.7.5. Show (by completing square of quadratics) that the minimum of

$$\iint_R [\frac{1}{2}(\nabla U)^2 - f(x,y)U]\, dA,$$

where U satisfies (6.7.3), occurs when $KU = F$.

.7.6. Consider a somewhat arbitrary triangle (as illustrated in Figure 6.7.5) with $P_1 = (0,0)$, $P_2 = (L,0)$, $P_3 = (D,H)$ and interior angles θ_i. The solution on the triangle will be linear $U = a + bx + cy$.

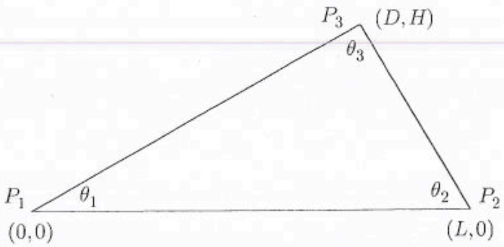

Figure 6.7.5 Triangular finite element.

(a) Show that $\iint_R (\nabla U)^2\, dA = (b^2 + c^2)\frac{1}{2}LH$.

(b) The coefficients a, b, c are determined by the conditions at the three vertices $U(P_i) = U_i$. Demonstrate that $a = U_1, b = \frac{U_2 - U_1}{L}$, and $c = \frac{U_3 - U_1 - \frac{D}{L}(U_2 - U_1)}{H}$.

(c) Show that $\frac{1}{\tan\theta_1} = \frac{D}{H}$, $\frac{1}{\tan\theta_2} = \frac{L-D}{H}$, and using $\tan\theta_3 = -\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{\tan\theta_1 \tan\theta_2 - 1}$ show that $\frac{1}{\tan\theta_3} = \frac{H}{L} - \frac{D}{H} + \frac{D^2}{HL}$.

(d) Using Exercise 6.7.4 and parts (a), (b), (c) of this exercise, show that for the contribution from this one triangle, $K_{12} = -\frac{1}{2\tan\theta_3}$. The other entries of the stiffness matrix follow in this way.

5.7.7. Continue with part (d) of Exercise 6.7.6 to obtain

(a) K_{11} (b) K_{22} (c) K_{33} (d) K_{23} (e) K_{13}

Chapter 7

Higher Dimensional Partial Differential Equations

7.1 Introduction

In our discussion of partial differential equations, we have solved many problems by the method of separation of variables, but all involved only two independent variables:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) \qquad \rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2}.$$

In this chapter we show how to extend the method of separation of variables to problems with more than two independent variables.

In particular, we discuss techniques to analyze the heat equation (with constant thermal properties) in two and three dimensions,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{(two dimensions)} \tag{7.1.1}$$

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \text{(three dimensions)} \tag{7.1.2}$$

for various physical regions with various boundary conditions. Also of interest will be the steady-state heat equation, Laplace's equation, in three dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

In all these problems, the partial differential equation has at least three independent variables. Other physical problems, not related to the flow of thermal energy, may also involve more than two independent variables. For example, the vertical displacement u of a vibrating membrane satisfies the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

It should also be mentioned that in acoustics, the perturbed pressure u satisfies the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

We will discuss and analyze some of these problems.

7.2 Separation of the Time Variable

We will show that similar methods can be applied to a variety of problems. We will begin by discussing the vibrations of a membrane of any shape, and follow that with some analysis for the conduction of heat in any two- or three-dimensional region.

7.2.1 Vibrating Membrane: Any Shape

Let us consider the displacement u of a vibrating membrane of any shape. Later (Secs. 7.3 and 7.7) we will specialize our result to rectangular and circular membranes. The displacement $u(x, y, t)$ satisfies the two-dimensional wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).} \tag{7.2.1}$$

The initial conditions will be

$$u(x, y, 0) = \alpha(x, y) \tag{7.2.2}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y), \tag{7.2.3}$$

but as usual they will be ignored at first when separating variables. A homogeneous boundary condition will be given along the entire boundary; $u = 0$ on the boundary is the most common condition. However, it is possible, for example, for the displacement to be zero on only part of the boundary and for the rest of the boundary to be "free." There are many other possible boundary conditions.

Let us now apply the method of separation of variables. We begin by showing that the time variable can be separated out from the problem for a membrane of

any shape by seeking product solutions of the following form:

$$\boxed{u(x, y, t) = h(t)\phi(x, y).} \tag{7.2.4}$$

Here $\phi(x, y)$ is an as yet unknown function of the two variables x and y. We do not (at this time) specify further $\phi(x, y)$ since we might expect different results in different geometries or with different boundary conditions. Later, we will show that for rectangular membranes $\phi(x, y) = F(x)G(y)$, while for circular membranes $\phi(x, y) = F(r)G(\theta)$; that is, the form of further separation depends on the geometry. It is for this reason that we begin by analyzing the general form (7.2.4). In fact, for most regions that are *not* geometrically as simple as rectangles and circles, $\phi(x, y)$ cannot be separated further. If (7.2.4) is substituted into the equation for a vibrating membrane, (7.2.1), then the result is

$$\phi(x, y)\frac{d^2 h}{dt^2} = c^2 h(t) \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right). \tag{7.2.5}$$

We will attempt to proceed as we did when there were only two independent variables. Time can be separated from (7.2.5) by dividing by $h(t)\phi(x, y)$ (and an additional division by the constant c^2 is convenient):

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2 h}{dt^2} = \frac{1}{\phi}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = -\lambda. \tag{7.2.6}$$

The left-hand side of the first equation is only a function of time, while the right-hand side is only a function of space (x and y). Thus, the two (as before) must equal a separation constant. Again, we must decide what notation is convenient for the separation constant, λ or $-\lambda$. A quick glance at the resulting ordinary differential equation for $h(t)$ shows that $-\lambda$ is more convenient (as will be explained). We thus obtain two equations, but unlike the case of two independent variables, one of the equations is itself still a partial differential equation:

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$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda\phi.} \tag{7.2.8}$$

The notation $-\lambda$ for the separation constant was chosen because the time-dependent differential equation (7.2.7) has oscillatory solutions if $\lambda > 0$. If $\lambda > 0$, then h is a linear combination of $\sin c\sqrt{\lambda}t$ and $\cos c\sqrt{\lambda}t$; it oscillates with frequency $c\sqrt{\lambda}$. The values of λ determine the natural frequencies of oscillation of a vibrating

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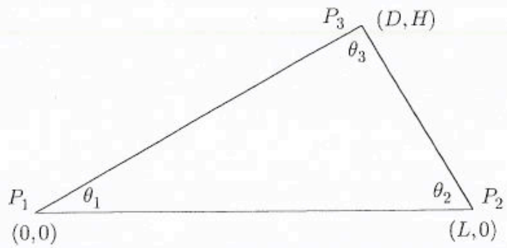

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7.2.2 Heat Conduction: Any Region

We will analyze the flow of thermal energy in any two-dimensional region. We begin by seeking product solutions of the form

$$\boxed{u(x,y,t) = h(t)\phi(x,y)} \tag{7.2.9}$$

for the two-dimensional heat equation, assuming constant thermal properties and no sources, (7.1.1). By substituting (7.2.9) into (7.1.1) and after dividing by $kh(t)\phi(x,y)$, we obtain

$$\frac{1}{k}\frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right). \tag{7.2.10}$$

A separation constant in the form $-\lambda$ is introduced so that the time-dependent part of the product solution exponentially decays (if $\lambda > 0$) as expected, rather than exponentially grows. Then, the two equations are

$$\boxed{\begin{aligned} \frac{dh}{dt} &= -\lambda kh \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= -\lambda\phi. \end{aligned}} \tag{7.2.11}$$

The eigenvalue λ relates to the decay rate of the time-dependent part. The eigenvalue λ is determined by the boundary value problem, again consisting of the partial differential equation (7.2.11) with a corresponding boundary condition on the entire boundary of the region.

For heat flow in any three-dimensional region, (7.1.2) is valid. A product solution,

$$\boxed{u(x,y,z,t) = h(t)\phi(x,y,z),} \tag{7.2.12}$$

may still be sought, and after separating variables, we obtain equations similar to

(7.2.11),

$$\boxed{\begin{aligned} \frac{dh}{dt} &= -\lambda kh \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} &= -\lambda\phi. \end{aligned}} \tag{7.2.13}$$

The eigenvalue λ is determined by finding those values of λ for which nontrivial solutions of (7.2.13) exist, subject to a homogeneous boundary condition on the entire boundary.

7.2.3 Summary

In situations described in this section the spatial part $\phi(x,y)$ or $\phi(x,y,z)$ of the solution of the partial differential equation satisfies the eigenvalue problem consisting of the partial differential equation,

$$\boxed{\nabla^2\phi = -\lambda\phi,} \tag{7.2.14}$$

with ϕ satisfying appropriate homogeneous boundary conditions, which may be of the form [see (1.5.2) and (4.5.5)]

$$\alpha\phi + \beta\nabla\phi\cdot\hat{n} = 0, \tag{7.2.15}$$

where α and β can depend on x, y, and z. If $\beta = 0$, (7.2.15) is the fixed boundary condition. If $\alpha = 0$, (7.2.15) is the insulated or free boundary condition. If both $\alpha \neq 0$ and $\beta \neq 0$, then (7.2.15) is the higher-dimensional version of Newton's law of cooling or the elastic boundary condition. In Sec. 7.4 we will describe general results for this two- or three-dimensional eigenvalue problem, similar to our theorems concerning the general one-dimensional Sturm-Liouville eigenvalue problem. However, first we will describe the solution of a simple two-dimensional eigenvalue problem in a situation in which $\phi(x,y)$ may be further separated, producing two one-dimensional eigenvalue problems.

EXERCISES 7.2

7.2.1. For a vibrating membrane of any shape that satisfies (7.2.1), show that (7.2.14) results after separating time.

7.2.2. For heat conduction in any two-dimensional region that satisfies (7.1.1), show that (7.2.14) results after separating time.

