Problem Set 3 Due Fri., Feb. 15, 2008

1. Concerning the operator T defined in Example 2 on page 34, prove the following directly, that is, without appealing to abstract theory of compact operators.

   (a) T is bounded, with $\|T\| = \max_{i=1}^{\infty} |\lambda_i|$,
   (b) $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty} \cup \{0\}$ and $\sigma_p(T) = \{\lambda_i\}_{i=1}^{\infty}$
   (c) T is normal, with T^* defined through $(T^*f)(\lambda_i) = \bar{\lambda}_i f(\lambda_i)$,
   (d) T is self-adjoint if and only if $\lambda_i \in \mathbb{R}\ \forall\, i$.

2. Define the right-shift operator S on $L^2(\mathbb{R}_+, dx)$ by $(\mathbb{R}_+ = (0,\infty))$

$$(Sf)(x) = \begin{cases} 0, & x \leq 1 \\ f(x-1), & 1 < x \end{cases}$$

   (a) Prove that S is an isometry but that it is not unitary.
   (b) Find an explicit expression for the adjoint of S.
   (c) Find $\sigma(S)$ and $\sigma(S^*)$. Distinguish between point spectrum, residual spectrum, and continuous spectrum.
   (b) Find an explicit expression for the resolvent of S at points of $\rho(S)$.