Problem Set 3  Due Fri., Feb. 15, 2008

1. Concerning the operator $T$ defined in Example 2 on page 34, prove the following directly, that is, without appealing to abstract theory of compact operators.

(a) $T$ is bounded, with $\|T\| = \max \{\|T \phi\| : \phi \neq 0\}$.

(b) $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \neq 0\}$ and $\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda \neq 0\}$.

(c) $T$ is normal, with $T^*$ defined through $(Tf)(x) = \overline{x}, f(x)$.

(d) $T$ is self-adjoint if and only if $\lambda \in \mathbb{R}$.

2. Define the right-shift operator $S$ on $L^2(\mathbb{R}, dx)$ by

$$ (Sf)(x) = \begin{cases} 0, & x \leq 1 \\ f(x-1), & 1 < x \end{cases} $$

(a) Prove that $S$ is an isometry but that it is not unitary.

(b) Find an explicit expression for the adjoint of $S$.

(c) Find $\sigma(S)$ and $\sigma(S^*)$. Distinguish between point spectrum, residual spectrum, and continuous spectrum.

(b) Find an explicit expression for the resolvent of $S$ at points of $\rho(S)$.