Locating Stopbands and Passbands in a Layered Material

April 9, 2008

1 Introduction

For electromagnetic waves propagating in vacuum (absence of material media, free currents and free charges), the relationship between the wave-length λ and the frequency f of the wave, or, equivalently, between the wave-vector $k = 2\pi/\lambda$ and the angular frequency $\omega = 2\pi f$, is $\omega = ck$. This relationship between k and ω is called the *dispersion relation* and it specifies the frequency for a given wave-vector. When one looks dispersion relations of non-homogeneous material media, expressions as simple as $\omega = ck$ are usually not what is found. Instead, one obtains some complicated expression which says that, for some choices of ω, k is a real number and, for others, it is a complex number. Values of ω with a complex k are *prohibited frequencies.* This is because insertion of $k = \alpha + i\beta$ into the wave form solution $A(x)e^{i(kx-\omega t)}$, makes the wave decay with $e^{-\beta x}$ as it attempts to travel through the medium. By determining all the allowed and prohibited frequencies, one obtains the *passbands* and *stopbands* of the material. In this write-up we calculate the stopbands and passbands for a layered medium supporting a polarized H-field.

2 The Maxwell Equations and Interface Conditions

Following Born and Wolf, we write the Maxwell equations as

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 & \nabla \times \mathbf{H} - \tfrac{1}{c}\tfrac{\partial \mathbf{D}}{\partial t} &= \tfrac{4\pi}{c}\mathbf{J}, \end{aligned} \tag{1}$$

where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. The two parmeters ϵ and μ are the electrical permittivity and magnetic permeability of the medium. They specify its electrical and magnetic properties. In the absence of free currents and free charges and assuming all fields are time-harmonic, e.g., $\mathbf{E} = e^{-i\omega t}\mathbf{E}(x, y, z)$, we obtain

$$\begin{aligned} \nabla \cdot \mathbf{D} = 0 \qquad & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} - ik_0\mathbf{B} = 0 \quad & \nabla \times \mathbf{H} + ik_0\mathbf{D} = 0, \end{aligned} \tag{2}$$

where $k_0 = \omega/c$. These equations are complemented by the following transmission conditions across any interface boundaries

$$\begin{aligned} \mathbf{n}_{12} \cdot (\mathbf{B^{(2)}} - \mathbf{B^{(1)}}) &= 0 \\ \mathbf{n}_{12} \cdot (\mathbf{D^{(2)}} - \mathbf{D^{(1)}}) &= 0 \\ \mathbf{n}_{12} \times (\mathbf{E^{(2)}} - \mathbf{E^{(1)}}) &= 0 \\ \mathbf{n}_{12} \times (\mathbf{H^{(2)}} - \mathbf{H^{(1)}}) &= 0 \end{aligned}$$

where $\mathbf{n}_{12}$ is the unit normal vector pointing from the region containing material 1 to the region containing material 2 and $\mathbf{B^{(2)}} - \mathbf{B^{(1)}}$, $\mathbf{E^{(2)}} - \mathbf{E^{(1)}}$ etc, represent the difference between the field values at the material 2 side and the material 1 side of the interface boundary. These equations and transmission conditions are valid no matter what the geometry and physical properties of the material medium. We now specialize to a non-magnetic layered material with a polarized magnetic field $\mathbf{H} = (0, 0, H)$.

3 Layered Medium

If we assume that $\mathbf{H} = (0, 0, H(x))$, then the full Maxwell system reduces to

$$dE/dx = ik_0\mu H \tag{3}$$

$$dH/dx = ik_0\epsilon E \tag{4}$$

where $\mathbf{E} = (0, E(x), 0)$.

Mathematically, the layered geometry enters via specification of the normal $\mathbf{n}$ in the jump conditions. Thus, insering $\mathbf{n} = (1, 0, 0)$, $\mathbf{B} = \mathbf{H} = (0, 0, H)$, $\mathbf{E} = (0, E, 0)$ and $\mathbf{D} = (0, \epsilon E, 0)$ in the jump conditions, we get

$$\begin{aligned} (1, 0, 0) \cdot (0, 0, B^{(2)} - B^{(1)}) &= 0 \\ (1, 0, 0) \cdot (0, \epsilon E^{(2)} - \epsilon E^{(1)}, 0) &= 0 \\ (1, 0, 0) \times (0, E^{(2)} - E^{(1)}, 0) &= 0 \\ (1, 0, 0) \times (0, 0, H^{(2)} - H^{(1)}) &= 0 \end{aligned}$$

Thus, the jump conditions specify that E and H must be continuous across the layer interfaces.

4 Floquet Theory

Using the second equation of (3), we can rewrite this continuity condition as H and $\epsilon^{-1}H$ continuous. Moreover, we can eliminate E in (3) to obtain

$$H'' + k_0^2\epsilon\mu H = 0. \tag{5}$$

This then completely specifies our problem: determine a function $H(x)$ satisfying (5) on $-\infty < x < \infty$, such that H and $\epsilon^{-1}H$ are continuous, where

$$\epsilon(x) = \begin{cases} \epsilon_1, & -\theta_1 d < x < 0 \\ \epsilon_2, & 0 < x < \theta_2 d. \end{cases}$$

an $\mu = 1$.

