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The leading behavior of $I_n(x)$ as $x \to +\infty$ does not depend on n; however, higher-order terms in the asymptotic expansion of $I_n(x)$ do depend on n. In fact, the complete asymptotic expansion of $I_n(x)$ as $x \to +\infty$ is given in (3.5.8) and (3.5.9) with $c_1 = 1/\sqrt{2\pi}$.

The only step that really needs justification is (6.4.30). The argument is nearly the same as that used to justify (6.4.24). The range of integration from $t = 0$ to $t = \varepsilon$ is broken up into the two ranges $0 \le t \le x^{-\alpha}$ and $x^{-\alpha} < t \le \varepsilon$, where $\frac{1}{4} < \alpha < \frac{1}{2}$. Now for fixed n, $\cos(nt)\, e^{x\cos t} \sim e^{x(1-t^2/2)}$ $(x \to +\infty)$ uniformly for all t satisfying $0 \le t \le x^{-\alpha}$ because $1 - t^2/2 \le \cos t \le 1 - t^2/2 + t^4/24$. Therefore, $\int_0^{x^{-\alpha}} \cos(nt)\, e^{x\cos t}\, dt \sim \int_0^{x^{-\alpha}} e^{x(1-t^2/2)}\, dt$ $(x \to +\infty)$. Also, when $x^{-\alpha} < t \le \varepsilon$, the integrands on both sides of (6.4.30) are subdominant with respect to e^x, so the contribution to (6.4.30) from the integration range $x^{-\alpha} < t \le \varepsilon$ is exponentially small compared to the contribution from the range $0 \le t \le x^{-\alpha}$.

Laplace's Method—Determination of Higher-Order Terms

The approach we have used to obtain the leading asymptotic behavior of integrals by Laplace's method can be extended to give the higher-order terms in the asymptotic expansion of the integral. To do this one would naturally expect to have to retain more terms in the expansions of $\phi(t)$ and $f(t)$ than those used to obtain (6.4.19). We illustrate the mechanics of this procedure for the case in which $\phi'(c) = 0$, $\phi''(c) < 0$, $f(c) \ne 0$, and $a < c < b$, where c is the location of the maximum of $\phi(t)$.

By (6.4.2), $I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{x\phi(t)}\, dt$ $(x \to +\infty)$ with exponentially small errors. The leading behavior of $I(x)$ given by (6.4.19c) is obtained by replacing $f(t)$ by $f(c)$ and $\phi(t)$ by $\phi(c) + \frac{1}{2}(t-c)^2\phi''(c)$. To compute the first correction to (6.4.19c) we must approximate $f(t)$ and $\phi(t)$ by two *more* terms in their Taylor series:

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} [f(c) + f'(c)(t-c) + \tfrac{1}{2}f''(c)(t-c)^2] \\ \times \exp\{x[\phi(c) + \tfrac{1}{2}(t-c)^2\phi''(c) + \tfrac{1}{6}(t-c)^3\phi'''(c) \\ + \tfrac{1}{24}(t-c)^4(d^4\phi/dt^4)(c)]\}\, dt, \qquad x \to +\infty. \qquad (6.4.32)$$

It is somewhat surprising that *two* additional terms in the series for $\phi(t)$ and $f(t)$ are required to compute just the next term in (6.4.19c). We will see shortly why this is so.

Because ε may be chosen small, we Taylor expand the integrand in (6.4.32) as follows:

$$\exp\{x[\tfrac{1}{6}(t-c)^3\phi'''(c) + \tfrac{1}{24}(t-c)^4(d^4\phi/dt^4)(c)]\} \\ = 1 + x[\tfrac{1}{6}(t-c)^3\phi'''(c) + \tfrac{1}{24}(t-c)^4(d^4\phi/dt^4)(c)] \\ + \tfrac{1}{72}x^2(t-c)^6[\phi'''(c)]^2 + \cdots.$$

Substituting this expansion into (6.4.32) and collecting powers of $t - c$ gives

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} e^{x\phi(c) + x(t-c)^2\phi''(c)/2} \\ \times \Big\{ f(c) + \frac{(t-c)^2}{2} f''(c) + (t-c)^4[\tfrac{1}{24}xf(c)(d^4\phi/dt^4)(c) + \tfrac{1}{6}xf'(c)\phi'''(c)] \\ + \tfrac{1}{72}(t-c)^6x^2f(c)[\phi'''(c)]^2 + \cdots \Big\}\, dt, \qquad x \to +\infty, \qquad (6.4.33)$$

where we have excluded odd powers of $t - c$ because they vanish upon integration. Only the displayed terms in (6.4.33) contribute to the next term in (6.4.19c). Notice that we do *not* Taylor expand $\exp[\frac{1}{2}x(t-c)^2\phi''(c)]$; we return to this point shortly.

