ticular, if $f \in C$ on a domain D of the $(t,x_1, \ldots, x_n)$ space, and P is a point of D, there exists a solution $\varphi \in C^n$ of (E_n) on some t interval and passing through P. If, in addition, $f \in$ Lip in D, that is, if

$$|f(t,x_1, \ldots, x_n) - f(t,\tilde{x}_1, \ldots, \tilde{x}_n)| \leq k \sum_{i=1}^{n} |x_i - \tilde{x}_i|$$

for some constant $k > 0$, then the solution through P is unique.

7. Dependence of Solutions on Initial Conditions and Parameters

A solution of a differential equation on an interval I can be considered as a function, not only of $t \in I$, but of the coordinates of a point through which the solution passes. For example, the first-order equation in one dimension $x' = x$ has the solution $\varphi(t) = \xi e^{t-\tau}$ through the point (τ,ξ). This determines a function of (t,τ,ξ), which is also called† φ, given by $\varphi(t,\tau,\xi) = \xi e^{t-\tau}$. In the general situation, it is important to know how φ behaves as a function of (t,τ,ξ) together, and, in particular, under what circumstances φ is continuous in (t,τ,ξ). In the following the behavior of the solutions as functions of the initial conditions will be investigated for the general case of a system.

Let D be a domain in the $(n + 1)$-dimensional real (t,x) space and suppose $f \in (C,\text{Lip})$ in D. Let ψ be a solution of the equation

$$\text{(E)} \qquad x' = f(t,x)$$

on some interval I. Thus $(t,\psi(t)) \in D$ for $t \in I$. It follows from the existence theorem that (E) has a unique solution through any point (τ,ξ) close enough to the given solution. However, the existence theorem assures the existence of the solution only over some short t interval containing τ. Actually, it can be shown that the solution exists over the whole interval I, and is a continuous function of (t,τ,ξ). The following theorem will be proved.

Theorem 7.1. *Let $f \in (C,\text{Lip})$ in a domain D of the $(n + 1)$-dimensional (t,x) space, and suppose ψ is a solution of* (E) *on an interval I: $a \leq t \leq b$. There exists a $\delta > 0$ such that for any $(\tau,\xi) \in U$, where*

$$U: \quad a < \tau < b \qquad |\xi - \psi(\tau)| < \delta$$

there exists a unique solution φ of (E) *on I with $\varphi(\tau,\tau,\xi) = \xi$. Moreover, $\varphi \in C$ on the $(n + 2)$-dimensional set*

$$V: \quad a < t < b \qquad (\tau,\xi) \in U$$

† There will be little chance of confusing these two functions. If φ is thought of as a function of (t,τ,ξ), then φ' will always mean $\partial\varphi/\partial t$.

REMARKS: In many applications τ is fixed, and in this case U can be considered as the set of all ξ satisfying $|\xi - \psi(\tau)| < \delta$, and V the domain $a < t < b$, $\xi \in U$. The proof for this case is contained in the proof of Theorem 7.1. An important consequence of the proof in this case is that the mapping T_t which associates with each point (τ,ξ), $\xi \in U$, the point $(t,\varphi(t,\tau,\xi))$ for some fixed t, $a < t < b$, is a topological mapping.† The uniqueness of the solutions guarantees that T_t is one-to-one, and the continuity of φ in ξ implies T_t is continuous. Since ξ can be considered as the point $\xi = \varphi(\tau,t,\tilde{\xi})$, where $\tilde{\xi} = \varphi(t,t,\tilde{\xi}) = \varphi(t,\tau,\xi)$, the continuity of φ again implies T_t^{-1} is continuous. Actually, the uniqueness of the solutions passing through (τ,ξ), $\xi \in U$, is sufficient for the continuity of φ in ξ; see Theorem 4.3, Chap. 2.

Often ψ can be continued outside of I, in which case U, V would include the end points a and b of I.

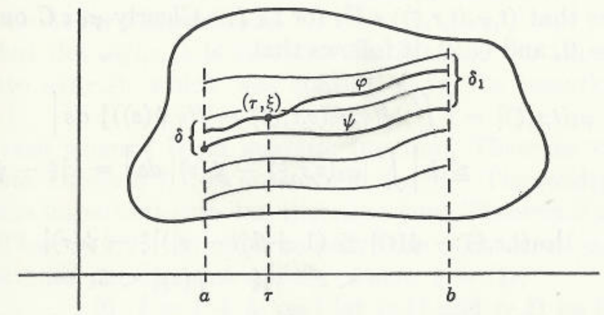

FIG. 2

Proof of Theorem 7.1. Choose $\delta_1 > 0$ so that the (t,x) region U_1, given by

$$U_1: \quad t \in I \qquad |x - \psi(t)| \leq \delta_1$$

is in D. Then let δ be chosen so that $\delta < e^{-k(b-a)}\delta_1$, where k is the Lipschitz constant. With this δ, define U as in the statement of the theorem; see Fig. 2 for the case $n = 1$. If $(\tau,\xi) \in U$, there exists a solution φ through (τ,ξ) locally, and this satisfies

$$\varphi(t,\tau,\xi) = \xi + \int_\tau^t f(s,\varphi(s,\tau,\xi))\, ds \tag{7.1}$$

as far as it exists. Moreover, for $t \in I$,

$$\psi(t) = \psi(\tau) + \int_\tau^t f(s,\psi(s))\, ds \tag{7.2}$$

† A topological mapping T of a set S onto a set $T(S)$ is a one-to-one mapping such that T and T^{-1} are continuous.

