2008.01.28

Example

Study of the multiplication operator T_g in L^2 defined through

$$g(x) = \begin{cases} x^2, & x \le -1, \\ 1, & -1 \le x \le 1, \\ 1/x^2, & 1 \le x, \end{cases}$$

by

$$\begin{cases} \mathcal{D}(T_g) = \{ f \in L^2(\mathbb{R}, dx) : gf \in L^2 \}, \\ T_g f = gf \quad \text{for all } f \in \mathcal{D}(T_g). \end{cases}$$

The purpose of this example is not to derive facts about T_g using the most abstract methods but to prove facts in more elementary ways that concretely demonstrate the nature of multiplication operators.

- The domain of T_g is defined as the largest subspace of L^2 on which multiplication by g makes sense concretely as an operator into L^2. Since

$$\int_{-1}^{\infty} |g(x) f(x)|^2 dx \le \int_{-1}^{\infty} |f(x)|^2 dx$$

and

$$\int_{-\infty}^{-1} |f(x)|^2 dx \le \int_{-\infty}^{-1} |f(x)|^2 x^4 dx = \int_{-\infty}^{-1} |f(x) g(x)|^2 dx,$$

we obtain

$$\mathcal{D}(T_g) = L^2(-\infty, -1; x^4 dx) \oplus L^2(-1, \infty; dx).$$

- T_g is unbounded: Let $f_n = \chi_{[-n, -n+1]}$ for $n = 1, 2, \ldots$. Then $(g f_n)(x) = x^2 \chi_{[-n,-n+1]} \in L^2$, so $f_n \in \mathcal{D}(T_g)$, and $T_g f_n = x^2 \chi_{[-n,-n-1]}$; also, $\| f_n \|_{L^2} = 1$. Now, we have

$$\| T_g f \|_{L^2}^2 = \int_{-n-1}^{-n} (x^2)^2 dx \ge \int_{-n-1}^{-n} n^4 dx = n^4 \to \infty \text{ as } n \to \infty,$$

which shows that T_g is unbounded.

- We show $\mathcal{D}(T_g)$ is dense in L^2. Let $f \in L^2$ be given. The Lebesgue integral has the property that

$$\int_{-n}^{\infty} |f|^2 dx \longrightarrow \int_{-\infty}^{\infty} |f|^2 dx = \|f\|_{L^2} < \infty \quad (n \to \infty).$$

If we define $f_n = \chi_{[-n,\infty]} f$, then we obtain

$$\|f - f_n\|_{L^2} = \int_{-\infty}^{-n} |f|^2 dx \to 0 \quad \text{as} \quad n \to \infty.$$

Since $|g(x)| \leq n^2$ for x in the support of f_n, we see that $gf_n \in L^2$, so that

$$f_n \in \mathcal{D}(T_g),$$

and this proves the assertion.

- We show $\mathcal{D}(T_g) \neq L^2$. Observe that the function

$$f(x) = \begin{cases} 1/x^2, & x \leq -1 \\ 0, & x > -1 \end{cases}$$

is in $L^2(\mathbb{R})$ but that $gf = \chi_{(-\infty,-1]}$ is not in L^2, so that $f \notin \mathcal{D}(T_g)$.

- We prove $\sigma(T_g) = [0,\infty) = \overline{\operatorname{Ran} g}$ (the closure of $\operatorname{Ran} g$ in $\mathbb{C}$):
First we prove that, if $\lambda \notin [0,\infty)$, then $\lambda \in \rho(T_g)$.
For $\lambda \notin [0,\infty)$, we have $d = \operatorname{dist}(\lambda, \overline{\operatorname{Ran} g}) > 0$ and for all $x \in \mathbb{R}$, $|\lambda - g(x)| \geq d > 0$. Therefore

$$0 < |(\lambda - g(x))^{-1}| \leq \frac{1}{d} < \infty \quad \forall \ x \in \mathbb{R}.$$

Since $|\lambda - g(x)| \geq d > 0$, the equality

$$(\lambda - g(x)) f(x) = 0 \quad \text{a.e.}, \quad f \in \mathcal{D}(T_g)$$

implies that $f(x) = 0$ in L^2 so that $(\lambda I - T_g)$ is injective.
To see that $\lambda I - T_g$ is surjective (onto L^2), let $f \in L^2$ be given. Since $|(\lambda - g(x))^{-1}| \leq \frac{1}{d}$, we have $h := (\lambda - g)^{-1} f \in L^2$, and since $f = (\lambda - g)h \in L^2$, we have $gh \in L^2$ so that $h \in \mathcal{D}(T_g)$.
We obtain $(\lambda I - T_g)h = f$.