7.2.3. (a) Obtain product solutions, $\phi = f(x)g(y)$, of (7.2.14) that satisfy $\phi = 0$ on the four sides of a rectangle. (*Hint*: If necessary, see Sec. 7.3.)

(b) Using part (a), solve the initial value problem for a vibrating rectangular membrane (fixed on all sides).

(c) Using part (a), solve the initial value problem for the two-dimensional heat equation with zero temperature on all sides.

7.3 Vibrating Rectangular Membrane

In this section we analyze the vibrations of a rectangular membrane, as sketched in Fig. 7.3.1. The vertical displacement $u(x, y, t)$ of the membrane satisfies the two-dimensional wave equation,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).} \tag{7.3.1}$$

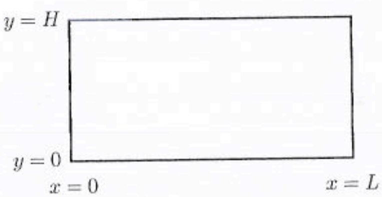

Figure 7.3.1 Rectangular membrane.

We suppose that the boundary is given such that all four sides are fixed with zero displacement:

$$u(0, y, t) = 0 \qquad u(x, 0, t) = 0 \tag{7.3.2}$$
$$u(L, y, t) = 0 \qquad u(x, H, t) = 0. \tag{7.3.3}$$

We ask what is the displacement of the membrane at time t if the initial position and velocity are given:

$$u(x, y, 0) = \alpha(x, y) \tag{7.3.4}$$
$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y). \tag{7.3.5}$$

As we indicated in Sec. 7.2.1, since the partial differential equation and the boundary conditions are linear and homogeneous, we apply the method of separation of variables. First, we separate only the time variable by seeking product solutions in the form

$$u(x, y, t) = h(t)\phi(x, y). \tag{7.3.6}$$

According to our earlier calculation, we are able to introduce a separation constant $-\lambda$, and the following two equations result:

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \tag{7.3.7}$$
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi. \tag{7.3.8}$$

We will show that $\lambda > 0$, in which case $h(t)$ is a linear combination of $\sin c\sqrt{\lambda}t$ and $\cos c\sqrt{\lambda}t$. The homogeneous boundary conditions imply that the eigenvalue problem is

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi} \tag{7.3.9}$$

$$\boxed{\begin{array}{ll} \phi(0, y) = 0 & \phi(x, 0) = 0 \\ \phi(L, y) = 0 & \phi(x, H) = 0; \end{array}} \tag{7.3.10}$$

that is, $\phi = 0$ along the entire boundary. We call (7.3.9)–(7.3.10) a two-dimensional eigenvalue problem.

The eigenvalue problem itself is a linear homogeneous PDE in two independent variables with homogeneous boundary conditions. As such (since the boundaries are simple), we can expect that (7.3.9)–(7.3.10) can be solved by separation of variables in Cartesian coordinates. In other words, we look for product solutions of (7.3.9)–(7.3.10) in the form

$$\phi(x, y) = f(x)g(y). \tag{7.3.11}$$

Before beginning our calculations, let us note that it follows from (7.3.6) that our assumption (7.3.11) is equivalent to

$$u(x, y, t) = f(x)g(y)h(t) \tag{7.3.12}$$

a product of functions of each independent variable. We claim, as we show in an appendix to this section, that we could obtain the same result by substituting (7.3.12) into the wave equation (7.3.1) as we now obtain by substituting (7.3.11) into the two-dimensional eigenvalue problem (7.3.9):

$$g(y)\frac{d^2 f}{dx^2} + f(x)\frac{d^2 g}{dy^2} = -\lambda f(x)g(y). \tag{7.3.13}$$

The x and y parts may be separated by dividing (7.3.13) by $f(x)g(y)$ and rearranging terms:

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2 g}{dy^2} = -\mu. \tag{7.3.14}$$

Since the first expression is only a function of x, while the second is only a function of y we introduce a *second* separation constant. We choose it to be $-\mu$ so that the easily solved equation, $d^2f/dx^2 = -\mu f$ has oscillatory solutions (as expected) if $\mu > 0$. Two ordinary differential equations result from separation of variables of a partial differential equation with two independent variables:

$$\boxed{\frac{d^2f}{dx^2} = -\mu f} \tag{7.3.15}$$

$$\boxed{\frac{d^2g}{dy^2} = -(\lambda - \mu)g.} \tag{7.3.16}$$

Equations (7.3.15) and (7.3.16) contain *two* separation constants λ and μ, both of which must be determined. In addition, $h(t)$ solves an ordinary differential equation:

$$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h.} \tag{7.3.17}$$

When we separate variables for a partial differential equation in three variables, $u(x, y, t) = f(x)g(y)h(t)$, we obtain three ordinary differential equations, one a function of each independent coordinate. However, there will be only two separation constants.

To determine the separation constants, we need to use the homogeneous boundary conditions (7.3.10). The product form (7.3.11) then implies that

$$\begin{aligned} f(0) &= 0 \text{ and } f(L) = 0 \\ g(0) &= 0 \text{ and } g(H) = 0. \end{aligned} \tag{7.3.18}$$

Of our three ordinary differential equations, only two will be eigenvalue problems. There are homogeneous boundary conditions in x and y. Thus,

$$\frac{d^2f}{dx^2} = -\mu f \quad \text{with} \quad f(0) = 0 \text{ and } f(L) = 0 \tag{7.3.19}$$

is a Sturm-Liouville eigenvalue problem in the x-variable, where μ is the eigenvalue and $f(x)$ is the eigenfunction. Similarly, the y-dependent problem is a regular Sturm-Liouville eigenvalue problem:

$$\frac{d^2g}{dy^2} = -(\lambda - \mu)g \quad \text{with} \quad g(0) = 0 \text{ and } g(H) = 0. \tag{7.3.20}$$

Here λ is the eigenvalue and $g(y)$ the corresponding eigenfunction.

Not only are both (7.3.19) and (7.3.20) Sturm-Liouville eigenvalue problems, but they are both ones we should be quite familiar with. Without going through the well-known details, the eigenvalues are

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots, \tag{7.3.21}$$

and the corresponding eigenfunctions are

$$f_n(x) = \sin\frac{n\pi x}{L}. \tag{7.3.22}$$

This determines the allowable values of the separation constant μ_n.

For each value of μ_n, (7.3.20) is still an eigenvalue problem. There is infinite number of eigenvalues λ for each n. Thus, λ should be double subscripted, λ_{nm}. In fact, from (7.3.20) the eigenvalues are

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3, \ldots, \tag{7.3.23}$$

where we *must* use a different index to represent the various y-eigenvalues (for each value of n). The corresponding y-eigenfunction is

$$g_{nm}(y) = \sin\frac{m\pi y}{H} \tag{7.3.24}$$

The separation constant λ_{nm} now can be determined from (7.3.23):

$$\boxed{\lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2,} \tag{7.3.25}$$

where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. The two-dimensional eigenvalue problem (7.3.9) has eigenvalues λ_{nm} given by (7.3.25) and eigenfunctions given by the product of the two one-dimensional eigenfunctions. Using the notation $\phi_{nm}(x, y)$ for the two-dimensional eigenfunction corresponding to the eigenvalue λ_{nm}, we have

$$\boxed{\phi_{nm}(x, y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H},} \quad \begin{array}{l} n = 1, 2, 3, \ldots \\ m = 1, 2, 3, \ldots. \end{array} \tag{7.3.26}$$

Note how easily the homogeneous boundary conditions are satisfied.

From (7.3.25) we have explicitly shown that all the eigenvalues are positive (for this problem). Thus, the time-dependent part of the product solutions are (as previously guessed) $\sin c\sqrt{\lambda_{nm}}t$ and $\cos c\sqrt{\lambda_{nm}}t$, oscillations with natural frequencies $c\sqrt{\lambda_{nm}} = c\sqrt{(n\pi/L)^2 + (m\pi/H)^2}, n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. In considering the displacement u, we have obtained two doubly infinite families of product solutions: $\sin n\pi x/L \sin m\pi y/H \sin c\sqrt{\lambda_{nm}}t$ and $\sin n\pi x/L \sin m\pi y/H \cos c\sqrt{\lambda_{nm}}t$. As with the vibrating string, each of these special product solutions is known as a mode of vibration. We sketch in Fig. 7.3.2 a representation of some of these modes. In each we sketch level contours of displacement in dotted lines at a fixed t. As time varies the shape stays the same, only the amplitude varies periodically. Each mode is a standing wave. Curves along which the displacement is always zero in a mode are called **nodal curves** and are sketched in solid lines. Cells are apparent with neighboring cells always being out of phase; that is, when one cell has a positive

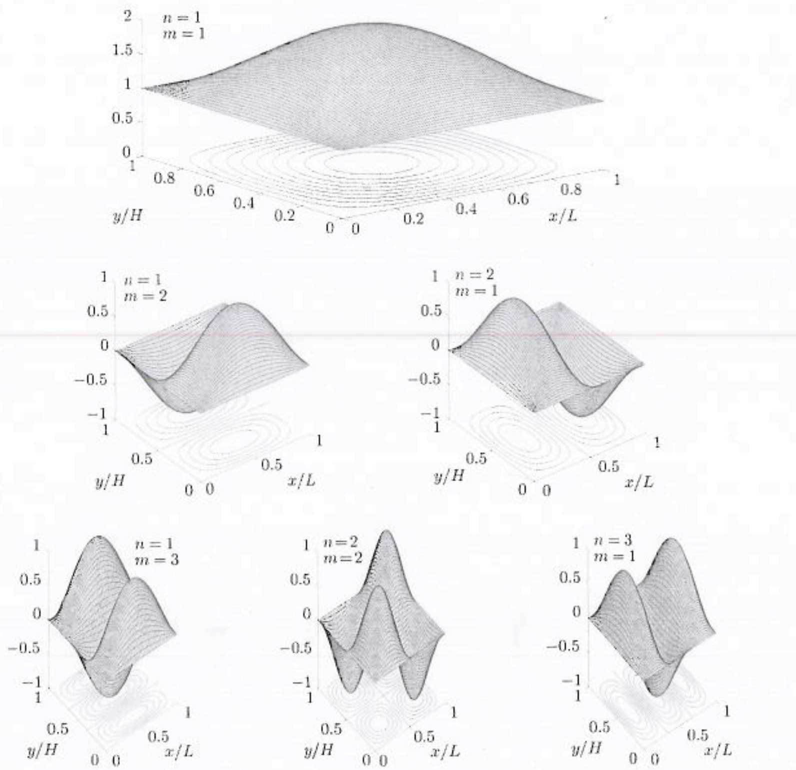

Figure 7.3.2 Nodal curves for modes of a vibrating rectangular membrane.

displacement the neighbor has negative displacement (as represented by the + and − signs).

