Continuation of the example of \( T_g : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R}) \) \( \text{if } f \mapsto g f \)
with \( g(x) = \begin{cases} 1, & x < 1 \\ \frac{1}{\sqrt{x^2}}, & 1 \leq x \end{cases} \).

Define \( S = \mathbb{R} \setminus [-1, 1] \) and \( \mathbb{R}_+ = (0, \infty) \).

We have the inclusions
\[
L^2(S, dx) \subset L^2(\mathbb{R}, dx)
\]
\[
L^2([1, \infty), dx) \subset L^2(\mathbb{R}_+, dx)
\]
defined through extension by zero, and we obtain
\[
L^2(\mathbb{R}, dx) \cong L^2(S, dx) \oplus L^2([1, \infty), dx)
\]

\( T_g \) acts on \( L^2(S, dx) \) by multiplication by \( g \) and on \( L^2([1, \infty), dx) \) as the identity. Note that, since \( g \) restricted to \( S \) is injective, \( T_g \) restricted to \( L^2(S, dx) \) has no eigenvalues.

This motivates the maps
\[
\Phi : S \to \mathbb{R}_+ : x \mapsto g(x)
\]
\[
\Phi : \mathbb{R}_+ \to S : y \mapsto \begin{cases} \frac{1}{\sqrt{y}}, & 0 < y \leq 1 \\ -\sqrt{y}, & 1 < y \end{cases}
\]
which are inverse to each other:
\[
\Phi \circ \Phi = \text{Id}_S, \quad \Phi \circ \Phi = \text{Id}_{\mathbb{R}_+}.
\]

Therefore, the pair \( \Phi, \Phi \) induces a 1-1 correspondence between functions on \( S \) and functions on \( \mathbb{R}_+ \).
Our basic observation is that, if we put \( g(y) = |X'(y)| d\mu_y \), \( g\) is a Borel measure on \( \mathbb{R}_+ \), and we have

\[
\begin{align*}
\int_{\mathbb{R}_+} f(x) g(x) \, dx &= \int_{\mathbb{R}_+} f(\xi(y)) g(\xi(y)) \, d\mu_{\xi(y)} \\
\int_{\mathbb{S}} h(\xi(x)) \xi(\xi(x)) \, dx &= \int_{\mathbb{S}} h(y) I(y) \, d\mu_{\xi(y)}
\end{align*}
\]

whenever these integrals are defined.

Based on this observation, one shows that there is a unitary operator \( U \)

\[
U : L^2(\mathbb{R}_+, \mathcal{A}_{\mu_y}) \to L^2(\mathbb{S}, \mathcal{M}_{\mu_{\xi(y)}}) : h \mapsto h \circ \xi
\]

\[
U^* : L^2(\mathbb{S}, \mathcal{M}_{\mu_{\xi(y)}}) \to L^2(\mathbb{R}_+, \mathcal{A}_{\mu_y}) : f \mapsto f \circ \xi
\]

The point of this construction is that \( T_g \) carries over to multiplication by the scalar but variable \( y \) on \( L^2(\mathbb{R}_+, \mathcal{A}_{\mu_y}) \):

\[
\hat{T}_g = U^* T_g U
\]

\[
\hat{T}_g h = U^* T_g (h \circ \xi) = U^* (g \circ (h \circ \xi)) = [g (h \circ \xi)] \circ \xi
\]

\[
= (g \circ \xi) (h \circ \xi (y)) = I_{\mathbb{S}} \cdot h
\]

\[
\Rightarrow (\hat{T}_g h)(y) = g(y) h(y)
\]

Now, let's pick a positive function \( \phi : \mathbb{R}_+ \to \mathbb{R} \) such that

\[
\int \phi(y) d\mu_y < \infty
\]

(\text{This is possible -- prove it.})
Now, if we define
\[ \phi_{g}(y) = g(y) \phi(y) \]
\[ (Vf)(y) = \phi(y)^{1/2} f(y) \]
we obtain a unitary operator
\[ V : L^{2}(\mathbb{R}_{+}, d\nu) \rightarrow L^{2}(\mathbb{R}_{+}, d\mu) \]

to wit:
\[ \int_{\mathbb{R}_{+}} (Vf)(y) \overline{g(y)} d\mu(y) = \int_{\mathbb{R}_{+}} f(y) \overline{g(y)} d\nu(y) \]

We see also that \( \tilde{T}_{g} \) is still represented by multiplication by \( y \):
\[ \tilde{T}_{g} = V^{-1} \tilde{T}_{g} V = V^{-1} U^{-1} \tilde{T}_{g} UV : L^{2}(\mathbb{R}_{+}, d\nu) \rightarrow L^{2}(\mathbb{R}_{+}, d\nu) \]
\[ (\tilde{T}_{g}f)(y) = y f(y) \]

Now we have an isomorphism induced by \( UV \):
\[ L^{2}(\mathbb{R}, d\nu) \cong L^{2}(\mathbb{R}_{+}, d\nu) \oplus L^{2}(I, d\alpha) \]

and \( T_{g} \) goes over to the action of multiplication by
the independent variable in the first component and the identity operator on the second:
\[ (f(x), g(x)) \mapsto (y f(x), g(x)) \]

Now think about how to write the second component \((\text{Id} \circ \tilde{T}_{g}) \) in a "standard" spectral representation, where \( T_{g} \) becomes multiplied by the independent variable. \( (\tilde{T}_{g} f)(y) \) has infinite multiplicity.