Since relation (4) is symmetric in T and T^*, we have

$$(T^*)^* = T.$$

The transformations I and O — the identity and the transformation carrying every element into the element 0 — coincide with their adjoints: $I^* = I$, $O^* = O$.

The following relations are obvious consequences of the definition of adjoint transformation:

$$(aT)^* = \bar{a}T^*, \quad (T_1 + T_2)^* = T_1^* + T_2^*, \quad (T_1T_2)^* = T_2^*T_1^*.$$

It follows from the equality of the norms (5) that $T_n \Rightarrow T$ implies $T_n^* \Rightarrow T^*$. On the other hand, it is obvious that $T_n \rightharpoonup T$ implies $T_n^* \rightharpoonup T$. However, in general $T_n \to T$ does *not* imply that $T_n^* \to T^*$.[3]

When T possesses an *inverse*, that is, when there is a linear transformation T^{-1} such that $T^{-1}T = TT^{-1} = I$, we have

$$T^*(T^{-1})^* = (T^{-1})^*T^* = I^* = I;$$

therefore T* also possesses an inverse and

$$(T^*)^{-1} = (T^{-1})^*.$$

The first member of relation (4) is a *bilinear form* of the two variables f and g, which means that it possesses the following properties [denoting it for the moment by $(f|g)$]:

$$(f_1 + f_2|g) = (f_1|g) + (f_2|g), \quad (f|g_1 + g_2) = (f|g_1) + (f|g_2),$$
$$(af|g) = a(f|g), \quad (f|ag) = \bar{a}(f|g);$$

furthermore, it is *bounded*, that is,

$$|(f|g)| \leq M \,\|f\|\, \|g\|,$$

and the smallest possible constant M is obviously equal to $\|T\|$. Conversely, *to every bounded bilinear form* $(f|g)$ *there corresponds a linear transformation* T *such that*

$$(f|g) = (Tf, g),$$

which we see by an argument analogous to that we used in the construction of the adjoint transformation.

[3] Let T_n be the linear transformation of the space l^2 defined by $T_n\{x_1, x_2, \ldots\} = \{x_{n+1}, x_{n+2}, \ldots\}$. Then we have $T^*\{x_1, x_2, \ldots\} = \{0, \ldots, 0, x_1, x_2, \ldots\}$, where there are n zeros before x_1. It is easy to see that, for every vector x, $T_n x \to 0$, while $\|T_n^* x\| = \|x\|$.

85. Completely Continuous Linear Transformations

Let us consider now the equations

$$f - Kf = g \quad \text{and} \quad f - K^*f = g,$$

which correspond, in the abstract Hilbert space $\mathfrak{H}$, to our integral equations with adjoint kernels in L^2; of course, it is no longer a question of either integrals or kernels, since K and K^* are two arbitrary linear transformations of the space $\mathfrak{H}$, adjoint to one another in the sense defined in the preceding section.

When the transformation K is of *finite rank*,

$$Kf = \sum_{i=1}^{r} (f, \psi_i)\varphi_i,$$

we have

$$K^*f = \sum_{i=1}^{r} (f, \varphi_i)\psi_i,$$

and the problem of our equations reduces to a problem of linear algebra, just as in the case of integral equations; see Sec. 70. We remark that, as we observed with regard to a particular problem in Sec. 78, the linear transformations of finite rank of the space $\mathfrak{H}$ can be defined as those which transform the entire space $\mathfrak{H}$ into a subspace of finite dimension.

For a linear transformation K of the most general type our methods do not suffice, which is not astonishing since in general the Fredholm alternative is no longer valid. In fact, consider the following linear transformation of the space L^2 of functions defined on the interval (0, 1):

$$Kf(x) = (1 - x)f(x);$$

this transformation is equal to its adjoint. But the equation $(I - K)f(x) = g(x)$, that is, the equation $xf(x) = g(x)$, cannot be solved in L^2 for all given functions $g(x)$ belonging to L^2, despite the fact that the homogeneous adjoint equation $xf(x) = 0$ possesses only the single solution $f(x) = 0$.