The principle of superposition implies that we should consider a linear combination of all possible product solutions. Thus, we must include both families, summing over both n and m,

$$\boxed{\begin{aligned} u(x,y,t) = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\cos c\sqrt{\lambda_{nm}}t \\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\sin c\sqrt{\lambda_{nm}}t. \end{aligned}} \tag{7.3.27}$$

The two families of coefficients A_{nm} and B_{nm} hopefully will be determined from

the two initial conditions. For example, $u(x,y,0) = \alpha(x,y)$ implies that

$$\alpha(x,y) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} A_{nm}\sin\frac{n\pi x}{L}\right)\sin\frac{m\pi y}{H}. \tag{7.3.28}$$

The series in (7.3.28) is an example of what is called a **double Fourier series**. Instead of discussing the theory, we show one method to calculate A_{nm} from (7.3.28). (In Sec. 7.4 we will discuss a simpler way.) For *fixed* x, we note that $\sum_{n=1}^{\infty} A_{nm}\sin n\pi x/L$ depends only on m; furthermore, it must be the coefficients of the Fourier sine series in y of $\alpha(x,y)$ over $0 < y < H$. From our theory of Fourier sine series, we therefore know that we may easily determine the coefficients:

$$\sum_{n=1}^{\infty} A_{nm}\sin\frac{n\pi x}{L} = \frac{2}{H}\int_0^H \alpha(x,y)\sin\frac{m\pi y}{H}\,dy, \tag{7.3.29}$$

for each m. Equation (7.3.29) is valid for all x; the right-hand side is a function of x (not y, because y is integrated from 0 to H). For each m, the left-hand side is a Fourier sine series in x; in fact, it is the Fourier sine series of the right-hand side, $2/H\int_0^H \alpha(x,y)\sin m\pi y/H\,dy$. The coefficients of this Fourier sine series in x are easily determined:

$$\boxed{A_{nm} = \frac{2}{L}\int_0^L\left[\frac{2}{H}\int_0^H \alpha(x,y)\sin\frac{m\pi y}{H}\,dy\right]\sin\frac{n\pi x}{L}\,dx.} \tag{7.3.30}$$

This may be simplified to one double integral over the entire rectangular region, rather than two iterated one-dimensional integrals. In this manner we have determined one set of coefficients from one of the initial conditions.

The other coefficients B_{nm} can be determined in a similar way. In particular, from (7.3.27), $\partial u/\partial t(x,y,0) = \beta(x,y)$, which implies that

$$\beta(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{7.3.31}$$

Thus, again using a Fourier sine series in y and a Fourier sine series in x, we obtain

$$\boxed{c\sqrt{\lambda_{nm}}B_{nm} = \frac{4}{LH}\int_0^L\int_0^H \beta(x,y)\sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}\,dy\,dx.} \tag{7.3.32}$$

The solution of our initial value problem is the doubly infinite series given by (7.3.27), where the coefficients are determined by (7.3.30) and (7.3.32).

We have shown that when all three independent variables separate for a partial differential equation, there results three ordinary differential equations, two of which are eigenvalue problems. In general, for a partial differential equation in N variables that completely separates, there will be N ordinary differential equations, $N-1$ of which are one-dimensional eigenvalue problems (to determine the $N-1$ separation constants). We have already shown this for $N = 3$ (this section) and $N = 2$.

EXERCISES 7.3

7.3.1. Consider the heat equation in a two-dimensional rectangular region $0 < x < L, 0 < y < H$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

* (a) $u(0,y,t) = 0,\quad u(L,y,t) = 0,\quad u(x,0,t) = 0,\quad u(x,H,t) = 0$
(b) $\frac{\partial u}{\partial x}(0,y,t) = 0,\quad \frac{\partial u}{\partial x}(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$
* (c) $\frac{\partial u}{\partial x}(0,y,t) = 0,\quad \frac{\partial u}{\partial x}(L,y,t) = 0,\quad u(x,0,t) = 0,\quad u(x,H,t) = 0$
(d) $u(0,y,t) = 0,\quad \frac{\partial u}{\partial x}(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$
(e) $u(0,y,t) = 0,\quad u(L,y,t) = 0,\quad u(x,0,t) = 0,$
$\frac{\partial u}{\partial y}(x,H,t) + hu(x,H,t) = 0,\quad (h > 0)$

7.3.2. Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L,\ 0 < y < H,\ 0 < z < W$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

subject to the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

(a) $u(0,y,z,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,z,t) = 0,\quad \frac{\partial u}{\partial z}(x,y,0,t) = 0,$
$u(L,y,z,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,z,t) = 0,\quad u(x,y,W,t) = 0$

* (b) $\frac{\partial u}{\partial x}(0,y,z,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,z,t) = 0,\quad \frac{\partial u}{\partial z}(x,y,0,t) = 0,$
$\frac{\partial u}{\partial x}(L,y,z,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,z,t) = 0,\quad \frac{\partial u}{\partial z}(x,y,W,t) = 0$

7.3.3 Solve

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}$$

on a rectangle $(0 < x < L, 0 < y < H)$ subject to

$$u(x,y,0) = f(x,y) \qquad \begin{aligned} u(0,y,t) &= 0 \quad & \frac{\partial u}{\partial y}(x,0,t) &= 0 \\ u(L,y,t) &= 0 \quad & \frac{\partial u}{\partial y}(x,H,t) &= 0. \end{aligned}$$

7.3.4. Consider the wave equation for a vibrating rectangular membrane $(0 < x < L,\ 0 < y < H)$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial conditions

$$u(x,y,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,0) = f(x,y).$$

Solve the initial value problem if

(a) $u(0,y,t) = 0,\quad u(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$
* (b) $\frac{\partial u}{\partial x}(0,y,t) = 0,\quad \frac{\partial u}{\partial x}(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$

7.3.5. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - k\frac{\partial u}{\partial t} \quad \text{with } k > 0.$$

(a) Give a *brief* physical interpretation of this equation.
(b) Suppose that $u(x,y,t) = f(x)g(y)h(t)$. What ordinary differential equations are satisfied by f, g, and h?

7.3.6. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is $z = H$ and the bottom is $z = 0$. Assume that

$$\begin{aligned} \frac{\partial}{\partial z}u(x,y,0) &= 0 \\ u(x,y,H) &= f(x,y) \end{aligned}$$

and $u = 0$ on the "lateral" sides.

(a) Separate the z-variable in general.
*(b) Solve for $u(x,y,z)$ if the region is a rectangular box, $0 < x < L, 0 < y < W, 0 < z < H$.

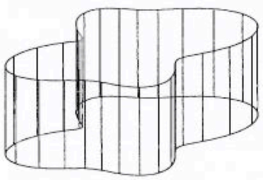

Figure 7.3.3

7.3.7. If possible, solve Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

in a rectangular-shaped region, $0 < x < L, 0 < y < W, 0 < z < H$, subject to the boundary conditions

(a)	$\frac{\partial u}{\partial x}(0, y, z) = 0,$	$u(x, 0, z) = 0,$	$u(x, y, 0) = f(x, y)$
	$\frac{\partial u}{\partial x}(L, y, z) = 0,$	$u(x, W, z) = 0,$	$u(x, y, H) = 0$
(b)	$u(0, y, z) = 0,$	$u(x, 0, z) = 0,$	$u(x, y, 0) = 0,$
	$u(L, y, z) = 0,$	$u(x, W, z) = f(x, z),$	$u(x, y, H) = 0$
*(c)	$\frac{\partial u}{\partial x}(0, y, z) = 0,$	$\frac{\partial u}{\partial y}(x, 0, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, 0) = 0$
	$\frac{\partial u}{\partial x}(L, y, z) = f(y, z),$	$\frac{\partial u}{\partial y}(x, W, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, H) = 0$
*(d)	$\frac{\partial u}{\partial x}(0, y, z) = 0,$	$\frac{\partial u}{\partial y}(x, 0, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, 0) = 0$
	$u(L, y, z) = g(y, z),$	$\frac{\partial u}{\partial y}(x, W, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, H) = 0$

Appendix to 7.3: Outline of Alternative Method to Separate Variables

An alternative (and equivalent) method to separate variables for

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{7.3.33}$$

is to assume product solutions of the form

$$u(x, y, t) = f(x)g(y)h(t). \tag{7.3.34}$$

By substituting (7.3.34) into (7.3.33) and dividing by $c^2 f(x)g(y)h(t)$, we obtain

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} = -\lambda, \tag{7.3.35}$$

after introducing a separation constant $-\lambda$. This shows that

$$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h.} \tag{7.3.36}$$

Equation (7.3.35) can be separated further,

$$\frac{1}{f}\frac{d^2f}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2g}{dy^2} = -\mu, \tag{7.3.37}$$

enabling a second separation constant $-\mu$ to be introduced:

$$\boxed{\frac{d^2f}{dx^2} = -\mu f} \tag{7.3.38}$$

$$\boxed{\frac{d^2g}{dy^2} = -(\lambda - \mu)g.} \tag{7.3.39}$$

In this way we have derived the same three ordinary differential equations (with two separation constants).