Since the coefficient ϵk_0^2 is periodic with period d, we should expect that

$$H(x) = \phi(x)e^{ikx}, \tag{6}$$

for some $k \in \mathbb{C}$, where ϕ is periodic with period d (this can be proved mathematically and is known as Floquet's theorem). On the other hand, we must have

$$H(x) = \begin{cases} Ae^{ik_0 n_1 x} + Be^{-ik_0 n_1 x}, & \text{inside layer } n_1 \\ Ce^{ik_0 n_2 x} + De^{-ik_0 n_2 x}, & \text{inside layer } n_2 \end{cases}$$

for complex constants A, B, C and D, where $n_1 = \sqrt{\epsilon_1}$ and $n_2 = \sqrt{\epsilon_2}$.

Putting these two differnt ways of writing the solution H together shall give us two algebraic equations for A, B, C and D. Indeed, from (6), we see that H is quasi-peiodic, namely $H(x+d) = e^{ikd}H(x)$, so that $H(\theta_2 d) = e^{ikd}H(-\theta_1 d)$ and $H'(\theta_2 d) = e^{ikd}H'(-\theta_1 d)$. Combined with (?), this gives

$$\begin{cases} H_1: & Ce^{ik_0 n_2 l_2} + De^{-ik_0 n_2 l_2} & = & e^{ikd}(Ae^{-ik_0 n_1 l_1} + Be^{ik_0 n_1 l_1}) \\ H_1': & n_2 Ce^{ik_0 n_2 l_2} - n_2 De^{-ik_0 n_2 l_2} & = & e^{ikd}(n_1 e^{-ik_0 n_1 l_1} - n_1 Be^{ik_0 n_1 l_1}). \end{cases}$$

The continuity of H_1 and $\epsilon^{-1}H_1'$ across the interface $x = 0$ gives two more conditions

$$\begin{cases} E_1: & A + B & = & C + D \\ E_1': & \frac{n_1}{\epsilon_1}(A - B) & = & \frac{n_2}{\epsilon_2}(C - D) \end{cases}$$

so that we have a homogeneous system of four equations in the four unknowns A, B, C and D (we went System of PDE's $\rightarrow$ ODE $\rightarrow$ Algebraic System). In order that this sytem have non-trivial solutions, its determinant must equal zero

$$0 = \begin{vmatrix} e^{ikd}e^{-ik_0 n_1 l_1} & e^{ikd}e^{ik_0 n_1 l_1} & -e^{ik_0 n_2 l_2} & -e^{-ik_0 n_2 l_2} \\ n_1 e^{ikd}e^{-ik_0 n_1 l_1} & -n_1 e^{ikd}e^{ik_0 n_1 l_1} & -n_2 e^{ik_0 n_2 l_2} & n_2 e^{-ik_0 n_2 l_2} \\ 1 & 1 & -1 & -1 \\ \frac{n_1}{\epsilon_1} & -\frac{n_1}{\epsilon_1} & -\frac{n_2}{\epsilon_2} & \frac{n_2}{\epsilon_2} \end{vmatrix}.$$

Computing this determinant we obtain a quadratic in e^{ikd}

$$e^{2ikd} - e^{ikd}\left(2c_1c_2 - \left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right)s_1s_2\right) + 1 = 0. \tag{7}$$

where $c_1 = \cos(n_1\theta_1 k_0 d)$, $c_2 = \cos(n_2\theta_2 k_0 d)$, $s_1 = \sin(n_1\theta_1 k_0 d)$ and $s_2 = \sin(n_2\theta_2 k_0 d)$. The independent coefficient of this quadratic is 1, so that its roots are reciprocals whose product equals 1, namely e^{ikd} and e^{-ikd}. We can also write (7) as

$$\cos(kd) = c_1 c_2 - \frac{1}{2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right) s_1 s_2. \tag{8}$$

Equation (8) gives in implicit form the dispersion relation $\omega = \omega(k)$ for the function E_1. Note the resemblance of (8) to the formula $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

5 The Homogenized Dispersion Relation

In the dispersion relation (8), as $k_0 d \to 0$ the right-hand side goes to 1, so that, for a sufficiently small neighborhood of $k_0 d = 0$, we can expand both the right- and left-hand sides in second-order Taylor series as follows

$$1 - \frac{k^2 d^2}{2} = \left(1 - \frac{(n_1\theta_1 k_0 d)^2}{2}\right)\left(1 - \frac{(n_2\theta_2 k_0 d)^2}{2}\right) - \frac{1}{2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right)(n_1\theta_1 k_0 d)(n_2\theta_2 k_0 d).$$

Disregrading the term in $(k_0 d)^4$ we obtain the "homogenized" dispersion relation

$$k^2 = (\theta_1\epsilon_1 + \theta_2\epsilon_2)k_0^2$$

or, writing $\omega = ck_0$,

$$\omega^2 = c^2(\epsilon_{eff}^{-1} k^2),$$

where $\epsilon_{eff} = \theta_1\epsilon_1 + \theta_2\epsilon_2$ is the weighted arithmetic mean of ϵ_1 and ϵ_2.

6 Plot of the Case $n_1 = 1$, $n_2 = iy$

Below is a contour-plot of the dispersion relation (8) when $n_1 = 1$, $n_2 = iy$, $\theta_1 = \theta_2 = 0.5$ and $d = 2\pi$. The horizontal axis is k_0 and the vertical axis is y. For pairs (k_0, y) in the black region $\cos k > 1$ and in the white region $\cos k < -1$, so that k is not pure real and there is no propagation in these regions. For (k_0, y) in the gray region $|\cos k| < 1$, so that this is the region of propagation.