Thus, using the fundamental inequality (2.2) with $\epsilon = 0$, there results

$$|\varphi(t,\tau,\xi) - \psi(t)| \leqq |\xi - \psi(\tau)|e^{k|t-\tau|} < \delta_1$$

This proves φ cannot leave U_1, and can therefore, by Theorem 4.1, be continued to the whole interval I.

The continuity of φ on V will be proved by showing that φ is the uniform limit of continuous functions on V. Note that φ satisfies (7.1) on I. Define the successive approximations $\{\varphi_j\}$ for (7.1) by

$$\begin{aligned} \varphi_0(t,\tau,\xi) &= \psi(t) + \xi - \psi(\tau) \\ \varphi_{j+1}(t,\tau,\xi) &= \xi + \int_\tau^t f(s,\varphi_j(s,\tau,\xi))\, ds \qquad (j = 0, 1, 2, \ldots) \end{aligned} \tag{7.3}$$

Then for $(\tau,\xi) \in U$

$$|\varphi_0(t,\tau,\xi) - \psi(t)| = |\xi - \psi(\tau)| < \delta_1$$

which shows that $(t,\varphi_0(t,\tau,\xi)) \in U_1$ for $t \in I$. Clearly $\varphi_0 \in C$ on V. From (7.3) for $j = 0$, and (7.2), it follows that

$$\begin{aligned} |\varphi_1(t,\tau,\xi) - \varphi_0(t,\tau,\xi)| &= \left| \int_\tau^t \{f(s,\varphi_0(s,\tau,\xi)) - f(s,\psi(s))\}\, ds \right| \\ &\leqq k \left| \int_\tau^t |\varphi_0(s,\tau,\xi) - \psi(s)|\, ds \right| = k|\xi - \psi(\tau)|\, |t - \tau| \end{aligned}$$

and hence

$$\begin{aligned} |\varphi_1(t,\tau,\xi) - \psi(t)| &\leqq (1 + k|t - \tau|)|\xi - \psi(\tau)| \\ &< e^{k|t-\tau|}|\xi - \psi(\tau)| < \delta_1 \end{aligned}$$

provided that $t \in I$, $(\tau,\xi) \in U$. Thus $(t,\varphi_1(t,\tau,\xi)) \in U_1$ and $\varphi_1 \in C$ on V. An induction shows that if $\varphi_0,\varphi_1, \ldots, \varphi_j$ are all in U_1 and continuous on V, then

$$|\varphi_{j+1}(t,\tau,\xi) - \varphi_j(t,\tau,\xi)| \leqq \frac{k^{j+1}|t - \tau|^{j+1}}{(j + 1)!} |\xi - \psi(\tau)| \tag{7.4}$$

if $t \in I$ and $(\tau,\xi) \in U$. This implies that

$$|\varphi_{j+1}(t,\tau,\xi) - \psi(t)| < e^{k|t-\tau|}|\xi - \psi(\tau)| < \delta_1$$

proving that $(t,\varphi_{j+1}(t,\tau,\xi)) \in U_1$. Also, from (7.3), $\varphi_{j+1} \in C$ on V. Thus by induction $(t,\varphi_j(t,\tau,\xi)) \in U_1$ and $\varphi_j \in C$ on V for all j.

Using (7.4), it follows that the φ_j converge uniformly on V to φ, which proves the continuity of φ on V. (Note that the uniform convergence of the φ_j also proves the existence of φ on I.)

Having established the existence and continuity of φ as a function of (t,τ,ξ), it is natural, and for purposes of application also important, to give reasonable sufficient conditions for the existence and continuity of the partial derivatives $\partial\varphi/\partial\tau$, $\partial\varphi/\partial\xi_j$ $(j = 1, \ldots, n)$, where the ξ_j are the components of ξ. Such a sufficient condition is the existence and continuity of the partial derivatives $\partial f/\partial x_j$ on D.

Let f_x denote the matrix (if it exists) with element $\partial f_i/\partial x_j$ in the ith row and jth column $(i,j = 1, \ldots, n)$. Also let φ_ξ be the matrix (if it exists) with element $\partial\varphi_i/\partial\xi_j$ in the ith row and jth column $(i,j = 1, \ldots, n)$. A matrix is said to be continuous if all its elements are. If $A = (a_{ij})$ is an n-by-n matrix, its determinant will be denoted by det A, and its trace, $\sum_{i=1}^{n} a_{ii}$, by tr A. The symbol exp u denotes e^u.

