Example

Study of the multiplication operator $T_g$ in $L^2$ defined through

\[ g(x) = \begin{cases} x^2, & x < -1, \\ 1, & -1 \leq x < 1, \\ 1/x^2, & 1 \leq x, \end{cases} \]

by \( \mathcal{D}(T_g) = \left\{ f \in L^2(\mathbb{R}, dx) : g f \in L^2 \right\} \)

\[ T_g f = g f \quad \text{for all} \quad f \in \mathcal{D}(T_g). \]

The purpose of this example is not to derive facts about $T_g$ using the most abstract methods but to prove facts in more elementary ways that concretely demonstrate the nature of multiplication operators.

• The domain of $T_g$ is defined as the largest subspace of $L^2$ on which multiplication by $g$ makes sense concretely as an operator into $L^2$.

Since

\[ \int_{-\infty}^{\infty} |g(x)|^2 \, dx = \int_{-1}^{1} |f(x)|^2 \, dx \]

and

\[ \int_{-\infty}^{\infty} |f(x)|^2 \, dx \leq \int_{-1}^{1} |f(x)|^2 \, dx + \int_{-\infty}^{-1} |f(x)g(x)|^2 \, dx, \]

we obtain

\[ \mathcal{D}(T_g) = L^2(-\infty, -1; \chi^4 \, dx) \oplus L^2(-1, \infty; \chi^2 \, dx). \]

• $T_g$ is unbounded: Let $f_n = \chi_{\{x \leq n \}}$ for $n = 1, 2, \ldots$. Then $g f_n(x) = x^2 \chi_{\{x \leq n \}} \in L^2$, so $f_n \in \mathcal{D}(T_g)$, and $T_g f_n = x^2 \chi_{\{x \leq n \}}$.

Also, $\|f_n\|_2 = 1$. Now, we have

\[ \|T_g f_n\|_2^2 = \int_{-n}^{n} (x^2)^2 \, dx \geq \int_{-n}^{n} n^4 \, dx = n^4 \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty, \]

which shows that $T_g$ is unbounded.
We show $\mathcal{D}(T_g)$ is dense in $L^2$. Let $f \in L^2$ be given. The Lebesgue integral has the property that

$$\int_{-\infty}^{\infty} |f|^2 \, dx \rightarrow \int_{-\infty}^{\infty} |f|^2 \, dx = \|f\|_2^2 < \infty \quad (n \to \infty).$$

If we define $f_n = \chi_{E_n} f$, then we obtain

$$\|f-f_n\|_2 = \int_{-\infty}^{\infty} |f|^2 \, dx \to 0 \quad \text{as} \quad n \to \infty.$$

Since $|f_n(x)| \leq n^2$ for $x$ in the support of $f_n$, we see that $g f_n \in L^2$, so that $f_n \in \mathcal{D}(T_g)$, and this proves the assertion.

We show $\mathcal{D}(T_g) \neq L^2$. Observe that the function

$$f(x) = \begin{cases} \sqrt{x^2} & x < -1 \\ 0 & x \geq -1 \end{cases}$$

is in $L^2(\mathbb{R})$ but that $gf = \chi_{(-\infty,1]}$ is not in $L^2$, so that $f \in \mathcal{D}(T_g)$.

We prove

$$\sigma(T_g) = [0, \infty) = \overline{\text{range of } Rg} \quad \text{(the closure of } \text{range of } Rg \text{ in } L^2).$$

First we prove that if $\lambda \not\in [0, \infty)$, then $\lambda \not\in \sigma(T_g)$.

For $\lambda \not\in [0, \infty)$, we have $d = \text{dist}(\lambda, \overline{\text{range of } Rg}) > 0$ and

for all $x \in \mathbb{R}$, $|x-g(x)| \geq d > 0$. Therefore

$$0 < |(\lambda-g(x))^{-1}| \leq \frac{1}{d} < \infty \quad \forall \, x \in \mathbb{R}.$$
Since \( \| x - g(x) \| \geq d > 0 \), the equality

\[(x - g(x))f(x) = 0 \quad \text{a.e.} \quad f \in \mathcal{D}(T_d)\]

implies that \( f(x) = 0 \) in \( L^2 \) so that \( (\mathcal{N} - T_d) \) is injective.

To see that \( (\mathcal{N} - T_d) \) is surjective (into \( L^2 \)), let \( f \in L^2 \) be given. Since \( \| (x - g(x))^{-1} \| \leq \frac{1}{d} \), we have \( h = (x - g(x))^{-1} f \in L^2 \), and since \( f = (x - g(x))h \in L^2 \), we have \( gh \in L^2 \) so that \( h \in \mathcal{D}(T_d) \).