Notice that we have obtained an explicit form of $(\lambda I - T_g)^{-1}$:

$$\left[(\lambda I - T_g)^{-1} f\right](x) = (\lambda - g(x))^{-1} f(x).$$

Let us write $r(x) = (\lambda - g(x))^{-1}$ and prove that

$$(\lambda I - T_g)^{-1} : L^2 \to \mathcal{D}(T_g) :: f \mapsto rf$$

is <u>bounded</u> and has norm $\|r\|_\infty = \frac{1}{d}$.
First, we see that, for $f \in L^2$,

$$\|rf\|_{L^2} \leq \|r\|_\infty \|f\|_{L^2}$$

so that $\|(\lambda I - T_g)^{-1}\| \leq \frac{1}{d}$. Since r is continuous, for each $\varepsilon > 0$, there is an interval I with $\mu(I) > 0$ such that

$$|r(x)| > \frac{1}{d} - \varepsilon \quad \forall \; x \in I.$$

Observe now that $\left|(\lambda I - T_g)^{-1} \chi_I (x)\right| = \left|r(x) \chi_I(x)\right| \geq \left(\frac{1}{d} - \varepsilon\right) \chi_I(x)$ $\forall\, x \in \mathbb{R}$, so that

$$\|(\lambda I - T_g)^{-1} \chi_I\|_{L^2} \geq \left(\tfrac{1}{d} - \varepsilon\right) \|\chi_I\|_{L^2}, \quad \|\chi_I\|_{L^2} \neq 0$$

Since ε is arbitrary, we have $\|(\lambda I - T_g)^{-1}\| \geq \frac{1}{d}$.
We have proved $\|(\lambda I - T_g)^{-1}\| = \frac{1}{d}$, and thus $\lambda \in \rho(T_g)$.

$= \overline{\text{Rang}}$

To prove that $[0,\infty) = \sigma(T_g)$, let $\lambda \in [0,\infty)$, and observe that, by the continuity of g, for each $n = 1, 2, \ldots$, there is an interval I_n with $|I_n| > 0$ such that

$$|\lambda - g(x)| < \frac{1}{n} \quad \text{for} \quad x \in I_n$$

Let us define

$$f_n = \chi_{I_n} \frac{1}{|I_n|^{1/2}}$$

so that $\|f_n\|_{L^2} = 1$. We have

$$\left| [(\lambda I - T_g) f_n](x) \right| = \left| (\lambda - g(x)) f_n(x) \right| \le \frac{1}{n} f_n(x) \quad \forall x \in \mathbb{R}$$

so that

$$\| (\lambda I - T_g) f_n \|_{L^2} \le \frac{1}{n} \| f_n \|_{L^2} = \frac{1}{n}.$$

Thus we see that $\lambda I - T_g$ cannot have a bounded inverse b/c we have demonstrated rather explicitly that <u>$\lambda I - T_g$ is not bounded from below</u> through the sequence f_n:

$$(\lambda I - T_g) f \to 0, \quad \|f_n\| = 1.$$

- We prove that $\lambda = 1$ is the only eigenvalue of T_g and that its multiplicity is infinite.

Let us look at the nullspace of $\lambda I - T_g$. Suppose that $(\lambda - g(x)) f(x) = 0$ a.e. for some $f \in L^2$. If $\lambda \neq 1$, then $\lambda - g(x) = 0$ at no more than one point so that $f = 0$ a.e. and λ is not an eigenvalue. Now suppose that

$$(1 - g(x)) f(x) = 0 \quad \text{a.e.}$$

For $x \notin [-1,1]$, $(1-g(x)) \neq 0$, so $f(x) = 0$ a.e. off $[-1,1]$.
Since $1-g(x) = 0$ for all $x \in [-1,1]$, we have

$$(1-g(x))f(x) = 0 \text{ a.e.}$$

for all f such that $f = 0$ a.e. off $[-1,1]$. Therefore, the eigenspace for $\lambda = 1$ is

$$\mathcal{N}ull(I - T_g) = L^2(-1,1),$$

naturally considered as a subspace of $L^2(\mathbb{R})$. Since $L^2(-1,1)$ is infinite-dimensional, $\lambda = 1$ has infinite multiplicity.