Next we extend the range of integration in (6.4.33) to $(-\infty, \infty)$ and substitute $s = \sqrt{x}(t - c)$:

$$I(x) \sim \frac{1}{\sqrt{x}} e^{x\phi(c)} \int_{-\infty}^{\infty} e^{s^2\phi''(c)/2} \\ \times \Big\{ f(c) + \frac{1}{x}[\tfrac{1}{2}s^2f''(c) + \tfrac{1}{24}s^4f(c)(d^4\phi/dt^4)(c) + \tfrac{1}{6}s^4f'(c)\phi'''(c) \\ + \tfrac{1}{72}s^6[\phi'''(c)]^2f(c)] \Big\}\, ds, \qquad x \to +\infty. \qquad (6.4.34)$$

Observe that all the displayed terms in (6.4.33) contribute to the coefficient of $1/x$ in (6.4.34); the additional terms that we have neglected in going from (6.4.32) to (6.4.34) contribute to the coefficients of $1/x^2$, $1/x^3$, and so on.

To evaluate the integrals in (6.4.34) we use integration by parts to derive the general formula $\int_{-\infty}^{\infty} e^{-s^2/2} s^{2n}\, ds = \sqrt{2\pi}(2n-1)(2n-3)(2n-5)\cdots(5)(3)(1)$. Thus, we have

$$I(x) \sim \sqrt{\frac{2\pi}{-x\phi''(c)}}\, e^{x\phi(c)} \Big\{ f(c) + \frac{1}{x}\Big[-\frac{f''(c)}{2\phi''(c)} + \frac{f(c)(d^4\phi/dt^4)(c)}{8[\phi''(c)]^2} \\ + \frac{f'(c)\phi'''(c)}{2[\phi''(c)]^2} - \frac{5[\phi'''(c)]^2f(c)}{24[\phi''(c)]^3} \Big] \Big\}, \qquad x \to +\infty. \qquad (6.4.35)$$

One aspect of the derivation of (6.4.35) requires explanation. In proceeding from (6.4.32) to (6.4.34) we did not Taylor expand $\exp[\frac{1}{2}x(t-c)^2\phi''(c)]$, but we did Taylor expand the cubic and quartic terms in the exponential. If we had Taylor expanded $\exp[\frac{1}{2}x(t-c)^2\phi''(c)]$ and retained only a finite number of terms, the resulting approximation to $I(x)$ would depend on ε (see Example 8). If we had not expanded the cubic and quartic terms and if $(d^4\phi/dt^4)(c)$ were nonnegative, then extending the range of integration from $(c - \varepsilon, c + \varepsilon)$ to $(-\infty, \infty)$ would yield a divergent integral which would be a poor approximation to $I(x)$ indeed! If we had not expanded the cubic and quartic terms and if $(d^4\phi/dt^4)(c) < 0$, then extending the range of integration from $(c - \varepsilon, c + \varepsilon)$ to $(-\infty, \infty)$ would yield a convergent integral. However, this convergent integral might not be asymptotic to $I(x)$ because replacing $\phi(t)$ by the four-term Taylor series in (6.4.32) can introduce new relative maxima which lie outside $(c - \varepsilon, c + \varepsilon)$ which would dominate the integral on the right side of (6.4.32). In summary, there are three reasons why we must Taylor expand the cubic and quartic terms in the exponential before we extend the range of integration to $(-\infty, \infty)$:

1. The resulting integrals are always convergent and depend on ε only through subdominant terms.
2. It is easy to evaluate the resulting Gaussian integrals.
3. We avoid introducing any spurious maxima into the integrand.