Theorem 7.2. *Let the hypothesis of Theorem 7.1 be satisfied, and suppose f_x exists and $f_x \in C$ on D. Then $\varphi \in C^1$ on V, and moreover*

$$\det \varphi_\xi(t,\tau,\xi) = \exp \int_\tau^t \operatorname{tr} f_x(s,\varphi(s,\tau,\xi))\, ds \tag{7.5}$$

REMARKS: The fact that $f_x \in C$ on D actually makes the explicit Lipschitz hypothesis for f superfluous.

Notice that det $\varphi_\xi(t,\tau,\xi)$ is just the Jacobian of the transformation, taking ξ into $\varphi(t,\tau,\xi)$, which was considered in the remarks following Theorem 7.1.

For the case where f is an analytic function, Theorem 7.2 is easily obtained from Theorem 7.1, as is shown in Sec. 8. The reader interested mainly in this important case can therefore omit Theorem 7.2.

Proof of Theorem 7.2. In order to prove the existence of φ_ξ, it is sufficient to consider the case of $\partial\varphi/\partial\xi_1$, where $\xi = (\xi_1, \ldots, \xi_n)$. Let $h = (h_1, 0, \ldots, 0)$, $\tilde{\xi} = \xi + h$, and let (τ,ξ) and $(\tau,\tilde{\xi})$ be in U. If χ is the function defined by

$$\chi(t,\tau,\xi,h) = \frac{\varphi(t,\tau,\tilde{\xi}) - \varphi(t,\tau,\xi)}{h_1}$$

for $(t,\tau,\xi) \in V$, then what has to be proved is that

$$\lim_{h\to 0} \chi(t,\tau,\xi,h) \tag{7.6}$$

exists. It will be shown that the limit in (7.6) exists uniformly on V and that the limit function is continuous on V. This will prove $\partial\varphi/\partial\xi_1$ exists and is continuous on V.

The motivation behind the proof is very simple: The solution φ satisfies (E), and so

$$\varphi'(t,\tau,\xi) = f(t,\varphi(t,\tau,\xi))$$

Thus, if φ and f are sufficiently differentiable,

$$\left(\frac{\partial\varphi}{\partial\xi_1}\right)' (t,\tau,\xi) = f_x(t,\varphi(t,\tau,\xi))\, \frac{\partial\varphi}{\partial\xi_1}\, (t,\tau,\xi)$$

where the latter product is an ordinary matrix product. Therefore $\partial\varphi/\partial\xi_1$ is a solution of a *linear* differential equation. All the following proof does is to justify this procedure.

Let

$$\theta(t,\tau,\xi,h) = \varphi(t,\tau,\tilde{\xi}) - \varphi(t,\tau,\xi)$$

Using the inequality (2.2), there results

$$|\theta(t,\tau,\xi,h)| \leqq |\theta(\tau,\tau,\xi,h)|e^{k|t-\tau|} \leqq |h_1|e^{k(b-a)} \tag{7.7}$$

Thus as $h_1 \to 0$, $\theta \to 0$ uniformly for $(t,\tau,\xi) \varepsilon V$.

Since φ is a solution of (E)

$$\theta'(t,\tau,\xi,h) = f(t,\varphi(t,\tau,\tilde{\xi})) - f(t,\varphi(t,\tau,\xi)) \tag{7.8}$$

Using the theorem of the mean on the right side of (7.8), and recalling that $f_x \varepsilon C$ on D, there exists a matrix $\Gamma = (\Gamma_{ij})$ such that

$$\theta'(t,\tau,\xi,h) = (f_x(t,\varphi(t,\tau,\xi)) + \Gamma)\theta(t,\tau,\xi,h) \tag{7.9}$$

where, given any $\epsilon_1 > 0$, there exists a δ_1, which depends on ϵ_1, such that $|\Gamma| = \sum_{i,j=1}^{n} |\Gamma_{ij}| < \epsilon_1$ if $|\theta| < \delta_1$ for $(t,\tau,\xi) \varepsilon V$.† By (7.7), then, $|\Gamma| \to 0$ as $h_1 \to 0$ uniformly for $(t,\tau,\xi) \varepsilon V$.

Since $\chi = \theta/h_1$, (7.9) yields

$$\chi'(t,\tau,\xi,h) = f_x(t,\varphi(t,\tau,\xi))\chi(t,\tau,\xi,h) + \gamma \tag{7.10}$$

where $\gamma = \Gamma\theta/h_1$ so that by (7.7)

$$|\gamma| \leqq |\Gamma|e^{k(b-a)}$$

Thus $\gamma \to 0$ as $h_1 \to 0$ uniformly on V. In particular, given any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that $|\gamma| < \epsilon$ if $|h_1| < \delta_\epsilon$. Thus (7.10) states that χ as a function of t is an ϵ-approximate solution of the linear equation

$$y' = f_x(t,\varphi(t,\tau,\xi))y \tag{7.11}$$

provided that $|h_1| < \delta_\epsilon$. The initial value $\chi(\tau,\tau,\xi,h)$ is e_1, where e_1 is the vector with components $(1, 0, \ldots, 0)$.