7.4 Statements and Illustrations of Theorems for the Eigenvalue Problem $\nabla^2\phi + \lambda\phi = 0$

In solving the heat equation and the wave equation in any two- or three-dimensional region R (with constant physical properties, such as density), we have shown that the spatial part $\phi(x, y, z)$ of product form solutions $u(x, y, z, t) = \phi(x, y, z)h(t)$ satisfies the following multidimensional eigenvalue problem:

$$\boxed{\nabla^2\phi + \lambda\phi = 0,} \tag{7.4.1}$$

with

$$a\phi + b\nabla\phi \cdot \hat{\mathbf{n}} = 0 \tag{7.4.2}$$

on the entire boundary. Here a and b can depend on x, y, and z. Equation (7.4.1) is known as the **Helmholtz equation**.

Equation (7.4.1) can be generalized to

$$\nabla \cdot (p\nabla\phi) + q\phi + \lambda\sigma\phi = 0, \tag{7.4.3}$$

where p, q and σ are functions of x, y, and z. This eigenvalue problem [with boundary condition (7.4.2)] is directly analogous to the one-dimensional regular Sturm-Liouville eigenvalue problem. We prefer to deal with a somewhat simpler case, (7.4.1), corresponding to $p = \sigma = 1$ and $q = 0$. We will state and prove results for (7.4.1). We leave the discussion of (7.4.3) to some exercises (in Sec. 7.5).

Only for very simple geometries (for example, rectangles, see Sec. 7.3, or circles, see Sec. 7.7) can (7.4.1) be solved explicitly. In other situations, we may have to rely on numerical treatments. However, certain general properties of (7.4.1) are quite useful, all analogous to results we understand for the one-dimensional Sturm-Liouville problem. The reasons for the analogy will be discussed in the next section. We begin by simply stating the theorems for the two-dimensional case of (7.4.1) and (7.4.2):

1. All the eigenvalues are real.
2. There exists an infinite number of eigenvalues. There is a smallest eigenvalue, but no largest one.
3. Corresponding to an eigenvalue, there *may* be many eigenfunctions (unlike regular Sturm-Liouville eigenvalue problems).
4. The eigenfunctions $\phi(x, y)$ form a "complete" set, meaning that any piecewise smooth function $f(x, y)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x, y) \sim \sum_\lambda a_\lambda \phi_\lambda(x, y). \tag{7.4.4}$$
Here $\sum_\lambda a_\lambda \phi_\lambda$ means a linear combination of all the eigenfunctions. The series converges in the mean if the coefficients a_λ are chosen correctly.
5. Eigenfunctions belonging to different eigenvalues (λ_1 and λ_2) are orthogonal relative to the weight $\sigma(\sigma = 1)$ over the entire region R. Mathematically,
$$\iint_R \phi_{\lambda_1}\phi_{\lambda_2}\, dx\, dy = 0 \quad \text{if } \lambda_1 \neq \lambda_2, \tag{7.4.5}$$
where $\iint_R dx\, dy$ represents an integral over the entire region R. Furthermore, different eigenfunctions belonging to the same eigenvalue can be made orthogonal by the Gram-Schmidt process (see Sec. 7.5). Thus, we may assume that (7.4.5) is valid even if $\lambda_1 = \lambda_2$ as long as ϕ_{λ_1} is independent of ϕ_{λ_2}.
6. An eigenvalue λ can be related to the eigenfunction by the Rayleigh quotient:
$$\lambda = \frac{-\oint \phi \nabla\phi \cdot \hat{n}\, dx + \iint_R |\nabla\phi|^2\, dx\, dy}{\iint_R \phi^2\, dx\, dy}. \tag{7.4.6}$$
The boundary conditions often simplify the boundary integral.

Here $\hat{n}$ is a unit outward normal and $\oint ds$ is a closed line integral over the entire boundary of the plane two-dimensional region, where ds is the differential arc length. The three-dimensional result is nearly identical; $\iint$ must be replaced by $\iiint$ and the boundary line integral $\oint ds$ must be replaced by the boundary surface integral $\oiint dS$, where dS is the differential surface area.

Example. We will prove some of these statements in Sec. 7.5. To understand their meaning, we will show how the example of Sec. 7.3 illustrates most of these theorems. For the vibrations of a rectangular ($0 < x < L$, $0 < y < H$) membrane with fixed zero boundary conditions, we have shown that the relevant eigenvalue problem is

$$\boxed{\begin{array}{c} \nabla^2\phi + \lambda\phi = 0 \\ \phi(0, y) = 0 \quad \phi(x, 0) = 0 \\ \phi(L, y) = 0 \quad \phi(x, H) = 0. \end{array}} \tag{7.4.7}$$

We have determined that the eigenvalues and corresponding eigenfunctions are

$$\boxed{\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2} \quad \begin{array}{l} n = 1, 2, 3, \ldots \\ m = 1, 2, 3, \ldots \end{array} \quad \text{with} \quad \boxed{\phi_{nm}(x, y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.} \tag{7.4.8}$$

1. **Real eigenvalues.** In our calculation of the eigenvalues for (7.4.7) we *assumed* that the eigenfunctions existed in a product form. Under that assumption, (7.4.8) showed the eigenvalues to be real. Our theorem guarantees that the eigenvalues will always be real.
2. **Ordering of eigenvalues.** There is a doubly infinite set of eigenvalues for (7.4.7), namely $\lambda_{nm} = (n\pi/L)^2 + (m\pi/H)^2$ for $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. There is a smallest eigenvalue, $\lambda_{11} = (\pi/L)^2 + (\pi/H)^2$, but no largest eigenvalue.
3. **Multiple eigenvalues.** For $\nabla^2\phi + \lambda\phi = 0$, our theorem states that, in general, it is possible for there to be more than one eigenfunction corresponding to the same eigenvalue. To illustrate this, consider (7.4.7) in the case in which $L = 2H$. Then
$$\lambda_{nm} = \frac{\pi^2}{4H^2}\left(n^2 + 4m^2\right) \tag{7.4.9}$$
with
$$\phi_{nm} = \sin\frac{n\pi x}{2H}\sin\frac{m\pi y}{H}. \tag{7.4.10}$$
We note that it is possible to have different eigenfunctions corresponding to the same eigenvalue. For example, $n = 4$, $m = 1$ and $n = 2$, $m = 2$, yield the same eigenvalue:
$$\lambda_{41} = \lambda_{22} = \frac{\pi^2}{4H^2}20.$$

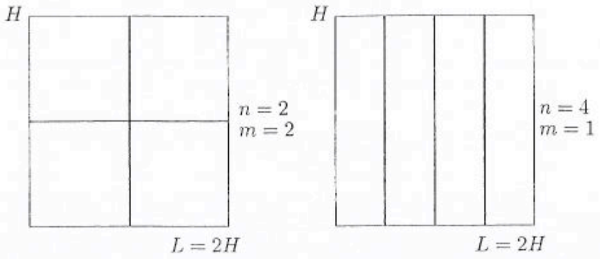

Figure 7.4.1 Nodal curves for eigenfunctions with the same eigenvalue (symmetric).

For $n = 4$, $m = 1$, the eigenfunction is $\phi_{41} = \sin 4\pi x/2H \sin \pi y/H$, while for $n = 2$, $m = 2$, $\phi_{22} = \sin 2\pi x/2H \sin 2\pi y/H$. The nodal curves for these eigenfunctions are sketched in Fig. 7.4.1. They are different eigenfunctions with the same eigenvalue, $\lambda = (\pi^2/4H^2)20$. It is not surprising that the eigenvalue is the same, since a membrane vibrating in these modes has cells of the same dimensions: one $H \times H/2$ and the other $H/2 \times H$. By symmetry they will have the same natural frequency (and hence the same eigenvalue since the natural frequency is $c\sqrt{\lambda}$). In fact, in general by symmetry [as well as by formula (7.4.9)] $\lambda_{(2n)m} = \lambda_{(2m)n}$.

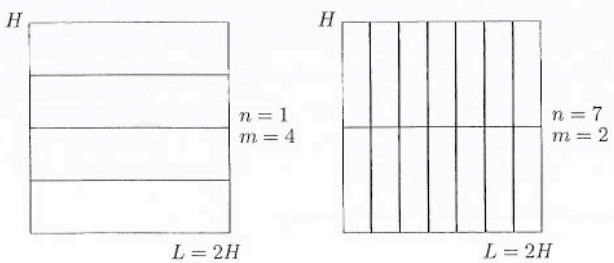

Figure 7.4.2 Nodal curves for eigenfunction with the same eigenvalue (asymmetric).

However, it is also possible for more than one eigenfunction to occur for reasons having nothing to do with symmetry. For example, $n = 1$, $m = 4$ and $n = 7$, $m = 2$ yield the same eigenvalue: $\lambda_{14} = \lambda_{72} = (\pi^2/4H^2)65$. The corresponding eigenfunctions are $\phi_{14} = \sin \pi x/2H \sin 4\pi y/H$ and $\phi_{72} = \sin 7\pi x/2H \sin 2\pi y/H$, which are sketched in Fig. 7.4.2. It is only coincidental that both of these shapes vibrate with the same frequency. In these situations, it is possible for two eigenfunctions to correspond to the same eigenvalue. We can find situations with even more multiplicities (or degeneracies). Since $\lambda_{14} = \lambda_{72} = (\pi^2/4H^2)65$, it is also true that $\lambda_{28} = \lambda_{(14)4} = (\pi^2/4H^2)260$.

However, by symmetry $\lambda_{28} = \lambda_{(16)1}$ and $\lambda_{(14)4} = \lambda_{87}$. Thus,

$$\lambda_{28} = \lambda_{(16)1} = \lambda_{(14)4} = \lambda_{87} = \left(\frac{\pi^2}{4H^2}\right) 260.$$

Here there are four eigenfunctions corresponding to the same eigenvalue.