However, if we restrict ourselves to the consideration of *completely continuous* linear transformations K, the method given in sections 77 to 80 for the space L^2 applies word for word to the abstract space $\mathfrak{H}$. We prove in particular the decomposition theorem (Sec. 78) and then, applying it, the Fredholm alternative (Sec. 73). Of course, we no longer pass from the equations $KK_1 = K_1K = K_1 - K$ to the equations $K_1^*K^* = K^*K_1^* = K_1^* - K^*$ by means of kernels (which no longer has meaning), but directly by basing our arguments on the definition and on the properties of adjoint transformations given in Sec. 84.

Instead of this "geometric" method, we can also use the "analytic" method of sections 71 and 72, at least for linear transformations K which can be approximated arbitrarily closely, in norm, by transformations of finite rank. Now this is true for all completely continuous linear transformations K. In fact, assuming at first that we are dealing with a separable Hilbert space and that $\{\varphi_k\}$ is a complete orthonormal sequence in it, we have the following theorem:

THEOREM. *If K is a completely continuous linear transformation, the "reduced" transformations K_n, defined by*

$$K_n f = \sum_{i,j=1}^{n} (f, \varphi_i)\,(K\varphi_i, \varphi_j)\varphi_j,$$

tend uniformly to the transformation K when $n \to \infty$.

We observe first that the transformation K_n can be written in the form $K_n = P_n K P_n$, where P_n denotes the orthogonal projection onto the subspace determined by $\varphi_1, \varphi_2, \ldots, \varphi_n$, that is

$$P_n f = \sum_{i=1}^{n} (f, \varphi_i)\varphi_i.$$

We note the following relations which we shall use:

$$P_n^* = P_n, \quad \|P_n\| \leq 1, \quad \|I - P_n\| \leq 1, \quad P_n \to I.$$

If the theorem were not true, that is, if $\|K - K_n\|$ did not tend to 0 with $1/n$, we could choose a sequence of elements f_n such that

(6) $$\|f_n\| = 1 \text{ and } \|(K - K_n)f_n\| \geq q,$$

where q is some positive quantity independent of n. In view of the complete continuity of K, we can assume without loss of generality that the sequences $\{Kf_n\}$ and $\{K(I - P_n)f_n\}$ are convergent; denote their limits by g and h respectively. Consider the decomposition

$$K - K_n = K - P_n K P_n = (I - P_n)K + P_n K(I - P_n);$$

we have

$$\|(I-P_n)Kf_n\| \leq \|(I-P_n)(Kf_n-g)\| + \|(I-P_n)g\| \leq \|Kf_n-g\| + \|g-P_n g\| \to 0,$$

$$\|P_n K(I-P_n)f_n\| \leq K(I-P_n)f_n\| \to \|h\|,$$

$$(h, h) = \lim_n (K(I-P_n)f_n, h) = \lim_n (f_n, (I-P_n)K^*h) \geq \lim_n \|(I-P_n)K^*h\| = 0,$$

and consequently

$$\|(K - K_n)f_n\| \to 0,$$

which is a contradiction of (6). Hence the theorem is proved.

We could just as well have used the transformations KP_n or $P_n K$ instead of $K_n = P_n K P_n$. The advantage of K_n is that it transforms the subspace determined by $\varphi_1, \varphi_2, \ldots, \varphi_n$ into itself and is zero on the orthogonal complement; hence it is essentially a transformation of an n-dimensional space into itself.

The case of a completely continuous linear transformation K of a *non-separable* space $\mathfrak{H}$ reduces to the preceding case by virtue of the fact that K is essentially the transformation of a separable subspace, that is, *there exists in $\mathfrak{H}$ a separable subspace $\mathfrak{H}_0$ which is transformed by K into itself and whose orthogonal complement is transformed by K into the element* 0.

In fact, since the transformation $A = K^*K$ is completely continuous and has the property that $A^* = A$, there exists, as we shall see in the following chapter, a sequence of elements $\varphi_1, \varphi_2, \ldots$ (the characteristic elements of A corresponding to non-zero characteristic values), such that for every element f orthogonal to all the φ_n we have $Af = 0$, and consequently

$$(Kf, Kf) = (K^*Kf, f) = (Af, f) = 0, \quad Kf = 0.$$

The denumerable set of elements

$$K^m \varphi_n \qquad (n = 1, 2, \ldots; m = 0, 1, \ldots)$$

then determines a separable subspace $\mathfrak{H}_0$ of $\mathfrak{H}$ which obviously is transformed by K into itself; since the orthogonal complement of $\mathfrak{H}_0$ is, in particular, orthogonal to all the φ_n, it is transformed by K into the element 0.