Consider now for fixed $(\tau,\xi) \varepsilon U$ the solution β of (7.11) which assumes the initial value e_1 at $t = \tau$. That this solution exists on I: $a \leqq t \leqq b$ follows from Theorem 5.1. The fact that χ is an ϵ-approximate solution of (7.11) for $|h_1| < \delta_\epsilon$ implies by Theorem 2.1 that

$$|\chi(t,\tau,\xi,h) - \beta(t,\tau,\xi)| \leqq \frac{\epsilon}{k}\,(e^{k(b-a)} - 1)$$

† Here use is made of the fact that for $(t,\tau,\xi) \varepsilon V$ the points $(t,\varphi(t,\tau,\xi)) \varepsilon U_1$, a closed bounded set. Thus f_x is uniformly continuous on U_1.

for (t,τ,ξ) on V. Clearly this implies that

$$\lim_{h\to 0} \chi(t,\tau,\xi,h) = \beta(t,\tau,\xi)$$

uniformly on V. This proves the existence of $\partial\varphi/\partial\xi_1$ and also proves that it is the solution of (7.11) which assumes the initial value e_1 at $t = \tau$. The uniformity of the convergence of χ as $h \to 0$, and the continuity of χ on V, imply the continuity of $\partial\varphi/\partial\xi_1$ on V.

An entirely similar argument proves the existence and continuity of $\partial\varphi/\partial\xi_j$, $j = 2, \ldots, n$, on V. Also if e_j is the vector with all components zero except the jth, which is 1,

$$\frac{\partial\varphi}{\partial\xi_j}(\tau,\tau,\xi) = e_j \qquad (j = 1, \ldots, n) \tag{7.12}$$

and $\partial\varphi/\partial\xi_j$ is a solution of (7.11). The columns of the matrix φ_ξ are precisely the vectors $\partial\varphi/\partial\xi_j$. Therefore the following matrix equation is valid:

$$\varphi_\xi'(t,\tau,\xi) = f_x(t,\varphi(t,\tau,\xi))\varphi_\xi(t,\tau,\xi) \tag{7.13}$$

where $\varphi_\xi' = \partial\varphi_\xi/\partial t$. The relation (7.12) may be written as

$$\varphi_\xi(\tau,\tau,\xi) = E \tag{7.14}$$

where E is the n-by-n unit matrix,

$$E = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

The relation (7.5) is a consequence of a general fact concerning matrix solutions of linear systems. Since this relation is of importance in itself, it will be proved in the next theorem. One obtains (7.5) from (7.18) below using (7.13) and (7.14) and the fact that det $E = 1$.

It is but a repetition of the previous arguments to show that $\partial\varphi/\partial\tau$ also satisfies the linear equation (7.11), once it is observed that it has the initial value given by

$$\frac{\partial\varphi}{\partial\tau}(\tau,\tau,\xi) = -f(\tau,\xi) \tag{7.15}$$

This is shown by a direct calculation as follows:

$$\begin{aligned}\varphi(\tau,\bar{\tau},\xi) - \varphi(\tau,\tau,\xi) &= \varphi(\tau,\bar{\tau},\xi) - \xi \\ &= \varphi(\tau,\bar{\tau},\xi) - \varphi(\bar{\tau},\bar{\tau},\xi) \\ &= \int_{\bar{\tau}}^{\tau} f(s,\varphi(s,\bar{\tau},\xi))\,ds\end{aligned}$$

Thus

$$\frac{\varphi(\tau,\bar\tau,\xi) - \varphi(\tau,\tau,\xi)}{\bar\tau - \tau} = -\frac{1}{\bar\tau - \tau}\int_\tau^{\bar\tau} f(s,\varphi(s,\bar\tau,\xi))\,ds$$

Since the integrand is continuous for $(s,\bar\tau,\xi) \in V$, it follows that the limit as $\bar\tau \to \tau$ exists for $(\tau,\xi) \in U$ and gives (7.15).

Theorem 7.3. *Let A be an n-by-n matrix with continuous elements on an interval I: $a \leqq t \leqq b$, and suppose Φ is a matrix of functions on I satisfying*

$$\Phi'(t) = A(t)\Phi(t) \qquad (t \in I) \tag{7.16}$$

Then $\det \Phi$ *satisfies on I the first-order equation*

$$(\det \Phi)' = (\operatorname{tr} A)(\det \Phi) \tag{7.17}$$

and thus for τ, $t \in I$

$$\det \Phi(t) = \det \Phi(\tau) \exp \int_\tau^t \operatorname{tr} A(s)\,ds \tag{7.18}$$

Proof. Let φ_{ij}, a_{ij} be the elements in the ith row and jth column of Φ and A, respectively. Then (7.16) says

$$\varphi_{ij}'(t) = \sum_{k=1}^n a_{ik}(t)\varphi_{kj}(t) \qquad (i,j = 1, \ldots, n) \tag{7.19}$$