We obtain \( (\mathcal{N} - T_d)h = f \).

Notice that we have obtained an explicit form of \( (\mathcal{N} - T_d)^{-1} \):

\[ [(\mathcal{N} - T_d)^{-1} f](x) = (x - g(x))^{-1} f(x). \]

Let us write \( \Gamma(x) = (x - g(x))^{-1} \) and prove that

\[(\mathcal{N} - T_d)^{-1} : L^2 \rightarrow \mathcal{D}(T_d) \quad f \mapsto \Gamma f \]

is bounded and has norm \( \| \Gamma \|_2 = \frac{1}{d} \).

First, we see that, for \( f \in L^2 \),

\[ \| \Gamma f \|_2 \leq \| f \|_2 + \| f \|_2 \]

so that \( \| (\mathcal{N} - T_d)^{-1} \| \leq \frac{1}{d} \). Since \( \Gamma \) is continuous, for each \( \varepsilon > 0 \), there is an interval \( I \) with \( m(I) > 0 \) such that

\[ |(x - g(x))^{-1}| \leq \frac{1}{d} - \varepsilon \quad \forall x \in I. \]

Observe now that

\[ (\mathcal{N} - T_d)^{-1} X_\varepsilon(x) = \Gamma(x) X_\varepsilon(x) = \left( \frac{1}{d} - \varepsilon \right) X_\varepsilon(x) \]

\( \forall x \in \mathbb{R} \), so that

\[ \| (\mathcal{N} - T_d)^{-1} X_\varepsilon \|_2 \leq \left( \frac{1}{d} - \varepsilon \right) \| X_\varepsilon \|_2, \quad \| X_\varepsilon \|_2 \neq 0 \]

Since \( \varepsilon \) is arbitrary, we have \( \| (\mathcal{N} - T_d)^{-1} \| \leq \frac{1}{d} \).

We have proved \( \| (\mathcal{N} - T_d)^{-1} \| = \frac{1}{d} \), and thus \( \lambda \in \mathcal{P}(T_d) \).
To prove that $E_0 = 0(T_N)$, let $\lambda \in [0, \infty)$, and observe that, by the continuity of $g$, for each $n = 1, 2, \ldots$

there is an interval $I_n$ with $|I_n| > 0$ such that

$$|\lambda - g(x)| < \frac{1}{n} \quad \text{for} \quad x \in I_n$$

Let us define

$$f_n = \chi_{I_n} \frac{1}{|I_n|^{1/2}}$$

so that $\|f_n\|_2 = 1$. We have

$$\left| \int (\lambda - T_N) f_n \phi \right| = \left| (\lambda - g(\phi)) f_n \phi \right| \leq \frac{1}{n} f_n(x) \quad \forall \phi \in \mathcal{D}$$

so that

$$\left\| (\lambda - T_N) f_n \right\|_2 \leq \frac{1}{n} \| f_n \|_2 = \frac{1}{n}.$$ 

Thus we see that $\lambda - T_N$ cannot have a bounded inverse because we have demonstrated rather explicitly that $\lambda - T_N$ is not bounded from below through the sequence $f_n$:

$$(\lambda - T_N)f_n \to 0, \quad \| f_n \|_2 = 1.$$ 

We prove that $\lambda = 1$ is the only eigenvalue of $T_N$ and that its multiplicity is infinite.

Let us look at the nullspace of $\lambda - T_N$. Suppose $\lambda - g(x) = 0$ a.e. for some $f \in L^2$. If $\lambda \neq 1$,

then $\lambda - g(x) = 0$ at no more than one point a.e. that $f = 0$ a.e.,

and $\lambda$ is not an eigenvalue. Now suppose that

$$(1 - g(x)) f(x) = 0 \quad \text{a.e.}$$
For \( x \in [-1, 1] \), \( (1 - g(x)) \neq 0 \), so \( f(x) = 0 \) a.e. off \([-1, 1]\).

Since \( 1 - g(x) = 0 \) for all \( x \in [-1, 1] \), we have

\[
(1 - g(x)) f(x) = 0 \quad \text{a.e.}
\]

for all \( f \) such that \( f = 0 \) a.e. off \([-1, 1]\). Therefore, the eigenspace for \( \lambda = 1 \) is

\[
\text{Null}(I - T_g) = L^2([-1, 1]),
\]

naturally considered as a subspace of \( L^2(\mathbb{R}) \). Since \( L^2([-1, 1]) \)

is infinite-dimensional, \( \lambda = 1 \) has infinite multiplicity.