4a. **Series of eigenfunctions.** According to this theorem, (7.4.4), the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$ can always be used to represent any piecewise smooth function $f(x, y)$. In our illustrative example, (7.4.7), $\sum_\lambda$ becomes a double sum,

$$f(x, y) \sim \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}. \tag{7.4.11}$$

5. **Orthogonality of eigenfunctions.** We will show that the multidimensional orthogonality of the eigenfunctions, as expressed by (7.4.5) for any two different eigenfunctions, can be used to determine the generalized Fourier coefficients in (7.4.4).[1] We will do this in exactly the way we did for one-dimensional Sturm-Liouville eigenfunction expansions. We simply multiply (7.4.4) by ϕ_{λ_i} and integrate over the entire region R:

$$\iint_R f\phi_{\lambda_i}\, dx\, dy = \sum_\lambda a_\lambda \iint_R \phi_\lambda \phi_{\lambda_i}\, dx\, dy. \tag{7.4.12}$$

Since the eigenfunctions are all orthogonal to each other (with weight 1 because $\nabla^2\phi + \lambda\phi = 0$), it follows that

$$\iint_R f\phi_{\lambda_i}\, dx\, dy = a_{\lambda_i} \iint_R \phi_{\lambda_i}^2\, dx\, dy, \tag{7.4.13}$$

or, equivalently,

$$\boxed{a_{\lambda_i} = \frac{\iint_R f\phi_{\lambda_i}\, dx\, dy}{\iint_R \phi_{\lambda_i}^2\, dx\, dy}.} \tag{7.4.14}$$

There is no difficulty in forming (7.4.14) from (7.4.13) since the denominator of (7.4.14) is necessarily positive.

For the special case that occurs for a rectangle with fixed zero boundary conditions, (7.4.7), the generalized Fourier coefficients a_{nm} are given by (7.4.14):

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dx\, dy}{\int_0^H \int_0^L \sin^2\frac{n\pi x}{L} \sin^2\frac{m\pi y}{H}\, dx\, dy}. \tag{7.4.15}$$

The integral in the denominator may be easily shown to equal $(L/2)(H/2)$ by calculating two one-dimensional integrals; in this way we rederive (7.3.30). In

[1] If there is more than one eigenfunction corresponding to the same eigenvalue, then we assume that the eigenfunctions have been made orthogonal (if necessary by the Gram-Schmidt process).

this case, (7.4.11), the generalized Fourier coefficient a_{nm} can be evaluated in two equivalent ways:

(a) Using one two-dimensional orthogonality formula for the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$

(b) Using two one-dimensional orthogonality formulas

4b. **Convergence.** As with any Sturm-Liouville eigenvalue problem (see Sec. 5.10), a *finite* series of the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$ may be used to approximate a function $f(x, y)$. In particular, we could show that if we measure error in the mean-square sense,

$$E \equiv \iint_R \left(f - \sum_\lambda a_\lambda \phi_\lambda\right)^2 dx\, dy, \tag{7.4.16}$$

with weight function 1, then this mean-square error is minimized by the coefficients a_λ being chosen by (7.4.14), the generalized Fourier coefficients. It is known that the approximation improves as the number of terms increases. Furthermore, $E \to 0$ as all the eigenfunctions are included. We say that the series $\sum_\lambda a_\lambda\phi_\lambda$ **converges in the mean** to f.

EXERCISES 7.4

7.4.1. Consider the eigenvalue problem

$$\nabla^2\phi + \lambda\phi = 0$$

$$\begin{aligned} \frac{\partial\phi}{\partial x}(0, y) &= 0 & \phi(x, 0) &= 0 \\ \frac{\partial\phi}{\partial x}(L, y) &= 0 & \phi(x, H) &= 0. \end{aligned}$$

*(a) Show that there is a doubly infinite set of eigenvalues.

(b) If $L = H$, show that most eigenvalues have more than one eigenfunction.

(c) Derive that the eigenfunctions are orthogonal in a two-dimensional sense using two one-dimensional orthogonality relations.

7.4.2. Without using the explicit solution of (7.4.7), show that $\lambda \geq 0$ from the Rayleigh quotient, (7.4.6).

7.4.3. If necessary, see Sec. 7.5:

(a) Derive that $\iint (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \oint (u\nabla v - v\nabla u)\cdot \hat{\boldsymbol{n}}\, ds$.

(b) From part (a), derive (7.4.5).

7.4.4. Derive (7.4.6). If necessary, see Sec. 7.6. [*Hint*: Multiply (7.4.1) by ϕ and integrate.]

7.5 Green's Formula, Self-Adjoint Operators and Multidimensional Eigenvalue Problems

Introduction. In this section we prove some of the properties of the multidimensional eigenvalue problem:

$$\boxed{\nabla^2\phi + \lambda\phi = 0} \tag{7.5.1}$$

with

$$\boxed{\beta_1\phi + \beta_2\nabla\phi\cdot\hat{\boldsymbol{n}} = 0} \tag{7.5.2}$$

on the entire boundary. Here β_1 and β_2 are real functions of the location in space. As with Sturm-Liouville eigenvalue problems, we will simply assume that there is an infinite number of eigenvalues for (7.5.1) with (7.5.2) and that the resulting set of eigenfunctions is complete. Proofs of these statements are difficult and beyond the intent of this text. The proofs for various other properties of the multidimensional eigenvalue problem are quite similar to corresponding proofs for the one-dimensional Sturm-Liouville eigenvalue problem. We let

$$L \equiv \nabla^2, \tag{7.5.3}$$

in which case the notation for the multidimensional eigenvalue problem becomes

$$L(\phi) + \lambda\phi = 0. \tag{7.5.4}$$

By comparing (7.5.4) to (7.4.3), we notice that the weight function for this multidimensional problem is expected to be 1.

Multidimensional Green's formula. The proofs for the one-dimensional Sturm-Liouville eigenvalue problem depended on $uL(v) - vL(u)$ being an exact differential (known as Lagrange's identity). The corresponding integrated form (known as Green's formula) was also needed. Similar identities will be derived for the Laplacian operator, $L = \nabla^2$, a multidimensional analog of the Sturm-Liouville differential operator. We will calculate $uL(v) - vL(u) = u\nabla^2 v - v\nabla^2 u$. We recall that $\nabla^2 u = \nabla\cdot(\nabla u)$ and $\nabla\cdot(a\boldsymbol{B}) = a\nabla\cdot\boldsymbol{B} + \nabla a\cdot\boldsymbol{B}$ (where a is a scalar and $\boldsymbol{B}$ a vector). Thus,

$$\begin{aligned} \nabla\cdot(u\nabla v) &= u\nabla^2 v + \nabla u\cdot\nabla v \\ \nabla\cdot(v\nabla u) &= v\nabla^2 u + \nabla v\cdot\nabla u. \end{aligned} \tag{7.5.5}$$

By subtracting these,

$$\boxed{u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v - v\nabla u).} \tag{7.5.6}$$

The differential form, (7.5.6), is the multidimensional version of Lagrange's identity, (5.5.7). Instead of integrating from a to b as we did in one-dimensional problems, we integrate over the entire two-dimensional region

$$\iint_R (u\nabla^2 v - v\nabla^2 u)\ dx\ dy = \iint_R \nabla \cdot (u\nabla v - v\nabla u)\ dx\ dy.$$

The right-hand side is in the correct form to apply the divergence theorem (recall that $\iint_R \nabla \cdot \boldsymbol{A}\ dx\ dy = \oint \boldsymbol{A} \cdot \hat{\boldsymbol{n}}\ ds$). Thus,

$$\iint_R (u\nabla^2 v - v\nabla^2 u)\ dx\ dy = \oint (u\nabla v - v\nabla u) \cdot \hat{\boldsymbol{n}}\ ds. \tag{7.5.7}$$

Equation (7.5.7) is analogous to Green's formula, (5.5.8). It is known as **Green's second identity**,[2] but we will just refer to it as **Green's formula.**

We have shown that $L = \nabla^2$ is a multidimensional **self-adjoint** operator in the following sense:

> If u and v are any two functions, such that
>
> $$\oint (u\nabla v - v\nabla u) \cdot \hat{\boldsymbol{n}}\ ds = 0, \tag{7.5.8}$$
>
> then
>
> $$\iint_R [u\nabla^2 v - v\nabla^2 u]\ dx\ dy = 0. \tag{7.5.9}$$
>
> where $L = \nabla^2$.

In many problems, prescribed homogeneous boundary conditions will cause the boundary term to vanish. For example, (7.5.8) and thus (7.5.9) is valid if u and v both vanish on the boundary. Again for three-dimensional problems, $\iint$ must be replaced by $\iiint$ and $\oint$ must be replaced by $\oiint$.

Orthogonality of the eigenfunctions . As with the one-dimensional Sturm-Liouville eigenvalue problem, we can prove a number of theorems directly from Green's formula (7.5.7). To show eigenfunctions corresponding to different eigenvalues are orthogonal, we consider two eigenfunctions ϕ_1 and ϕ_2 corresponding to the eigenvalues λ_1 and λ_2:

$$\begin{aligned} \nabla^2\phi_1 + \lambda_1\phi_1 &= 0 \quad \text{or} \quad L(\phi_1) + \lambda_1\phi_1 = 0 \\ \nabla^2\phi_2 + \lambda_2\phi_2 &= 0 \quad \text{or} \quad L(\phi_2) + \lambda_2\phi_2 = 0. \end{aligned} \tag{7.5.10}$$

[2] *Green's first identity* arises from integrating (7.5.5) [rather than (7.5.6)] with $v = u$ and applying the divergence theorem.

If both ϕ_1 and ϕ_2 satisfy the same homogeneous boundary conditions, then (7.5.8) is satisfied, in which case (7.5.9) follows. Thus

$$\iint_R (-\phi_1\lambda_2\phi_2 + \phi_2\lambda_1\phi_1)\ dx\ dy = (\lambda_1 - \lambda_2)\iint_R \phi_1\phi_2\ dx\ dy = 0.$$

If $\lambda_1 \neq \lambda_2$, then

$$\iint_R \phi_1\phi_2\ dx\ dy = 0, \tag{7.5.11}$$

which means that eigenfunctions corresponding to different eigenvalues are orthogonal (in a multidimensional sense with weight 1). If two or more eigenfunctions correspond to the same eigenvalue, they can be made orthogonal to each other (as well as all other eigenfunctions) by a procedure shown in the appendix of this section known as the Gram-Schmidt method.