The derivative of $\det \Phi$ is a sum of n determinants

$$(\det \Phi)' = \begin{vmatrix} \varphi_{11}' & \varphi_{12}' & \cdots & \varphi_{1n}' \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21}' & \varphi_{22}' & \cdots & \varphi_{2n}' \\ \cdot & \cdot & \cdots & \cdot \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \varphi_{n1}' & \varphi_{n2}' & \cdots & \varphi_{nn}' \end{vmatrix}$$

Using (7.19) in the first determinant on the right, one gets

$$\begin{vmatrix} \sum_k a_{1k}\varphi_{k1} & \sum_k a_{1k}\varphi_{k2} & \cdots & \sum_k a_{1k}\varphi_{kn} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix}$$

and this determinant is unchanged if one subtracts from the first row a_{12} times the second row plus a_{13} times the third row up to a_{1n} times the nth row. This gives

$$\begin{vmatrix} a_{11}\varphi_{11} & a_{11}\varphi_{12} & \cdots & a_{11}\varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix}$$

which is just $a_{11} \det \Phi$. Carrying out a similar procedure with the remaining determinants, one obtains finally (7.17). The equation (7.17) is of the form $u' - \alpha(t)u = 0$ from which follows

$$u \exp\left[-\int_\tau^t \alpha(s)\,ds\right] = \text{constant}$$

which gives (7.18).

The case where the right member f of (E) contains a parameter vector μ can be readily dealt with. Suppose μ space has k real dimensions, and let I_μ be the domain of μ space, $|\mu - \mu_0| < c$, where μ_0 is fixed and $c > 0$. As above, D is a domain of (t,x) space. Let D_μ be the domain of (t,x,μ) space

$$D_\mu: \quad (t,x) \in D \qquad \mu \in I_\mu$$

and let $f \in C$ on D_μ and satisfy a Lipschitz condition in x uniformly on D_μ. The differential equation

$$(\mathrm{E}_\mu) \qquad x' = f(t,x,\mu)$$

will be considered here. For a fixed given $\mu = \mu_0$, let ψ be a solution of (E_μ) on an interval $a \leqq t \leqq b$. Then the following theorem, which includes Theorem 7.1 as a special case, will be proved:

Theorem 7.4. *Let ψ be the solution of (E_μ) described above. There exists a $\delta > 0$ such that for any $(\tau,\xi,\mu) \in U_\mu$, where*

$$U_\mu: \quad a < \tau < b \qquad |\xi - \psi(\tau)| + |\mu - \mu_0| < \delta$$

there exists a unique solution φ of (E_μ) on $a \leqq t \leqq b$ satisfying

$$\varphi(\tau,\tau,\xi,\mu) = \xi$$

Moreover, $\varphi \in C$ on the $(n + k + 2)$-dimensional domain

$$V_\mu: \quad a < t < b \qquad (\tau,\xi,\mu) \in U_\mu$$

REMARK: An alternative proof of this theorem under slightly more restrictive hypotheses is given in the course of proving Theorem 7.5 below.

Proof of Theorem 7.4. The proof is like that of Theorem 7.1. As remarked there, the successive-approximations procedure can be used to prove the whole theorem. Choose $\delta_1 > 0$ so that the (t,x,μ) region $U_{1\mu}$ given by

$$U_{1\mu}: \quad a \leqq t \leqq b \qquad |x - \psi(t)| + |\mu - \mu_0| \leqq \delta_1$$

is in D_μ. Define the approximations $\{\varphi_j\}$ by

$$\varphi_0(t,\tau,\xi,\mu) = \psi(t) + \xi - \psi(\tau)$$
$$\varphi_{j+1}(t,\tau,\xi,\mu) = \xi + \int_\tau^t f(s,\varphi_j(s,\tau,\xi,\mu),\mu)\,ds$$

Clearly

$$|\varphi_0(t,\tau,\xi,\mu) - \psi(t)| = |\xi - \psi(\tau)|$$

and

$$|\varphi_1(t,\tau,\xi,\mu) - \varphi_0(t,\tau,\xi,\mu)| = \left| \int_\tau^t \{f(s,\varphi_0(s,\tau,\xi,\mu),\mu) - f(s,\psi(s),\mu_0)\}\,ds \right| \tag{7.20}$$

