathbb{R},\mu_\alpha) \;:\; \sum_{\alpha\in\mathcal{A}} \int_{\mathbb{R}} |\lambda f_\alpha(\lambda)|^2 \mu_\alpha(d\lambda) < \infty \right\}$

- For each $f = \{f_\alpha\} \in \mathcal{D}(\tilde{T})$, $(\tilde{T} f)_\alpha(\lambda) = \lambda f_\alpha(\lambda)$

Notes on the notation

If $\{\mathcal{H}_\alpha\}_{\alpha\in\mathcal{A}}$ is a family of Hilbert spaces, then their direct product $\sum_{\alpha\in\mathcal{A}} \mathcal{H}_\alpha$ is the set of elements $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, with $f_\alpha \in \mathcal{H}_\alpha$, such that $\sum_{\alpha\in\mathcal{A}} \|f_\alpha\|^2 < \infty$. $\sum_{\alpha\in\mathcal{A}} \mathcal{H}_\alpha$ is a Hilbert space with inner product

$$\langle \{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{A}} \rangle = \sum_{\alpha\in\mathcal{A}} \langle f_\alpha, g_\alpha \rangle_{\mathcal{H}_\alpha}.$$

[For this rendition of the theorem, see Dunford/Schwartz Part II § XII Def. 4; thm 5.]

Often, this form of the spectral theorem is too restrictive and we prefer a more abstract form, in which a self-adjoint operator T is guaranteed to be unitarily equivalent to a multiplication operator in a single L^2 space. See Reed/Simon, Vol. 1 p. 260.

<u>The Spectral Theorem</u> Let T be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T)$. Then there is a measure space (X, μ), with μ a finite measure (μ subsumes the family of measurable sets), a unitary operator $\mathcal{U} : L^2(X, d\mu) \to \mathcal{H}$, and a <u>real-valued</u> function f on X such that, if we define $\tilde{T} = \mathcal{U}^{-1} T \mathcal{U}$ on $\mathcal{D}(\tilde{T}) = \mathcal{U}^{-1}[\mathcal{D}(T)]$, then

- $\mathcal{D}(\tilde{T}) = \{ g \in L^2(X, d\mu) : fg \in L^2(X, d\mu) \}$,
- $\tilde{T}(g) = fg$ for all $g \in \mathcal{D}(\tilde{T})$.

There are many other versions of the spectral theorem, the most elegant of which involves the "spectral resolution of the identity." This can be found in many books, such as Reed/Simon (Thm. VIII.6 p. 263 of Vol. 1), Akhiezer/Glazman (§61 p. 16 and §66 Thm. 1 and 2 p 36-38)

We will not pursue the topic of resolutions of the identity at this point.

We outline below *a* proof of the spectral theorem for a <u>bounded self-adjoint</u> operator T in $\mathcal{H}$ of <u>multiplicity one</u>. The outline is essentially that followed in Reed/Simon, Vol 1, and proofs of the necessary lemmas and theorems can be found there.

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\Psi"'] \arrow[dr, "\Phi"] & \\ \mathcal{L}(\mathcal{H}) & & C(\sigma(T)) \arrow[ll, dashed, "\varphi"'] \end{tikzcd}$$

1. By the self-adjointness of T, $\sigma(T) \subset \mathbb{R}$ (as is the case in finite dimension) [Theorem VI.8, p.194]. The spectrum $\sigma(T)$ is a <u>closed</u> subset of $\mathbb{R}$ (it could be any one). As in the structure we established for the finite-dimensional case, Φ denotes restriction of a polynomial to $\sigma(T)$ and Ψ denotes evaluation of a polynomial at T.

2. One proves, from elementary principles, the <u>spectral mapping theorem</u> $\sigma(p(T)) = p(\sigma(T))$ and that $\|\Phi(p)\| = \|\Psi(p)\|$ [Lemma 1, p. 223 and Theorem VI.6 p. 192]. This gives rise to an <u>isometric algebraic $*$-homomorphism $\varphi : \mathrm{Ran}(\Phi) \to \mathcal{L}(\mathcal{H})$</u>, so each polynomial on $\sigma(T)$ is associated to a linear operator on $\mathcal{H}$ with the same norm.