We can now prove that the eigenvalues will be real. The proof is left for an exercise since the proof is identical to that used for the one-dimensional Sturm-Liouville problem (see Sec. 5.5).

EXERCISES 7.5

7.5.1. The vertical displacement of a nonuniform membrane satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

where c depends on x and y. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

(a) Show that the time variable can be separated by assuming that

$$u(x, y, t) = \phi(x, y)h(t).$$

Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \quad \text{with} \quad \phi = 0 \quad \text{on the boundary.} \tag{7.5.12}$$

What is $\sigma(x, y)$?

(b) If the eigenvalues are known (and $\lambda > 0$), determine the frequencies of vibration.

7.5.2. See Exercise 7.5.1. Consider the two-dimensional eigenvalue problem given in (7.5.12).

(a) Prove that the eigenfunctions belonging to different eigenvalues are orthogonal (with what weight?).

(b) Prove that all the eigenvalues are real.

(c) Do Exercise 7.6.1.

7.5.3. Redo Exercise 7.5.2 if the boundary condition is instead

(a) $\nabla\phi \cdot \hat{n} = 0$ on the boundary

(b) $\nabla\phi \cdot \hat{n} + h(x,y)\phi = 0$ on the boundary

(c) $\phi = 0$ on part of the boundary and $\nabla\phi \cdot \hat{n} = 0$ on the rest of the boundary

7.5.4. Consider the heat equation in three dimensions with no sources but with nonconstant thermal properties

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u),$$

where $c\rho$ and K_0 are functions of x, y, and z. Assume that $u = 0$ on the boundary. Show that the time variable can be separated by assuming that

$$u(x,y,z,t) = \phi(x,y,z)h(t).$$

Show that $\phi(x,y,z)$ satisfies the eigenvalue problem

$$\nabla \cdot (p\nabla\phi) + \lambda\sigma(x,y,z)\phi = 0 \quad \text{with } \phi = 0 \text{ on the boundary.} \tag{7.5.13}$$

What are $\sigma(x,y,z)$ and $p(x,y,z)$?

7.5.5. See Exercise 7.5.4. Consider the three-dimensional eigenvalue problem given in (7.5.13).

(a) Prove that the eigenfunctions belonging to different eigenvalues are orthogonal (with what weight?).

(b) Prove that all the eigenvalues are real.

(c) Do Exercise 7.6.3.

7.5.6. Derive an expression for

$$\iint [uL(v) - vL(u)]\, dx\, dy$$

over a two-dimensional region R, where

$$L = \nabla^2 + q(x,y) \quad [\text{i.e., } L(u) = \nabla^2 u + q(x,y)u].$$

7.5.7. Consider Laplace's equation $\nabla^2 u = 0$ in a three-dimensional region R (where u is the temperature). Suppose that the heat flux is given on the boundary (not necessarily a constant).

(a) Explain *physically* why $\oint \nabla u \cdot \hat{n}\, dS = 0$.

(b) Show this mathematically.

7.5.8. Suppose that in a three-dimensional region R

$$\nabla^2\phi = f(x,y,z)$$

with f given and $\nabla\phi \cdot \hat{n} = 0$ on the boundary.

(a) Show mathematically that (if there is a solution)

$$\iiint_R f\, dx\, dy\, dz = 0.$$

(b) Briefly explain physically (using the heat flow model) why condition (a) must hold for a solution. What happens in a heat flow problem if

$$\iiint_R f\, dx\, dy\, dz > 0?$$

7.5.9. Show that the boundary term (7.5.8) vanishes if both u and v satisfy (7.5.2):

(a) Assume that $\beta_2 \neq 0$.

(b) Assume $\beta_2 = 0$ for part of the boundary.

Appendix to 7.5: Gram-Schmidt Method

We wish to show in general that eigenfunctions *corresponding to the same eigenvalue* can be made orthogonal. The process is known as **Gram-Schmidt orthogonalization**. Let us suppose that $\phi_1, \phi_2, \ldots, \phi_n$ are independent eigenfunctions corresponding to the *same eigenvalue*. We will form a set of n-independent eigenfunctions denoted $\psi_1, \psi_2, \ldots, \psi_n$ which are mutually orthogonal, even if $\phi_1, \ldots, \phi_n$ are not. Let $\psi_1 = \phi_1$ be any one eigenfunction. Any linear combination of the eigenfunctions is also an eigenfunction (since they satisfy the *same* linear homogeneous differential equation and boundary conditions). Thus, $\psi_2 = \phi_2 + c\psi_1$ is also an eigenfunction (automatically independent of ψ_1), where c is an arbitrary constant. We choose c so that $\psi_2 = \phi_2 + c\psi_1$ is orthogonal to ψ_1: $\iint_R \psi_1\psi_2\, dx\, dy = 0$ becomes

$$\iint_R (\phi_2 + c\psi_1)\psi_1\, dx\, dy = 0.$$

c is uniquely determined:

$$c = \frac{-\iint_R \phi_2\psi_1\, dx\, dy}{\iint_R \psi_1^2\, dx\, dy}. \tag{7.5.14}$$

Since there may be more than two eigenfunctions corresponding to the same eigenvalue, we continue this process.

A third eigenfunction is $\psi_3 = \phi_3 + c_1\psi_1 + c_2\psi_2$, where we choose c_1 and c_2 so that ψ_3 is orthogonal to the previous two: $\iint_R \psi_3 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dx\, dy = 0$. Thus,

$$\iint_R (\phi_3 + c_1\psi_1 + c_2\psi_2) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dx\, dy = 0.$$

However, ψ_2 is already orthogonal to ψ_1, and hence

$$\begin{aligned} \iint_R \phi_3\psi_1 \, dx\, dy + c_1 \iint_R \psi_1^2 \, dx\, dy &= 0 \\ \iint_R \phi_3\psi_2 \, dx\, dy + c_2 \iint_R \psi_2^2 \, dx\, dy &= 0, \end{aligned}$$

easily determining the two constants. This process can be used to determine n orthogonal eigenfunctions. In general,

$$\psi_j = \phi_j - \sum_{i=1}^{j-1} \left(\frac{\iint_R \phi_j \psi_i \, dx\, dy}{\iint_R \psi_i^2 \, dx\, dy} \right) \psi_i.$$

We have shown that even in the case of a multiple eigenvalue, we are always able to restrict our attention to orthogonal eigenfunctions, if necessary by this Gram-Schmidt construction.

7.6 Rayleigh Quotient and Laplace's Equation

7.6.1 Rayleigh Quotient

In Sec. 5.6 we obtained the Rayleigh quotient, for the one-dimensional Sturm-Liouville eigenvalue problem. The result was obtained by multiplying the differential equation by ϕ, integrating over the entire region, solving for λ, and simplifying using integration by parts. We will derive a similar result for

$$\nabla^2\phi + \lambda\phi = 0. \tag{7.6.1}$$

We proceed as before by multiplying (7.6.1) by ϕ. Integrating over the entire two-dimensional region and solving for λ yields

$$\lambda = \frac{-\iint_R \phi \nabla^2 \phi \, dx\, dy}{\iint_R \phi^2 \, dx\, dy}. \tag{7.6.2}$$

Next, we want to generalize integration by parts to multidimensional functions. Integration by parts is based on the product rule for the derivative, $d/dx(fg) = f\, dg/dx + g\, df/dx$. Instead of using the derivative, we use a product rule for the divergence, $\nabla \cdot (fg) = f\nabla \cdot g + g \cdot \nabla f$. Letting $f = \phi$ and $g = \nabla\phi$, it follows that $\nabla \cdot (\phi\nabla\phi) = \phi\nabla \cdot (\nabla\phi) + \nabla\phi \cdot \nabla\phi$. Since $\nabla \cdot (\nabla\phi) = \nabla^2\phi$ and $\nabla\phi \cdot \nabla\phi = |\nabla\phi|^2$,

$$\phi\, \nabla^2\phi = \nabla \cdot (\phi\, \nabla\phi) - |\nabla\phi|^2. \tag{7.6.3}$$

Using (7.6.3) in (7.6.2) yields an alternative expression for the eigenvalue,

$$\lambda = \frac{-\int\int_R \nabla \cdot (\phi\nabla\phi)\, dx\, dy + \int\int_R |\nabla\phi|^2 \, dx\, dy}{\int\int_R \phi^2 \, dx\, dy}. \tag{7.6.4}$$

Now we use (again) the divergence theorem to evaluate the first integral in the numerator of (7.6.4). Since $\iint_R \nabla \cdot \boldsymbol{A}\, dx\, dy = \oint \boldsymbol{A} \cdot \hat{\boldsymbol{n}}\, ds$, it follows that

$$\boxed{\lambda = \frac{-\oint \phi\nabla\phi \cdot \hat{\boldsymbol{n}}\, ds + \iint_R |\nabla\phi|^2 \, dx\, dy}{\iint_R \phi^2 \, dx\, dy},} \tag{7.6.5}$$

known as the **Rayleigh quotient.** This is quite similar to the Rayleigh quotient for Sturm-Liouville eigenvalue problems. Note that there is a boundary contribution for each: $-p\phi\, d\phi/dx|_a^b$ for (5.6.3) and $-\oint \phi\nabla\phi \cdot \hat{\boldsymbol{n}}\, ds$ for (7.6.5).