The uniform continuity of f in $U_{1\mu}$ implies that, given any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$|f(s,\varphi_0(s,\tau,\xi,\mu),\mu) - f(s,\psi(s),\mu_0)| < \epsilon$$

provided that $a \leqq s \leqq b$, $(\tau,\xi,\mu) \in U_{1\mu}$, and

$$|\xi - \psi(\tau)| + |\mu - \mu_0| < \delta_\epsilon \tag{7.21}$$

Thus (7.20) implies

$$|\varphi_1(t,\tau,\xi,\mu) - \varphi_0(t,\tau,\xi,\mu)| < \epsilon|t - \tau|$$

provided (7.21) is valid. Proceeding as before, there now results

$$|\varphi_{j+1}(t,\tau,\xi,\mu) - \varphi_j(t,\tau,\xi,\mu)| \leqq \frac{\epsilon|t-\tau|^{j+1}k^j}{(j+1)!}$$

where k is the Lipschitz constant. Let ϵ be chosen so that

$$\frac{\epsilon}{k}(e^{k(b-a)} - 1) < \frac{\delta_1}{2}$$

and let $\delta = \delta_\epsilon < \delta_1/2$ be chosen as above for this ϵ. Then it follows easily by induction that, for all j, $(t,\varphi_j(t,\tau,\xi,\mu))$ is in the region $U_{1\mu}$ for all $(\tau,\xi,\mu) \in U_\mu$. The continuity and the uniform convergence of the φ_j on V_μ lead to the result of the theorem.

The generalization of Theorem 7.2 to $(\mathbf{E}_\mu)$ is valid. In fact, it follows directly from Theorem 7.2 itself.

Theorem 7.5. *Let the hypothesis of Theorem 7.4 be satisfied and suppose that $f_x \in C$, $f_\mu \in C$, on D_μ. Then the solution φ defined in Theorem 7.4 is of class C^1 on V_μ.*

Proof. Consider the $(n + k)$-dimensional u space consisting of points with coordinates

$$u_i = x_i \qquad (i = 1, \ldots, n)$$
$$u_{i+n} = \mu_i \qquad (i = 1, \ldots, k)$$

and define the vector function $F = (F_1, \ldots, F_{n+k})$ on D_μ by

$$F_i(t,u) = f_i(t,x,\mu) \qquad (i = 1, \ldots, n)$$
$$F_{i+n}(t,u) = 0 \qquad (i = 1, \ldots, k)$$

Then, by Theorem 7.1, the system of equations

$$u' = F(t,u) \tag{7.22}$$

has for a solution the vector $\chi = (\chi_1, \ldots, \chi_{n+k})$ given by

$$\chi_i(t) = \varphi_i(t,\tau,\xi,\mu) \qquad (i = 1, \ldots, n)$$
$$\chi_{i+n}(t) = \mu_i \qquad (i = 1, \ldots, k)$$

since χ has the initial value given by

$$\chi_i(\tau) = \xi_i \qquad (i = 1, \ldots, n)$$
$$\chi_{i+n}(\tau) = \mu_i \qquad (i = 1, \ldots, k)$$

Thus $\mu_1, \ldots, \mu_k$ may be thought of as forming part of the components of an initial-value vector for the system (7.22), and the F in (7.22) satisfies the conditions in Theorem 7.2. Therefore the first partial derivatives of χ with respect to τ, ξ_i, and μ_i exist and are continuous on V_μ, and from the definition of χ this implies the theorem.

From

$$\varphi(t,\tau,\xi,\mu) = \xi + \int_\tau^t f(s,\varphi(s,\tau,\xi,\mu),\mu)\,ds$$

it follows that

$$\frac{\partial\varphi}{\partial\mu_j}(t,\tau,\xi,\mu) = \int_\tau^t \left[f_x(s,\varphi(s,\tau,\xi,\mu),\mu)\frac{\partial\varphi}{\partial\mu_j}(s,\tau,\xi,\mu) + \frac{\partial f}{\partial\mu_j}(s,\varphi(s,\tau,\xi,\mu),\mu) \right] ds$$

This shows that $\partial\varphi/\partial\mu_j$ is the solution of the initial-value problem

$$y' = f_x(t,\varphi(t,\tau,\xi,\mu),\mu)y + \frac{\partial f}{\partial\mu_j}(t,\varphi(t,\tau,\xi,\mu),\mu) \qquad y(\tau) = 0$$

Hypotheses under which the existence of higher derivatives of φ with respect to τ, ξ_i, or μ_i can be shown to exist are readily ascertained from the fact that the first-order derivatives are solutions of a linear equation. For example, $\partial\varphi/\partial\xi_i$ is the solution β_i of

$$y' = f_x(t,\varphi(t,\tau,\xi,\mu),\mu)y \tag{7.23}$$

with the initial value e_i. Clearly $\partial^2\varphi/\partial\xi_j\partial\xi_i$ is $\partial\beta_i/\partial\xi_j$, if it exists. But ξ enters (7.23) as a parameter. If τ and μ are held fixed in (7.23), then ξ in (7.23) plays the role of μ in Theorem 7.5. Thus, if $f_x(t,\varphi(t,\tau,\xi,\mu),\mu)$ has a continuous derivative with respect to ξ_j, then $\partial\beta_i/\partial\xi_j$ exists. If f has continuous partial derivatives of the second order with respect to the components of x, then $f_x(t,\varphi(t,\tau,\xi,\mu),\mu)$ will have continuous first-order partial derivatives with respect to ξ_j.