The "analytic" method has the further advantage that it can also be applied, at least partially, to linear transformations K which *are not completely continuous*. Let us define the *Fredholm radius* of the linear transformation K to be the least upper bound Ω of the values $\omega > 0$ for which there exists a linear transformation of finite rank L such that

$$\|K - L\| \leq \frac{1}{\omega}.$$

As we have just seen, $\Omega = \infty$ for completely continuous linear transformations. Choosing $L = 0$, we see that for every transformation

$$\Omega \geq \frac{1}{\|K\|}.$$

According to sections 71—73, it follows that the resolvent transformation K_λ behaves in the interior of the circle

$$|\lambda| = \Omega$$

exactly as if K were completely continuous: hence it has only polar singularities which cannot have an accumulation point in the interior of this circle, and the Fredholm alternative holds for the functional equations

$$f - \lambda K f = g, \; f' - \bar{\lambda} K^* f' = g'.$$

*

Let us mention several other variants of the definition of *complete continuity*. Recall the definition given in Sec. 76:[4]

DEFINITION 1. *A linear transformation K is said to be completely continuous if it transforms every infinite and bounded set into a compact set, that is, if for every infinite sequence of elements f_n such that $\|f_n\| \leq C$, the sequence $\{Kf_n\}$ contains a subsequence which converges in the strong sense to an element of the space $\mathfrak{H}$.*

Originally, HILBERT introduced the notion of complete continuity for numerical-valued functions $F(f, g, \ldots, v)$, where $f, g, \ldots, v$ are variable elements of the space $\mathfrak{H}$; the function F is completely continuous if

$$F(f_n, g_n, \ldots, v_n) \to F(f, g, \ldots, v)$$

when the elements $f_n, g_n, \ldots, v_n$ tend *weakly* to the elements $f, g, \ldots, v$.[5]

With the aid of this notion we can define complete continuity for a linear transformation in the following manner:

DEFINITION 2. *A linear transformation K is said to be completely continuous if the bilinear form $(f|g) = (Kf, g)$ is a weakly continuous function of the elements f and g: $f_n \rightharpoonup f$, $g_n \rightharpoonup g$ imply $(f_n|g_n) \to (f|g)$.*

Two other definitions, which are more convenient in certain applications, are:

DEFINITION 3. *A linear transformation K is said to be completely continuous if it transforms every weakly convergent sequence of elements into a strongly convergent sequence, that is, if*

$$f_n \rightharpoonup f \quad \textit{implies} \quad Kf_n \to Kf.$$

DEFINITION 4. *A linear transformation K is said to be completely continuous if from every bounded infinite sequence of elements we can select a subsequence $\{f_n\}$ for which*

$$(f_n - f_m|f_n - f_m) = (K(f_n - f_m), f_n - f_m) \to 0 \text{ for } m, n \to \infty.$$

We shall show that all these definitions are equivalent.

$1 \to 2$. We assume that K is completely continuous according to definition 1, and show that then $f_n \rightharpoonup f$, $g_n \rightharpoonup g$ imply $(Kf_n, g_n) \to (Kf, g)$. If this were not the case, there would be a positive quantity q such that

$$|(Kf_n, g_n) - (Kf, g)| \geq q$$

for an infinite number of indices $n = n_1, n_2, \ldots$. Since the sequence $\{f_n\}$ is weakly convergent and hence bounded, we can also require, without loss of generality, that the sequence $\{Kf_{n_k}\}$ be convergent in the strong sense. On the other hand, $f_n \rightharpoonup f$ implies $Kf_n \rightharpoonup Kf$, since

$$(Kf_n, h) = (f_n, K^*h) \to (f, K^*h) = (Kf, h),$$

for every element h; hence we necessarily have $Kf_{n_k} \to Kf$. From this it follows that

$$|(Kf_{n_k}, g_{n_k}) - (Kf, g)| = |(Kf_{n_k} - Kf, g_{n_k}) + (Kf, g_{n_k} - g)| \leq$$
$$\leq \|Kf_{n_k} - Kf\| \, \|g_{n_k}\| + |(Kf, g_{n_k} - g)| \to 0$$

for $k \to \infty$. Since on the other hand the first member is $\geq q > 0$, we have encountered a contradiction. Therefore K is also completely continuous in the sense of Definition 2.