Example. We consider any region in which the boundary condition is $\phi = 0$ on the entire boundary. Then $\oint \phi\nabla\phi \cdot \hat{\boldsymbol{n}}\, ds = 0$, and hence from (7.6.5), $\lambda \geq 0$. If $\lambda = 0$, then (7.6.5) implies that

$$0 = \iint_R |\nabla\phi|^2 \, dx\, dy. \tag{7.6.6}$$

Thus,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} = \mathbf{0} \tag{7.6.7}$$

everywhere. From (7.6.7) it follows that $\partial\phi/\partial x = 0$ and $\partial\phi/\partial y = 0$ everywhere. Thus, ϕ is a constant everywhere, but since $\phi = 0$ on the boundary, $\phi = 0$ everywhere. $\phi = 0$ everywhere is not an eigenfunction, and thus we have shown that $\lambda = 0$ is not an eigenvalue. In conclusion, $\lambda > 0$.

7.6.2 Time-Dependent Heat Equation and Laplace's Equation

Equilibrium solutions of the time-dependent heat equation satisfy Laplace's equation. Solving Laplace's equation $\nabla^2\phi = 0$ subject to homogeneous boundary conditions corresponds to investigating whether $\lambda = 0$ is an eigenvalue for (7.6.1).

Zero temperature boundary condition. Consider $\nabla^2\phi = 0$ subject to $\phi = 0$ along the entire boundary. It can be concluded from (7.6.6) that $\phi = 0$ everywhere inside the region (since $\lambda = 0$ is not an eigenvalue). For an object of any shape subject to the zero temperature boundary condition on the entire boundary, the steady-state (equilibrium) temperature distribution is zero temperature, which is somewhat obvious physically. For the time-dependent heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$, (7.6.1) arises by separation of variables, and $\lambda > 0$ (from the Rayleigh quotient) proves that $u(x, y, t) \to 0$ as $t \to \infty$, the time dependent temperature approaches the equilibrium temperature distribution for large time.

Insulated boundary condition. Consider $\nabla^2\phi = 0$ subject to $\nabla\phi \cdot \hat{n} = 0$ along the entire boundary. It can be concluded from (7.6.6) that $\phi = c$ =arbitrary constant everywhere inside the region (since $\lambda = 0$ is an eigenvalue and $\phi = c$ is the eigenfunction). The constant equilibrium solution can be determined from the initial value problem for the time-dependent diffusion (heat) equation $\frac{\partial u}{\partial t} = k\nabla^2 u$. The fundamental integral conservation law (see Section 1.5) using the entire region is $\frac{d}{dt}\iint u\,dx\,dy = k\oint \nabla u \cdot \hat{n}\,ds = 0$, where we have used the insulated boundary condition. Thus, the total thermal energy $\iint u\,dx\,dy$ is constant in time, and its equilibrium value ($t \to \infty$), $\iint c\,dx\,dy = cA$, equals its initial value ($t = 0$), $\iint f(x, y)\,dx\,dy$. In this way, $c = \frac{1}{A}\iint f(x, y)\,dx\,dy$, so that the constant equilibrium temperature with insulated boundaries must be the average of the initial temperature distribution. Here A is the area of the two-dimensional region. For the time-dependent heat equation with insulated boundary conditions, it can be shown that $u(x, y, t) \to c = \frac{1}{A}\iint f(x, y)\,dx\,dy$ as $t \to \infty$ since $\lambda \geq 0$ (with $\phi = 1$ corresponding to $\lambda = 0$ from the Rayleigh quotient) (i.e., the time-dependent temperature approaches the equilibrium temperature distribution for large time).

Similar results hold in three dimensions.

EXERCISES 7.6

7.6.1. See Exercise 7.5.1. Consider the two-dimensional eigenvalue problem with $\sigma > 0$

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \text{ with } \phi = 0 \text{ on the boundary.}$$

(a) Prove that $\lambda \geq 0$.

(b) Is $\lambda = 0$ an eigenvalue, and if so, what is the eigenfunction?

7.6.2. Redo Exercise 7.6.1 if the boundary condition is instead

(a) $\nabla\phi \cdot \hat{n} = 0$ on the boundary

(b) $\nabla\phi \cdot \hat{n} + h(x, y)\phi = 0$ on the boundary

(c) $\phi = 0$ on part of the boundary and $\nabla\phi \cdot \hat{n} = 0$ on the rest of the boundary

7.6.3. Redo Exercise 7.6.1 if the differential equation is

$$\nabla \cdot (p\nabla\phi) + \lambda\sigma(x, y, z)\phi = 0$$

with boundary condition

(a) $\phi = 0$ on the boundary

(b) $\nabla\phi \cdot \hat{n} = 0$ on the boundary

7.6.4. (a) If $\nabla^2\phi = 0$ with $\phi = 0$ on the boundary, prove that $\phi = 0$ everywhere. (*Hint*: Use the fact that $\lambda = 0$ is not an eigenvalue for $\nabla^2\phi = -\lambda\phi$.)

(b) Prove that there cannot be two different solutions of the problem

$$\nabla^2 u = f(x, y, z)$$

subject to the given boundary condition $u = g(x, y, z)$ on the boundary. [*Hint*: Consider $u_1 - u_2$ and use part (a).]

7.7 Vibrating Circular Membrane and Bessel Functions

7.7.1 Introduction

An interesting application of both one-dimensional (Sturm-Liouville) and multidimensional eigenvalue problems occurs when considering the vibrations of a circular membrane. The vertical displacement u satisfies the two-dimensional wave equation,

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.} \tag{7.7.1}$$

The geometry suggests that we use polar coordinates, in which case $u = u(r, \theta, t)$. We assume that the membrane has zero displacement at the circular boundary, $r = a$:

$$\text{BC:} \quad \boxed{u(a, \theta, t) = 0.} \tag{7.7.2}$$

The initial position and velocity are given:

$$\text{IC:} \quad \boxed{\begin{aligned} u(r, \theta, 0) &= \alpha(r, \theta) \\ \frac{\partial u}{\partial t}(r, \theta, 0) &= \beta(r, \theta). \end{aligned}} \tag{7.7.3}$$

7.7.2 Separation of Variables

We first separate out the time variable by seeking product solutions,

$$u(r, \theta, t) = \phi(r, \theta)h(t). \tag{7.7.4}$$

Then, as shown earlier, $h(t)$ satisfies

$$\frac{d^2h}{dt^2} = -\lambda c^2 h, \tag{7.7.5}$$

where λ is a separation constant. From (7.7.5), the natural frequencies of vibration are $c\sqrt{\lambda}$ (if $\lambda > 0$). In addition, $\phi(r,\theta)$ satisfies the two-dimensional eigenvalue problem

$$\boxed{\nabla^2\phi + \lambda\phi = 0,} \tag{7.7.6}$$

with $\phi = 0$ on the entire boundary, $r = a$:

$$\phi(a,\theta) = 0 \tag{7.7.7}$$

We will attempt to obtain solutions of (7.7.6) in the product form appropriate for polar coordinates,

$$\phi(r,\theta) = f(r)g(\theta), \tag{7.7.8}$$

since for the circular membrane $0 < r < a$, $-\pi < \theta < \pi$. This is equivalent to originally seeking solutions to the wave equation in the form of a product of functions of each independent variable, $u(r,\theta,t) = f(r)g(\theta)h(t)$. We substitute (7.7.8) into (7.7.6); in polar coordinates $\nabla^2\phi = 1/r\ \partial/\partial r(r\ \partial\phi/\partial r) + 1/r^2\ \partial^2\phi/\partial\theta^2$, and thus $\nabla^2\phi + \lambda\phi = 0$ becomes

$$\frac{g(\theta)}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{f(r)}{r^2}\frac{d^2g}{d\theta^2} + \lambda f(r)g(\theta) = 0. \tag{7.7.9}$$

r and θ may be separated by multiplying by r^2 and dividing by $f(r)g(\theta)$:

$$-\frac{1}{g}\frac{d^2g}{d\theta^2} = \frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \lambda r^2 = \mu. \tag{7.7.10}$$

We introduce a second separation constant in the form μ because our experience with circular regions (see Secs. 2.4.2 and 2.5.2) suggests that $g(\theta)$ must oscillate in order to satisfy the periodic conditions in θ. Our three differential equations, with two separation constants, are thus

$$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h} \tag{7.7.11}$$

$$\boxed{\frac{d^2g}{d\theta^2} = -\mu g} \tag{7.7.12}$$

$$\boxed{r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu)\,f = 0.} \tag{7.7.13}$$

Two of these equations must be eigenvalue problems. However, ignoring the initial conditions, the only given boundary condition is $f(a) = 0$, which follows from

$u(a,\theta,t) = 0$ or $\phi(a,\theta) = 0$. We must remember that $-\pi < \theta < \pi$ and $0 < r < a$. Thus, both θ and r are defined over finite intervals. As such there should be boundary conditions at both ends. The periodic nature of the solution in θ implies that

$$g(-\pi) = g(\pi) \tag{7.7.14}$$

$$\frac{dg}{d\theta}(-\pi) = \frac{dg}{d\theta}(\pi). \tag{7.7.15}$$

We already have a condition at $r = a$. Polar coordinates are singular at $r = 0$; a singularity condition must be introduced there. Since the displacement must be finite, we conclude that

$$|f(0)| < \infty.$$

7.7.3 Eigenvalue Problems (One Dimensional)

After separating variables, we have obtained two eigenvalue problems. We are quite familiar with the θ-eigenvalue problem, (7.7.12) with (7.7.14) and (7.7.15). Although it is not a regular Sturm-Liouville problem due to the periodic boundary conditions, we know that the eigenvalues are

$$\mu_m = m^2, \quad m = 0, 1, 2, \ldots. \tag{7.7.16}$$

The corresponding eigenfunctions are both

$$g(\theta) = \sin m\theta \quad \text{and} \quad g(\theta) = \cos m\theta, \tag{7.7.17}$$

although for $m = 0$ this reduces to one eigenfunction (not two as for $m \neq 0$). This eigenvalue problem generates a full Fourier series in θ, as we already know. m is the number of crests in the θ-direction.