In much the same way, if f has continuous partial derivatives of the second order with respect to the components of (x,μ), then $\partial^2\varphi/\partial\mu_i\partial\mu_j$

exists as do the mixed derivatives $\partial^2\varphi/\partial\mu_i\partial\xi_j$. The case where the partial derivatives of φ are taken with respect to the components of (τ,ξ,μ) is left to the reader as an exercise.

8. Complex Systems

So far it has been assumed in the equation (E) that t,x,f were all real. If f is a continuous complex-valued function on an open connected set D in the (t,w) space, where t is real and w is complex n-dimensional (real $2n$-dimensional), then the equation

$$(E_1) \qquad w' = f(t,w)$$

is defined to be the problem of finding an interval I on the real t line and a (complex) differentiable function φ on I such that

(i) $\quad (t,\varphi(t)) \in D \qquad (t \in I)$

(ii) $\quad \varphi'(t) = f(t,\varphi(t)) \qquad \left(t \in I, \ ' = \dfrac{d}{dt}\right)$

It is an easy task to see that all the existence, uniqueness, continuation, and dependence theorems proved in Secs. 1 to 7 are valid for (E_1) as well, if one defines the norm $|w|$ of a complex vector $w = (w_1, \ldots, w_n)$ formally as before, namely,

$$|w| = \sum_{i=1}^{n} |w_i|$$

Here, of course, $|w_i| = ((\Re w_i)^2 + (\Im w_i)^2)^{\frac{1}{2}}$, where $\Re w_i$ and $\Im w_i$ are the real and imaginary parts of w_i. Moreover, the Theorems 7.4 and 7.5 concerning the equation

$$(E_\mu) \qquad x' = f(t,x,\mu)$$

can be extended in an obvious way to the case where μ is a complex parameter vector, if f is defined for complex x and μ. Linear systems are an important case where the above remarks apply.

Usually a function defined on a set of complex numbers that occurs in a differential equation is analytic. Let F be a vector function defined on a domain (open connected set) D of the complex n-dimensional w space. Then F is said to be *analytic* at a point $\omega \in D$ if in some neighborhood $|w - \omega| < \rho$, $\rho > 0$, each component F_j of F is continuous in

$$w = (w_1, \ldots, w_n)$$

and is analytic in each w_k when all other w_l, $l \neq k$, are held fixed. An equivalent definition is that each F_j is representable by a convergent power series

$$F_j(w_1, \ldots, w_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} A_{m_1 \ldots m_n}(w_1 - \omega_1)^{m_1} \cdots (w_n - \omega_n)^{m_n}$$

in some neighborhood $|w - \omega| < \rho$, $\rho > 0$. The $A_{m_1 \ldots m_n}$ are complex constants. A function F is said to be analytic in a domain D if it is analytic at each point of D.

It will be recalled that an analytic function in a domain D possesses derivatives of all orders on D. A basic property of analytic functions is that, if a sequence of analytic functions converges uniformly on a domain D, then the limit function is analytic in D.

It is evident that since an analytic function F in D is represented locally by a power series it is locally single-valued, that is, for every point $\omega \in D$ there is a $\rho > 0$ such that F is single-valued on $|w - \omega| < \rho$. However, in the large, it need not be single-valued. For example, the function F given by $F(w) = w^{\frac{1}{2}}$, where w has one complex dimension, is analytic in the ring $1 < |w| < 2$ but is double-valued there. If $w^{\frac{1}{2}}$ is taken as positive and real on the interval $1 < \Re w < 2$ and w is followed around a closed path ($|w| = \frac{3}{2}$, for example), then $w^{\frac{1}{2}}$ assumes negative real values when w again reaches the positive real axis. The function $F(w) = w^\alpha$, α real and irrational, assumes infinitely many values in the ring.

An important extension of the problem (E) is to the case where t may be complex. Suppose that f is an analytic complex-valued vector function defined on a domain D in the complex (z,w) space, where the z space has one complex dimension, and the w space is complex n-dimensional. Then the equation

$$(E_2) \qquad w' = f(z,w)$$

is defined to be the problem of finding a domain H in the complex z plane and a (complex) differentiable locally single-valued function φ [a solution of (E_2)] on H such that

(i) $\quad (z,\varphi(z)) \in D \qquad (z \in H)$

(ii) $\quad \varphi'(z) = f(z,\varphi(z)) \qquad \left(z \in H, \ ' = \dfrac{d}{dz}\right)$

The existence and uniqueness of solutions of (E_2) can be inferred from the method of successive approximations. Indeed, suppose f has components $f_1, \ldots, f_n$, and $w = (w_1, \ldots, w_n)$, and f is analytic on the domain

$$R_2: \qquad |z - z_0| < a \qquad |w - w_0| < b \qquad (a,b > 0)$$

which will be called a rectangle, although it is $n + 1$ complex dimensional. Note that w_0 is a vector here and not a component.

Theorem 8.1. *Suppose f is analytic and bounded on the open rectangle R_2, and let*

$$M = \sup_{(z,w) \in R_2} |f(z,w)| \qquad \alpha = \min\left(a, \frac{b}{M}\right)$$

Then there exists on $|z - z_0| < \alpha$ a unique analytic function φ which is a solution of (E_2) satisfying $\varphi(z_0) = w_0$.