3. Since polynomials are dense in $C(\sigma(T))$ [the Stone-Weierstraß Theorem], <u>φ is extended to $C(\sigma(T))$</u>, $\varphi : C(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ [Theorem VII.1 p. 222, for properties of φ]. <u>Denote</u> $\varphi(f)$ by $f(T)$ for $f \in C(\sigma(T))$. <u>φ is positive</u>, meaning that $f > 0 \Rightarrow f(T) > 0$. (Recall $f(T) > 0$ means $\langle f(T)v, v\rangle > 0$ for all $v \in \mathcal{H}$.

4. The <u>spectral measures</u>. Choose $v \in \mathcal{H}$ and consider the <u>positive</u> linear functional $C(\sigma(T)) \to \mathbb{C} :: f \mapsto \langle f(T)v, v\rangle$. By the <u>Riesz-Markov Theorem</u> [Theorem IV.14 p. 107 and remark (4) p. 106], there exists a unique Borel measure μ_v that represents this functional, that is,

$$\langle f(T)v, v\rangle = \int_{\mathbb{R}} f(\lambda)\,\mu_v(d\lambda).$$

5. <u>Identification of Hilbert spaces</u> (dependent on the choice of v)
Define a map <u>$\mathcal{U} : C(\sigma(T)) \to \mathcal{H}$ by $\mathcal{U}(f) = f(T)v$</u>.
Then consider $C(\sigma(T))$ as a subspace of $L^2(\mathbb{R}, \mu_v)$ and observe that

$$\int f\bar{g}\,d\mu_v = \langle f(T)\bar{g}(T)v, v\rangle = \langle f(T)v, g(T)v\rangle = \langle \mathcal{U}(f), \mathcal{U}(g)\rangle,$$

so that <u>$\mathcal{U}$ is an isometry</u>.

6. Since $C(\sigma(T))$ is dense in $L^2(\mathbb{R}, \mu_v)$ (in the L^2-norm).
[remark (3) p. 106 OR Theorem 3.14 of Rudin p. 69]
$\mathcal{U}$ is extended to $L^2(\mathbb{R}, \mu_v)$,

$$\mathcal{U} : L^2(\mathbb{R}, \mu_v) \longrightarrow \mathcal{H}.$$

Now, $\mathrm{Ran}(\mathcal{U})$ is not necessarily all of $\mathcal{H}$, but only a sub-Hilbert space of $\mathcal{H}$ called the <u>orbit</u> $\mathcal{O}(v)$ of v under T.
If $\mathcal{O}(v) = \mathcal{H}$, then T is of multiplicity one on $\mathcal{H}$.
We see that multiplication by λ corresponds to T under $\mathcal{U}$:

$$\mathcal{U}(\lambda f(\lambda)) = (Tf(T))v = T(f(T)v) = T(\mathcal{U}f(\lambda)).$$

Properties of self-adjoint operators.

The properties we present here are easily proved using the spectral theorem. They can be proved from elementary principles also, and in fact some of them are used in proving the spectral theorem.

Let $T : \mathcal{D}(T) \to \mathcal{H}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then the following statements hold.

a. $\sigma(T) \subset \mathbb{R}$.

b. If λ and μ are eigenvalues of T, with $\lambda \neq \mu$, then their eigenspaces are orthogonal, that is $\mathcal{N}ull(\lambda I - T) \perp \mathcal{N}ull(\mu I - T)$.

c. If $\lambda \in \sigma(T)$, then either one of the following hold:

i. λ is an eigenvalue of T, that is,

$$(\lambda I - T)v = 0, \quad \|v\| = 1$$

for some $v \in \mathcal{H}$, OR

ii. There exists a sequence $\{x_n\}$ from $\mathcal{H}$ such that

$$(\lambda I - T)v_n \to 0, \quad \|v\| = 1.$$

In this case, $\mathcal{R}an(\lambda I - T)$ is dense in $\mathcal{H}$ but not equal to $\mathcal{H}$.

In particular, T has no residual spectrum.

d. The graph of T is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. This is the same as saying that T is a closed operator.