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Compact operators, or "completely continuous" operators.

References: Naylor/Sell §5.24 (general theory)
§6.10 - 6.11 (spectral theory)
§6.13 (applications)
R. Kress §2.4
Reed/Simon §VI.5
Akhiezer/Glazman § Ch. V
Riesz / Sz-Nagy §85

There are several equivalent definitions of a compact, or completely continuous, operator, as you will see by consulting the references cited above (or many others). We'll use one of the most common and give two equivalent definitions as theorems.

Defn Let X and Y be normed linear spaces and $T: X \to Y$ a linear operator. T is compact if, for each bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X, $\{Tx_n\}_{n=1}^{\infty}$ admits a convergent subsequence in Y.

Theorem $T: X \to Y$ is compact if and only if, for each bounded subset S of X, the closure of $T[S]$ is compact in Y.

Before giving any further theorems on compact operators, we give an example that illustrates the most general form of a (nonsingular) normal compact operator in Hilbert space.

Example 2

Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{C}$ that converges to 0 as $i \to \infty$ and such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Define the measure μ by

$$\mu = \sum_{i=1}^{\infty} \delta(\lambda - \lambda_i),$$

or, equivalently,

$$\mu(E) = \left| E \cap \{\lambda_i\}_{i=1}^{\infty} \right| \quad \text{for } E \subset \mathbb{C}.$$

We will work in the Hilbert space $\mathcal{H} = L^2(\mathbb{C}, d\mu)$.

Notice that, if $f \in L^1(\mathbb{C}, d\mu)$, then

$$\int f \, d\mu = \sum_{i=1}^{\infty} f(\lambda_i).$$

We see that any function f is equivalent, with respect to the measure μ, to a function supported on the set $\{\lambda_i\}_{i=1}^{\infty}$. We also have

$$f \in \mathcal{H} \iff \sum_{i=1}^{\infty} |f(\lambda_i)|^2 < \infty,$$

$$f, g \in \mathcal{H} \implies \int f \bar{g} \, d\mu = \sum_{i=1}^{\infty} f(\lambda_i) \overline{g(\lambda_i)},$$

and from this we conclude that there is a unitary operator from $\mathcal{H}$ to ℓ^2:

$$\mathcal{H} \to \ell^2 :: f \longmapsto \{f(\lambda_i)\}_{i=1}^{\infty}.$$

Let $T : \mathcal{H} \to \mathcal{H}$ be defined by

$$(Tf)(\lambda_i) = \lambda_i f(\lambda_i) \quad , \; i = 1 \ldots \infty .$$

This rule defines T well because $f \in \mathcal{H}$ is characterized uniquely by its values at the λ_i. We also have

$$\mathcal{D}(T) = \mathcal{H}$$

because $\sum_{i=1}^{\infty} |\lambda_i f(\lambda_i)|^2 \leq \max_{i=1}^{\infty}\{|\lambda_i|^2\} \cdot \sum_{i=1}^{\infty} |f(\lambda_i)|^2 < \infty$

if $f \in \mathcal{H}$. You can prove the following fact by using ideas analogous to those in Problem Set 2:

Fact
- T is bounded, with $\|T\| = \max\{|\lambda_i|\}$.
- $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty} \cup \{0\}$ and $\sigma_p(T) = \{\lambda_i\}_{i=1}^{\infty}$.
- T is normal, with T^* defined by $(Tf)(\lambda_i) = \overline{\lambda_i} f(\lambda_i)$
- T is self-adjoint if and only if $\lambda_i \in \mathbb{R} \; \forall \, i$.

The following theorem can be proved by applying abstract theory of compact operators, but it is useful to prove it directly, using the definition of compactness.

Theorem T is compact. (as defined above)

In the proof, if $\{f_n\}_{n=1}^{\infty}$ and $\{g_m\}_{m=1}^{\infty}$ are sequences, we use the notation $\{g_m\} \subset \{f_n\}$ to express the statement that "$\{g_m\}$ is a subsequence of $\{f_n\}$." $\mathbb{N} = \{1, 2, 3, \ldots\}$.

<u>Proof</u> Define $T_j : \mathcal{H} \to \mathcal{H}$ by $(T_j f)(\lambda_i) = \begin{cases} \lambda_i f(\lambda_i), & i \le j \\ 0, & j < i \end{cases}$

for each $j \in \mathbb{N}$. Observe that, for each bounded sequence $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{H}$, $\{T_j g_n\}_{n=1}^{\infty}$ is a bounded sequence that is contained in a finite-dimensional subspace of $\mathcal{H}$, and therefore there is a subsequence $\{h_m\}_{m=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that $\{T_j h_m\}_{m=1}^{\infty}$ converges.