Proof. Since the matrix $f_w = (\partial f_i/\partial w_j)$ is bounded on any closed rectangle $\bar{R}_2 \subset R_2$, it follows that f satisfies a Lipschitz condition on $\bar{R}_2$. Therefore one can construct the successive approximations

$$\varphi_0(z) = w_0$$
$$\varphi_{k+1}(z) = w_0 + \int_{z_0}^{z} f(\zeta,\varphi_k(\zeta))\, d\zeta \qquad (k = 0, 1, 2, \ldots) \tag{8.1}$$

where the integrals can be taken along a straight line joining z_0 to z. Applying the argument in Theorem 3.1, one obtains the existence of a unique solution φ on the circle $|z - z_0| < \alpha$ which satisfies $\varphi(z_0) = w_0$. Clearly φ_0 is analytic in z on $|z - z_0| < \alpha$, and thus the function f_0 defined by $f_0(z) = f(z,\varphi_0(z))$, being an analytic function of an analytic function, is analytic on $|z - z_0| < \alpha$. From (8.1) it follows that φ_1 is analytic on $|z - z_0| < \alpha$, and an easy induction proves that all the approximations φ_k are analytic on $|z - z_0| < \alpha$. Since the solution φ is the uniform limit of the sequence $\{\varphi_k\}$ of analytic functions, it is itself analytic on $|z - z_0| < \alpha$. This completes the proof.

REMARK: Unless other restrictive assumptions are made on f, the circle of analyticity $|z - z_0| < \alpha$ cannot be improved. For $a \leq b/M$, this is illustrated by the case where f is independent of w, and has singularities on the circle $|z - z_0| = a$. For $a > b/M$ the example

$$w' = f(w) = M\left[\frac{1}{2}\left(1 + \frac{w}{b}\right)\right]^{1/m}$$

where w is one dimensional, illustrates this. The solution φ of this equation for which $\varphi(0) = 0$ (here $z_0 = w_0 = 0$), is

$$\varphi(z) = b\left[\left(1 + \frac{z}{c_m}\right)^{m/(m-1)} - 1\right]$$

where

$$c_m = \left(\frac{m2^{1/m}}{m-1}\right)\frac{b}{M}$$

Clearly f is analytic and bounded in the circle $|w| < b$, and $\sup |f(w)| = M$ there. The solution φ has a singular point at $z = -c_m < -b/M$, and this tends to $z = -b/M$ as $m \to \infty$. Therefore, for any given $r > b/M$, the solution φ has a singularity in the region

$$\frac{b}{M} < |z| < r$$

if m is made large enough.

The analogue of Theorem 7.1 for the equation (E_2) is the following result:

Theorem 8.2. *Let f be analytic in a domain D of the (z,w) space, and suppose ψ is a solution of (E_2) on H, where H is a closed convex domain of the z plane. There exists a $\delta > 0$ such that for any $(\zeta,\omega) \in U$, where*

$$U: \qquad \zeta \in H \qquad |\omega - \psi(\zeta)| < \delta$$

there exists a unique solution $\varphi = \varphi(z,\zeta,\omega)$ of (E_2) on H with $\varphi(\zeta,\zeta,\omega) = \omega$. Moreover, φ is analytic on the $n + 2$ complex dimensional domain

$$V: \qquad z \in H \qquad (\zeta,\omega) \in U$$

REMARK: Actually H need not be convex. It is sufficient if H is simply connected and if there is some constant $c > 0$ such that any two points of H may be joined by a polygonal arc of length less than c.

Proof of Theorem 8.2. The proof follows that part of the proof of Theorem 7.1 that deals with the successive approximations. The path of integration from ζ to z in the successive approximations can be taken as a straight line if H is convex. In any case, the path can be taken as a polygonal path of length less than c. The argument of Theorem 7.1 carries over with the obvious modifications necessary to meet the requirements that the variables are complex. The approximations φ_j are all analytic on V. Thus the limit function, to which the approximations converge uniformly on V, must be analytic on V.

Since φ has all derivatives with respect to z,ζ,ω on V, the equation

$$\varphi'(z,\zeta,\omega) = f(z,\varphi(z,\zeta,\omega))$$

can be differentiated with respect to ω_j, giving

$$\frac{\partial \varphi'}{\partial \omega_j}(z,\zeta,\omega) = f_w(z,\varphi(z,\zeta,\omega))\frac{\partial \varphi}{\partial \omega_j}$$

Thus $\partial\varphi/\partial\omega_j$ is the solution of the linear equation

$$y' = f_w(z,\varphi(z,\zeta,\omega))y \tag{8.2}$$

with initial condition $(\partial\varphi/\partial\omega_j)(\zeta,\zeta,\omega) = e_j$. Thus the analogue of the main result of Theorem 7.2 is proved. The result analogous to (7.5) follows in much the same way as (7.5). The result here is