$2 \to 3$. We have seen that $f_n \rightharpoonup f$ implies $Kf_n \rightharpoonup Kf$ for every linear transformation K. When in addition K is completely continuous in the sense of definition 2 we have that $h_n = f_n - f \rightharpoonup 0$ and $g_n = Kf_n - Kf \rightharpoonup 0$ imply

$$\|Kf_n - Kf\|^2 = (Kh_n, g_n) \to 0;$$

hence

$$Kf_n \to Kf,$$

which proves that K is also completely continuous in the sense of Definition 3.

$3 \to 4$. Let $\{h_n\}$ be a bounded infinite sequence of elements of $\mathfrak{H}$, $\|h_n\| \leq C$. By virtue of the theorem of choice proved in sections 32 and 35, we can select a weakly convergent subsequence $\{f_n\}$ from the sequence $\{h_n\}$. Since the transformation K is assumed completely continuous in the sense of Definition 3, the sequence $\{Kf_n\}$ will be strongly convergent and hence

$$|(K(f_n - f_m), f_n - f_m)| \leq \|Kf_n - Kf_m\| \cdot 2C \to 0$$

when $m, n \to \infty$. Therefore K is also completely continuous in the sense of Definition 4.

$4 \to 1$. We assume that the linear transformation K is completely continuous in the sense of Definition 4. Let $\{h_n\}$ be a bounded infinite sequence of elements of $\mathfrak{H}$; the sequences

$$h_{1n} = h_n + Kh_n, \quad h_{2n} = h_n - Kh_n, \quad h_{3n} = h_n + iKh_n, \quad h_{4n} = h_n - iKh_n$$

are then also bounded. Hence we can determine a sequence of integers $\{n_k\}$ such that, denoting h_{n_k} by f_k and h_{rn_k} by f_{rk} $(r = 1, 2, 3, 4)$, we have

$$(K(f_{rk} - f_{rj}), f_{rk} - f_{rj}) \to 0 \qquad (j, k \to \infty;\ r = 1, 2, 3, 4).$$

Inasmuch as

$$(K(f_{1k} - f_{1j}), f_{1k} - f_{1j}) - (K(f_{2k} - f_{2j}), f_{2k} - f_{2j}) +$$
$$+ i(K(f_{3k} - f_{3j}), f_{3k} - f_{3j}) - i(K(f_{4k} - f_{4j}), f_{4k} - f_{4j}) =$$
$$= 4(K(f_k - f_j), K(f_k - f_j)),$$

[4] Cf. F. RIESZ [9] (p. 74).

[5] HILBERT [*] (Note 4) and RIESZ [*] (p. 96).

we shall have

$$\|Kf_k - Kf_j\| \to 0 \quad \text{when} \quad j, k \to \infty;$$

therefore the sequence $\{Kf_k\}$ is (strongly) convergent. The transformation K is consequently also completely continuous in the sense of Definition 1.

This completes the proof of the equivalence of the four definitions.

86. Biorthogonal Sequences. A Theorem of Paley and Wiener

We say that the sequences $\{f_n\}$, $\{g_n\}$ of elements of the Hilbert space $\mathfrak{H}$ form a *normalized biorthogonal system* if

$$(f_n, g_m) = 0 \text{ for } n \neq m \text{ and } (f_n, g_n) = 1.$$

This biorthogonal system is said to be *complete* if each of the systems $\{f_n\}$, $\{g_n\}$ is complete in $\mathfrak{H}$, that is, if the linear combinations of the f_n, as well as those of the g_n, are everywhere dense in $\mathfrak{H}$. Then we have the biorthogonal developments

$$f = \sum_{n=1}^{\infty} (f, g_n) f_n, \qquad f = \sum_{n=1}^{\infty} (f, f_n) g_n,$$

valid whenever the series in the second member converges. It clearly suffices to consider the first series. Denoting its sum by f', we shall have

$$(f', g_m) = \sum_{n=1}^{\infty} (f, g_n)(f_n, g_m) = (f, g_m),$$

hence

$$(f' - f, g_m) = 0 \qquad (m = 1, 2, \ldots),$$

which implies that $f' - f = 0$, $f' = f$.