Now let $\{f_n^{(0)}\}_{n=1}^{\infty}$ be a bounded sequence from $\mathcal{H}$, say $\sum_{\lambda=1}^{\infty} |f_n^{(0)}(\lambda_i)|^2 < M$. We shall prove that $\{Tf_n^{(0)}\}_{n=1}^{\infty}$ has a convergent subsequence. By the remark above and the principle of induction, we obtain a sequence of sequences $\{f_n^{(i)}\}_{n=1}^{\infty}$, $i \in \mathbb{N}$, with $\{f_n^{(i)}\}_{n=1}^{\infty} \subset \{f_n^{(j)}\}_{n=1}^{\infty}$ for $i > j$ and such that $\{T_i f_n^{(i)}\}_{n=1}^{\infty}$ converges. Define $\{f_n^{\infty}\}_{n=1}^{\infty}$ to be the "diagonal" of $f_n^{(i)}$, that is $f_n^{\infty} = f_n^{(n)}$. Since $\{f_n^{\infty}\}_{n=1}^{\infty} \subset \{f_n^{(i)}\}_{n=1}^{\infty}$ for each $i \in \mathbb{N}$, we see that $\{T_i f_n^{\infty}\}_{n=1}^{\infty}$ converges for each i.

Let $\varepsilon > 0$ be given. By the convergence of $\{\lambda_i\}_{i=1}^{\infty}$ to 0, $\exists\, I$ such that $i \ge I \Rightarrow |\lambda_i| < \frac{\varepsilon}{3M}$. By the convergence of $\{T_I f_n^{\infty}\}_{n=1}^{\infty}$, $\exists\, M \ge I$ such that

$$\|T_I f_n^{\infty} - T_I f_m^{\infty}\| < \frac{\varepsilon}{3} \quad \text{for } n, m > M$$

By definition of T_I and T, we have

$$[(T - T_I) f](\lambda_i) = \begin{cases} 0, & i \le j \\ \lambda_i f(\lambda_i), & j < i, \end{cases}$$

and hence, for all n,

$$\|Tf_n^{\infty} - T_I f_n^{\infty}\| \le \max_{i \ge I} |\lambda_i| \, \|f_n^{\infty}\| \le \frac{\varepsilon}{3M} M = \frac{\varepsilon}{3}.$$

Finally, we obtain, for $n, m \geq I$,

$$\|Tf_n^\infty - Tf_m^\infty\| \leq \|Tf_n^\infty - T_I f_n^\infty\| + \|T_I f_n^\infty - T_I f_m^\infty\| + \|T_I f_m^\infty - Tf_m^\infty\|$$

$$\leq \varepsilon .$$

This proves that $\{Tf_n^\infty\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$, and since $\mathcal{H}$ is complete, this subsequence of $\{Tf_n^{(0)}\}_{n=1}^\infty$ converges.

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We will state a theorem to the effect that <u>all</u> normal compact operators are unitarily equivalent to operators essentially like that of Example 2, except that the eigenspaces for $|\lambda_i| > 0$ may be of arbitrary finite dimension and 0 may also be an eigenvalue.

First, we state some basic theorems on compact operators:

<u>Thm</u> Each compact operator is bounded.

<u>Thm</u> If $A: X \to Y$ and $B: Y \to Z$ are bounded linear transformations and A or B is compact, then $BA: A \to Z$ is compact.

<u>Thm</u> Let $\mathcal{H}$ be a Hilbert space. A linear operator $T: \mathcal{H} \to \mathcal{H}$ is compact if and only if there exists a sequence $\{T_j\}_{j=1}^\infty$ of finite-rank operators ($\dim(\text{Range } T_j) < \infty$) such that $\|T_j - T\| \to 0$ as $j \to \infty$.
[See Riesz/Sz-Nagy, Thm. in §85 or Reed/Simon Thm VI.13]

Spectral Theory for Compact Operators (on a separable Hilbert space)

The Fredholm Alternative (for compact operators)
If $T: \mathcal{H} \to \mathcal{H}$ is compact and $\lambda \neq 0$, then exactly one of the following holds:

- λ is an eigenvalue of T $\quad (\lambda \in \sigma_p(T))$
- $\lambda I - T$ has a bounded inverse $\quad (\lambda \in \rho(T))$

Theorem Let $T: \mathcal{H} \to \mathcal{H}$ be compact. Then $\sigma(T)$ is a countable set that has no nonzero limit point. Each nonzero element of $\sigma(T)$ is an eigenvalue of T with finite-dimensional eigenspace.

Theorem (Spectral Theorem for compact operators)
Let $T: \mathcal{H} \to \mathcal{H}$ be compact and normal. Then there exists a complete orthonormal Hilbert-space basis $\{\varphi_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty}$ from $\mathbb{C}$ such that $\lambda_n \to 0$ as $n \to \infty$ and $T\varphi_n = \lambda_n \varphi_n$.

This theorem tells us that every normal self-adjoint operator on a separable Hilbert space is unitarily equivalent to a direct sum of spectral representations of the type we have seen in Example 2.

!! Check the website for Problem Set 3, due Fri., Feb. 15.