Math 7380 - 3
Final Exam
Due Sat., Dec. 12, 2009

You may use notes and written literature, but collaboration or consultation with other people is not allowed. If you have questions about the meaning of a problem, please talk to me (Stephen Shipman) about it.
There are four problems.

1. For the linearized KdV equation

$$u_t + a u_x = b u_{xxx},$$

(a) find the dispersion relation, relating ω to k in the solutions

$$u(x,t) = e^{i(kx-\omega t)}.$$

(b) For initial data $u(x,0) = e^{-x^2}$, write the solution in the form of a Fourier integral.

(c) Find the leading-order asymptotics of $u(mt,t)$ as $t \to \infty$, where $m \in \mathbb{R}$.

2. Consider the nonlinear advection equation

$$u_t + c(u) u_x = 0, \qquad c(u) = u - u^3,$$

with initial data

$$u(x,0) = e^{-\alpha|x|},$$

where $\alpha > 0$.

(a) Determine α such that a shock is initiated at time $t = 3$.

(b) Find the value of x at which the shock is initiated.

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3. The scalar curl of a vector field $F(x,y) = \langle F_1(x,y), F_2(x,y) \rangle$ in an open set Ω of $\mathbb{R}^2$ is defined by

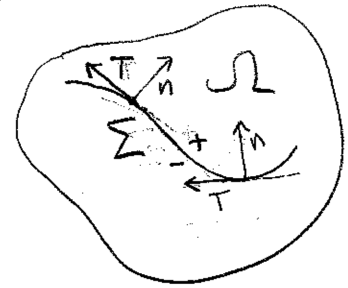
$$\operatorname{curl} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

as long as F is of class C^1, and integration by parts leads to the following definition of the curl of a vector distribution $F = \langle F_1, F_2 \rangle$:

$$\langle \operatorname{curl} F, \phi \rangle = -\int \nabla\phi \cdot \langle F_2; F_1 \rangle$$

for all $\phi \in C_c^\infty(\Omega)$.

Let Σ be a smooth curve contained in Ω, let u be of class C^2 in $\Omega \setminus \Sigma$ and let u and ∇u be continuous up to Σ from each side.



Prove that, if u satisfies

$$\operatorname{curl}(\sigma \nabla u) = f$$

in the distributional sense in Ω, where σ is a 2×2 matrix with C^1 entries in Ω and f is a measurable function in Ω, then the tangential component of ∇u is continuous across Σ, that is,

$$\nabla u \cdot T \big|_+ = \nabla u \cdot T \big|_-$$

more →

4. Denote by Ω the subset $[0,P]\times[0,L]$ of $\mathbb{R}^2$, and let ε be a measurable function on Ω such that $0<\varepsilon_-<\varepsilon(x,y)<\varepsilon_+$. Consider the following form in $L^2(\Omega,\varepsilon)$ $\left((u,v)=\int_\Omega \varepsilon u\bar{v}\right)$:

$$\begin{cases} Q(a)=\left\{u\in H^1(\Omega):\ u(x,0)=0,\ u(P,y)=e^{ik}u(0,y)\right\}, \\ a(u,v)=\displaystyle\int_\Omega \sigma\nabla u\cdot\nabla\bar{v}, \quad \text{for } u,v\in Q(a), \end{cases}$$

where $0<\sigma_-<\sigma(x,y)<\sigma_+$ in Ω.

(a) Find the associated self-adjoint operator A in $L^2(\Omega,\varepsilon)$ such that $a(u,v)=(Au,v)$ $\forall$ u s.th. $a(u,v)\le C_u\|v\|_{L^2}$ for some $C_u>0$. Of course, you must specify the domain and action of A.

(b) Find all the eigenvalues λ_j and corresponding eigenfunctions ψ_j of A, that is,

$$(A-\lambda_j)\psi_j=0,$$

with $\varepsilon\equiv 1$ and $\sigma\equiv 1$ (all of them are separable functions).

(c) Solve $Au-\lambda u=e^{i\frac{k}{P}x}$ as explicitly as possible, where λ is not an eigenvalue of A.