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Continuation of the example of $T_g : \mathcal{D}(T_g) \to L^2(\mathbb{R}) :: f \to gf$ with
$$g(x) = \begin{cases} x^2 , & x \le -1 , \\ 1 , & -1 \le x \le 1 , \\ 1/x^2 , & 1 \le x . \end{cases}$$

Define $S = \mathbb{R} \setminus [-1,1)$ and $\mathbb{R}_+ = (0,\infty)$.
We have the inclusions
$$L^2(S,dx) \hookrightarrow L^2(\mathbb{R},dx)$$
$$L^2(I,dx) \hookrightarrow L^2(\mathbb{R},dx)$$
defined through extension by zero, and we obtain
$$L^2(\mathbb{R},dx) \cong L^2(S,dx) \oplus L^2(I,dx) .$$
T_g acts on $L^2(S,dx)$ by multiplication by g and on $L^2(I,dx)$ as the identity. Notice that, since g restricted to S is injective, T_g restricted to $L^2(S,dx)$ has no eigenvalues.
This motivates the maps
$$\Psi : S \to \mathbb{R}_+ :: x \longmapsto g(x)$$
$$X : \mathbb{R}_+ \to S :: y \longmapsto \begin{cases} 1/\sqrt{y} , & 0 < y \le 1 \\ -\sqrt{y} , & 1 < y \end{cases} ,$$
which are inverse to each other:
$$X \circ \Psi = \mathrm{Id}_S , \quad \Psi \circ X = \mathrm{Id}_{\mathbb{R}_+} .$$

Therefore the pair X, Ψ induces a 1-1 correspondence between functions on S and functions on $\mathbb{R}_+$

Our basic observation is that, if we put $d\mu(y) = |X'(y)|\,dy$, $d\mu$ is a Borel measure on $\mathbb{R}_+$, and we have

$$\begin{cases} \int_S f(x)\,\bar{g}(x)\,dx = \int_{\mathbb{R}_+} f(X(y))\,\bar{g}(X(y))\,d\mu(y) \\ \int_S h(X(x))\,\bar{k}(X(x))\,dx = \int_{\mathbb{R}_+} h(y)\,\bar{k}(y)\,d\mu(y) \end{cases}$$

whenever these integrals are defined.

Based on this observation, one shows that there is a unitary operator $\mathcal{U}$

$$\mathcal{U} : L^2(\mathbb{R}_+, d\mu(y)) \to L^2(S, dx) :: h \mapsto h \circ X$$

$$\mathcal{U}^{-1} : L^2(S, dx) \to L^2(\mathbb{R}_+, d\mu(y)) :: f \mapsto f \circ X$$

The point of this construction is that T_g carries over to multiplication by the independent variable y on $L^2(\mathbb{R}_+, d\mu(y))$:

$$\hat{T}_g = \mathcal{U}^{-1} T_g \mathcal{U}$$

$$\hat{T}_g h = \mathcal{U}^{-1} T_g (h \circ X) = \mathcal{U}^{-1}(g\,(h \circ X)) = [g\,(h \circ Y)] \circ X$$

$$= (g \circ X)(h \circ Y \circ X) = Id_S \cdot h$$

$$\Rightarrow (\hat{T}_g h)(y) = y\,h(y).$$

Now let's pick a positive function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\int \phi(y)\,d\mu(y) < \infty$$

(This is possible — prove it.)

Now, if we define

$$d\nu(y) = \phi(y)\,d\mu(y)$$
$$(Vf)(y) = \phi(y)^{1/2} f(y)$$

we obtain a <u>unitary</u> operator

$$V : L^2(\mathbb{R}_+, d\nu) \to L^2(\mathbb{R}_+, d\mu) ;$$

to wit: $\displaystyle\int_{\mathbb{R}_+} (Vf)(y)\overline{(Vg)(y)}\,d\mu(y) = \int_{\mathbb{R}_+} f(y)\overline{g(y)}\,d\nu(y)$.

We see also that $\hat{T}_g$ is still represented by multiplication by y:

$$\tilde{\hat{T}}_g = V^{-1}\hat{T}_g V = V^{-1}U^{-1}T_g U V : L^2(\mathbb{R}_+, d\nu) \to L^2(\mathbb{R}_+, d\nu),$$
$$(\tilde{\hat{T}}_g f)(y) = y f(y) .$$

Now we have an isomorphism mediated by UV:

$$L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}_+, d\nu) \oplus L^2(I, dx) ,$$

and T_g goes over to the action of <u>multiplication by the independent variable</u> in the first component and the identity operator on the second:

$$(f(y), g(x)) \longrightarrow (y f(y), g(x)) .$$

Now think about how to write the second component (Id on $L^2(I)$) in a "standard" spectral representation, where T_g becomes mult. by the independent var. ($\lambda = 1$ has infinite multiplicity.)