- To prove that \( \lambda I - T_g \) has dense range for all \( \lambda \neq 1 \)
  (i.e., there is no residual spectrum), one can work with
  the inverse \( (\lambda I - g(x))^{-1} \) and examine the domain of the
  associated multiplication operator. [See the Problem set 2.]

- The adjoint of \( T_g \)

  As in problem 1, set 2, we must show that \( \mathcal{B}(T_g^*) = \mathcal{B}(T_g) \).

  The main role in showing \( T_g^* = T_g \) is simply

  \[
  \int (g \bar{f}) \overline{h} = \int f \overline{(g \bar{h})}.
  \]

- \( T_g \) is normal. One shows that

  \[
  \mathcal{D}(TT^*) = \{ f \in L^2 : \overline{g} f \in \mathcal{B}(T_g) \} = \{ f \in L^2 : g^2 f \in L^2 \}
  \]

  \[
  = \mathcal{D}(T^*T),
  \]

  that this domain is dense in \( L^2(\mathbb{R}) \), and that

  on \( \mathcal{D}(TT^*) \), we have

  \[
  T_g T_g^* = T_g^* T_g = T_g \|_2.
  \]
A study of multiplication operators on $L^2(\mathbb{R})$. This exercise asks you to prove properties of the most general multiplication operators on $L^2(\mathbb{R})$ without appealing to the abstract study of unbounded operators in Hilbert space. Thus you get a concrete feeling for the concepts of spectrum, adjoint, closedness, etc. that is needed in working with the spectral theory in mathematical physics.

Let $g: \mathbb{R} \to \mathbb{C}$ be a Lebesgue-measurable function (denote Lebesgue measure by $\mu$). The essential range of $g$ is

$$E(g) = \{ y \in \mathbb{C} : \exists \epsilon > 0, \mu \{ x : |g(x) - y| < \epsilon \} > 0 \}.$$

Define the operator $T_g$ in $L^2(\mathbb{R}, \mu)$ by

$$\mathcal{D}(T_g) = \{ f \in L^2(\mathbb{R}) : g f \in L^2(\mathbb{R}) \},$$

$$T_g f = g f \quad \text{for all } f \in \mathcal{D}(T_g).$$

a. Prove that $T_g$ is bounded if and only if $g \in L^\infty(\mathbb{R})$ and that, in this case, $\|T_g\| = \|g\|_{\infty}$.

b. Prove that $\mathcal{D}(T_g)$ is dense in $L^2(\mathbb{R})$ and that $\mathcal{D}(T_g) = L^2(\mathbb{R})$ if and only if $T_g$ is bounded.

c. Prove that $\sigma(T_g) = \mathcal{E}(g)$ and that $\sigma(T_g)$ is closed in $\mathbb{C}$.

d. For $\lambda \notin \sigma(T_g)$, find an explicit form for the resolvent $(\lambda I - T_g)^{-1}$, and prove $\| (\lambda I - T_g)^{-1} \| = \left[ \text{dist}(\lambda, \sigma(T_g)) \right]^{-1}$.

e. Characterize the eigenvalues of $T_g$ in relation to the function $g$. 

g. Prove that $D(T_g^*) = D(T_g)$ and that, for $f \in D(T_g^*)$, $T_g^* f = \overline{g f}$ (that is, $T_g^* = T_g$). Prove that $T_g^*$ is a closed operator. [Use the fact that it is the adjoint of an operator; proof by its explicit form is more difficult.]

Conclude that $T_g$ is also closed.

h. For $x \in \sigma(T_g)$, prove that there exists a sequence $f_n \in D(T_g)$ such that $(T_g - \lambda x) f_n \to 0$ and $\|f_n\| = 1$. Construct this sequence as explicitly as possible. [This will show you that you can prove existence of the via directly, without appealing to the closedness of $T_c$ and the result of Problem 1.

(i. below)

j. Let $E$ be a $\mu$-measurable subset of $\mathbb{R}$. Prove that the operator $P_E: L^2 \to L^2$ by $f \mapsto \chi_E f$ is an orthogonal projection. Find its nullspace and its range. Prove that $P_E$ commutes with $T_g$.

k. Find a choice of $g$ such that $T_g$ is unitary.

For arbitrary $g$, find a continuous function of $T_g$ that is unitary. [For $h \in C(\mathbb{R})$, $h(T_g)$ is given by $[h(T_g) f](x) = h(g(x)) f(x)$.

l. Find a choice of $g$ such that $T_g$ is anti-self-adjoint, that is, $T_g^* = -T_g$.

m. $D(AB) = \{ f \in D(B) : B(f) \in D(A) \}$ is the natural domain for $AB$.

Prove that $T_g$ is normal, that is, $T_g T_g^* = T_g^* T_g$ (pay att to domains).