The following theorem is very useful in the theories of various different series of functions. Its proof will be based on the fact that, for every linear transformation K such that $\|K\| < 1$, the transformation $(I - K)^{-1}$ exists.

THEOREM.[6] *Assume that the sequence $\{f_n\}$ differs only slightly from the complete orthonormal sequence $\{\varphi_n\}$, in the sense that there exists a constant θ, $0 \le \theta < 1$, such that*

$$\left\| \sum a_n(\varphi_n - f_n) \right\|^2 \le \theta^2 \sum |a_n|^2$$

for every finite sequence $\{a_n\}$ of complex numbers. Then there exists a sequence $\{g_n\}$ which, with $\{f_n\}$, forms a complete normalized biorthogonal system; furthermore, every element f of Hilbert space has convergent developments

$$f = \sum (f, g_n) f_n, \; f = \sum (f, f_n) g_n$$

[6] PALEY and WIENER [*] (p. 100). The above proof is due to SZ.-NAGY [5]; moreover, a more general theorem is to be found there.

and we have

$$(1 + \theta)^{-1} \|f\| \le \left(\sum |(f, g_n)|^2\right)^{\frac{1}{2}} \le (1 - \theta)^{-1} \|f\|,$$

$$(1 - \theta) \|f\| \le \left(\sum |(f, f_n)|^2\right)^{\frac{1}{2}} \le (1 + \theta) \|f\|.$$

To prove this theorem, we observe first that under the given hypothesis, the series

$$\sum (f, \varphi_n)(\varphi_n - f_n)$$

converges for every f and, denoting its sum by Kf, that the transformation K thus defined is linear and bounded, $\|K\| \le \theta$. In fact, we have

$$\left\| \sum_{k=m}^{n} (f, \varphi_k)(\varphi_k - f_k) \right\|^2 \le \theta^2 \sum_{k=m}^{n} |(f, \varphi_k)|^2 \to 0$$

for $m, n \to \infty$, which assures the existence of Kf; the linearity of K is evident; finally, since

$$\|Kf\|^2 = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} (f, \varphi_k)(\varphi_k - f_k) \right\|^2 \le \lim_{n\to\infty} \theta^2 \sum_{k=1}^{n} |(f, \varphi_k)|^2 = \theta^2 \|f\|^2,$$

we have $\|K\| \le \theta$ and hence also $\|K^*\| \le \theta$.

The transformation $T = I - K$ therefore has an inverse and we have

$$(1 - \theta)\|f\| \le \|Tf\| \le (1 + \theta)\|f\|, \; (1 - \theta)\|f\| \le \|T^*f\| \le (1 + \theta)\|f\|,$$

$$(1 - \theta)\|T^{-1}g\| \le \|g\| \le (1 + \theta)\|T^{-1}g\|.$$

Since $T\varphi_n$ is clearly equal to f_n, we show that the elements $g_n = (T^{-1})^*\varphi_n$ satisfy the theorem. In fact, we have

$$(f_n, g_m) = (T\varphi_n, (T^{-1})^*\varphi_m) = (T^{-1}T\varphi_n, \varphi_m) = (\varphi_n, \varphi_m) = \delta_{nm}$$

and, whatever be the element f of $\mathfrak{H}$,

$$f = T(T^{-1}f) = T\sum_n (T^{-1}f, \varphi_n)\varphi_n = \sum_n (f, (T^{-1})^*\varphi_n) T\varphi_n = \sum_n (f, g_n) f_n,$$

$$f = (T^*)^{-1}T^*f = (T^*)^{-1} \sum_n (T^*f, \varphi_n)\varphi_n = \sum_n (f, T\varphi_n)(T^*)^{-1}\varphi_n = \sum_n (f, f_n) g_n;$$

consequently the systems $\{f_n\}$ and $\{g_n\}$ are complete. Furthermore, we have

$$\sum_n |(f, g_n)|^2 = \sum_n |(T^{-1}f, \varphi_n)|^2 = \|T^{-1}f\|^2,$$

$$\sum_n |(f, f_n)|^2 = \sum_n |(T^*f, \varphi_n)|^2 = \|T^*f\|^2,$$

which completes the proof.

We give as an example the following application of the theorem to *non-harmonic Fourier series*. Consider, in the space $L^2(-\pi, \pi)$, the functions

$$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{i\lambda_n x} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$