- To prove that $\lambda I - T_g$ has dense range for all $\lambda \neq 1$ (i.e., there is no residual spectrum), one can work with the inverse $(\lambda - g(x))^{-1}$ and examine the domain of the associated multiplication operator. [See the Problem set 2]

- <u>The adjoint of T_g</u>
As in problem 1, set 2, we must show that $\mathcal{D}(T_g^*) = \mathcal{D}(T_{\bar{g}})$.
The main idea in showing $T_g^* = T_{\bar{g}}$ is simply

$$\int (gf)\bar{h} = \int f\,\overline{(\bar{g}h)}.$$

- <u>T_g is normal</u>. One shows that

$$\mathcal{D}(TT^*) := \{ f \in L^2 : \bar{g}f \in \mathcal{D}(T_g) \} = \{ f \in L^2 : |g|^2 f \in L^2 \}$$
$$= \mathcal{D}(T^*T),$$

that this domain is dense in $L^2(\mathbb{R})$, and that, on $\mathcal{D}(TT^*)$, we have

$$T_g T_g^* = T_g^* T_g = T_{|g|^2}.$$

PROBLEM SET 2, due Fri., Feb. 8

1. A study of multiplication operators on $L^2(\mathbb{R})$. This exercise asks you to prove properties of the most general multiplication operators on $L^2(\mathbb{R})$ without appealing to the abstract study of unbounded operators in Hilbert space. Thus you get a concrete feeling for the concepts of spectrum, adjoint, closedness, etc. that is needed in working with the spectral theory in mathematical physics.

Let $g: \mathbb{R} \to \mathbb{C}$ be a Lebesgue-measurable function (denote Lebesgue measure by μ). The essential range of g is

$$ER(g) = \{\lambda \in \mathbb{C} : \forall \varepsilon > 0, \mu\{x : |g(x) - \lambda| < \varepsilon\} > 0\}.$$

Define the operator T_g in $L^2(\mathbb{R}, \mu)$ by

$$\mathcal{D}(T_g) = \{f \in L^2(\mathbb{R}) : gf \in L^2(\mathbb{R})\},$$

$$T_g f = gf \quad \text{for all } f \in \mathcal{D}(T_g).$$

a. Prove that T_g is bounded if and only if $g \in L^\infty(\mathbb{R})$ and that, in this case, $\|T_g\| = \|g\|_\infty$.

b. Prove that $\mathcal{D}(T_g)$ is dence in $L^2(\mathbb{R})$ and that $\mathcal{D}(T_g) = L^2(\mathbb{R})$ if and only if T_g is bounded.

c. Prove that $\sigma(T_g) = ER(g)$ and that $\sigma(T_g)$ is closed in $\mathbb{C}$.

d. For $\lambda \notin \sigma(T_g)$, find an explicit form for the resolvent $(\lambda I - T_g)^{-1}$, and prove $\|(\lambda I - T_g)^{-1}\| = [\text{dist}(\lambda, \sigma(T_g))]^{-1}$

e. Characterize the eigenvalues of T_g in relation to the function g.

[Problem set 2 # 1, cont.]

f. Prove that T_g has no residual spectrum.

g. Prove that $\mathcal{D}(T_g^*) = \mathcal{D}(T_g)$ and that, for $f \in \mathcal{D}(T_g^*)$, $T_g^* f = \bar{g} f$ (that is, $T_g^* = T_{\bar{g}}$). Prove that T_g^* is a closed operator [use the fact that it is the adjoint of an operator; proof by its explicit form is more difficult]. Conclude that T_g is also closed.

h. For $\lambda \in \sigma(T_g)$, prove that there exists a sequence $f_n \in \mathcal{D}(T_g)$ such that $(\lambda I - T_g) f_n \to 0$ and $\|f_n\| = 1$. Construct this sequence as explicitly as possible. [This will show you that you can prove existence of the f_n directly, without appealing to the closedness of T_g and the result of Problem Set 1.]

(i. below)

j. Let E be a μ-measurable subset of $\mathbb{R}$. Prove that the operator $P_E : L^2 \to L^2 :: f \mapsto \chi_E f$ is an orthogonal projection. Find its nullspace and its range. Prove that P_E commutes with T_g.

k. Find a choice of g such that T_g is unitary.
For arbitrary g, find a continuous function of T_g that is unitary
[For $h \in C(\mathbb{R})$, $h(T_g)$ is given by $[h(T_g) f](\lambda) = h(g(\lambda)) f(\lambda)$.

l. Find a choice of g such that T_g is anti-self-adjoint, that is $T_g^* = -T_g$.

i. $\mathcal{D}(AB) := \{ f \in \mathcal{D}(B) : B(f) \in \mathcal{D}(A) \}$ is the natural domain for AB.
Prove that T_g is normal, that is, $T_g T_g^* = T_g^* T_g$ (pay attn to domains).