For each integral value of m, (7.7.13) helps to define an eigenvalue problem for λ:

$$\boxed{r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - m^2)f = 0} \tag{7.7.18}$$

$$\boxed{f(a) = 0} \tag{7.7.19}$$

$$\boxed{|f(0)| < \infty.} \tag{7.7.20}$$

Since (7.7.18) has nonconstant coefficients, it is not surprising that (7.7.18) is somewhat difficult to analyze. Equation (7.7.18) can be put in the Sturm-Liouville form by dividing it by r:

$$\boxed{\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0,} \tag{7.7.21}$$

or $Lf + \lambda r f = 0$, where $L = d/dr\,(r\, d/dr) - m^2/r$. By comparison to the general Sturm-Liouville differential equation,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q\phi + \lambda\sigma\phi = 0,$$

with independent variable r, we have that $x = r$, $p(r) = r$, $\sigma(r) = r$, and $q(r) = -m^2/r$. Our problem is not a regular Sturm-Liouville problem due to the behavior at the origin ($r = 0$):

1. The boundary condition at $r = 0$, (7.7.20), is not of the correct form.
2. $p(r) = 0$ and $\sigma(r) = 0$ at $r = 0$ (and hence is not positive everywhere).
3. $q(r)$ approaches ∞ as $r \to 0$ [and hence $q(r)$ is not continuous] for $m \neq 0$.

However, we claim that all the statements concerning regular Sturm-Liouville problems are still valid for this important singular Sturm-Liouville problem. To begin with there are an infinite number of eigenvalues (for each m). We designate the eigenvalues as λ_{nm}, where $m = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$, and the eigenfunctions $f_{nm}(r)$. *For each fixed m,* these eigenfunctions are orthogonal with weight r [see (7.7.21)], since it can be shown that the boundary terms vanish in Green's formula (see Exercise 5.5.1). Thus,

$$\int_0^a f_{mn_1} f_{mn_2} r\, dr = 0 \quad \text{for } n_1 \neq n_2. \tag{7.7.22}$$

Shortly, we will state more explicit facts about these eigenfunctions.

7.7.4 Bessel's Differential Equation

The r-dependent separation of variables solution satisfies a "singular" Sturm-Liouville differential equation, (7.7.21). An alternative form is obtained by using the product rule of differentiation and by multiplying by r:

$$\boxed{r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + \left(\lambda r^2 - m^2\right) f = 0.} \tag{7.7.23}$$

There is some additional analysis of (7.7.23) that can be performed. Equation (7.7.23) contains two parameters, m and λ. We already know that m is an integer, but the allowable values of λ are as yet unknown. It would be quite tedious to solve numerically (7.7.23) for various values of λ (for different integral values of m). Instead, we might notice that the simple scaling transformation,

$$\boxed{z = \sqrt{\lambda} r,} \tag{7.7.24}$$

removes the dependence of the differential equation on λ:

$$\boxed{z^2\frac{d^2 f}{dz^2} + z\frac{df}{dz} + \left(z^2 - m^2\right) f = 0.} \tag{7.7.25}$$

We note that the change of variables (7.7.24) may be performed[3] since we showed in Sec. 7.6 from the multidimensional Rayleigh quotient that $\lambda > 0$ (for $\nabla^2\phi + \lambda\phi = 0$) anytime $\phi = 0$ on the entire boundary, as it is here. We can also show that $\lambda > 0$ for this problem using the one-dimensional Rayleigh quotient based on (7.7.18)–(7.7.20) (see Exercise 7.7.13). Equation (7.7.25) has the advantage of not depending on λ; less work is necessary to compute solutions of (7.7.25) than of (7.7.23). However, we have gained more than that since (7.7.25) has been investigated for over 150 years. It is now known as **Bessel's differential equation of order** m.

7.7.5 Singular Points and Bessel's Differential Equation

In this subsection we *briefly* develop some of the properties of Bessel's differential equation. Equation (7.7.25) is a second-order linear differential equation with variable coefficients. We will not be able to obtain an exact closed-form solution of (7.7.25) *involving elementary functions.* To analyze a differential equation, one of the first things we should do is search for any special values of z that might cause some difficulties. $z = 0$ is a singular point of (7.7.25).

Perhaps we should define a singular point of a differential equation. We refer to the standard form:

$$\frac{d^2 f}{dz^2} + a(z)\frac{df}{dz} + b(z) f = 0.$$

If $a(z)$ and $b(z)$ and all their derivatives are finite at $z = z_0$, then $z = z_0$ is called an ordinary point. Otherwise, $z = z_0$ is a singular point. For Bessel's differential equation, $a(z) = 1/z$ and $b(z) = 1 - m^2/z^2$. All finite[4] z except $z = 0$ are ordinary points. $z = 0$ is a singular point [since, for example, $a(0)$ does not exist].

In the neighborhood of any ordinary point, it is known from the theory of differential equations that all solutions of the differential equation are well behaved [i.e., $f(z)$ and all its derivatives exist at any ordinary point]. We thus are guaranteed that all solutions of Bessel's differential equation are well behaved at every finite point except possibly at $z = 0$. The only difficulty can occur in the neighborhood of $z = 0$. We will investigate the expected behavior of solutions of Bessel's differential equation in the neighborhood of $z = 0$. We will describe a crude (but important) approximation. If z is very close to 0, then we should expect that $z^2 f$ in Bessel's

[3]In other problems, if $\lambda = 0$, then the transformation (7.7.24) is invalid. However, (7.7.24) is unnecessary for $\lambda = 0$ since in this case (7.7.23) becomes an equidimensional equation and can be solved (as in Sec. 2.5.2).

[4]With an appropriate definition, it can be shown that $z = \infty$ is not an ordinary point for Bessel's differential equation.

differential equation can be ignored, since it is much smaller than $m^2 f$.[5] We do not ignore $z^2\ d^2f/dz^2$ or $z\ df/dz$ because although z is small, it is possible that derivatives of f are large enough so that $z\ df/dz$ is as large as $-m^2 f$. Dropping $z^2 f$ yields

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} - m^2 f \approx 0, \tag{7.7.26}$$

a valid approximation near $z = 0$. The advantage of this *approximation* is that (7.7.26) is exactly solvable, since it is an equidimensional (also known as a Cauchy or Euler) equation (see Sec. 2.5.2). Equation (7.7.26) can be solved by seeking solutions in the form

$$f \approx z^s. \tag{7.7.27}$$

By substituting (7.7.27) into (7.7.26) we obtain a quadratic equation for s,

$$s(s-1) + s - m^2 = 0, \tag{7.7.28}$$

known as the **indicial equation**. Thus, $s^2 = m^2$, and the two roots (indices) are $s = \pm m$. If $m \neq 0$ (in which case we assume $m > 0$), then we obtain two independent approximate solutions.

$$f \approx z^m \quad \text{and} \quad f \approx z^{-m} \quad (m > 0). \tag{7.7.29}$$

However, if $m = 0$, we only obtain one independent solution $f \approx z^0 = 1$. A second solution is easily derived from (7.7.26). If $m = 0$,

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} \approx 0 \quad \text{or} \quad z\frac{d}{dz}\left(z\frac{df}{dz}\right) \approx 0.$$

Thus, $z\ df/dz$ is constant and, in addition to $f \approx 1$, it is also possible for $f \approx \ln z$. In summary, for $m = 0$, two independent solutions have the expected behavior near $z = 0$,

$$f \approx 1 \quad \text{and} \quad f \approx \ln z \quad (m = 0). \tag{7.7.30}$$

The general solution of Bessel's differential equation will be a linear combination of two independent solutions, satisfying (7.7.29) if $m \neq 0$ and (7.7.30) if $m = 0$. We have only obtained the expected approximate behavior near $z = 0$. More will be discussed in the next subsection. Because of the singular point at $z = 0$, it is possible for solutions not to be well behaved at $z = 0$. We see from (7.7.29) and (7.7.30) that independent solutions of Bessel's differential equation can be chosen such that one is well behaved at $z = 0$ and one solution is not well behaved at $z = 0$ [note that for one solution $\lim_{z\to 0} f(z) = \pm\infty$].

7.7.6 Bessel Functions and Their Asymptotic Properties (near $z = 0$)

We continue to discuss Bessel's differential equation of order m,

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + \left(z^2 - m^2\right) f = 0. \tag{7.7.31}$$

[5]Even if $m = 0$, we still claim that $z^2 f$ can be neglected near $z = 0$ and the result will give a reasonable approximation.

As motivated by the previous discussion, we claim there are two types of solutions, solutions that are well behaved near $z = 0$ and solutions that are singular at $z = 0$. Different values of m yield a different differential equation. Its corresponding solution will depend on m. We introduce the standard notation for a *well-behaved solution* of (7.7.31), $J_m(z)$, called the **Bessel function of the first kind of order m**. In a similar vein, we introduce the notation for a singular solution of Bessel's differential equation, $Y_m(z)$, called the **Bessel function of the second kind of order m**. You can solve a lot of problems using Bessel's differential equation by just remembering that $Y_m(z)$ *approaches* $\pm\infty$ *as* $z \to 0$.

The general solution of any linear homogeneous second-order differential equation is a linear combination of two independent solutions. Thus, the general solution of Bessel's differential equation (7.7.31) is

$$\boxed{f = c_1 J_m(z) + c_2 Y_m(z).} \tag{7.7.32}$$

Precise definitions of $J_m(z)$ and $Y_m(z)$ are given in Sec. 7.8. However, for our immediate purposes, we simply note that they satisfy the following asymptotic properties *for small* z ($z \to 0$):

$$\boxed{\begin{array}{lll} J_m(z) & \sim & \begin{cases} 1 & m = 0 \\ \frac{1}{2^m m!} z^m & m > 0 \end{cases} \\ Y_m(z) & \sim & \begin{cases} \frac{2}{\pi}\ln z & m = 0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m > 0. \end{cases} \end{array}} \tag{7.7.33}$$

It should be seen that (7.7.33) is consistent with our approximate behavior, (7.7.29) and (7.7.30). We see that $J_m(z)$ is bounded as $z \to 0$ whereas $Y_m(z)$ is not.

7.7.7 Eigenvalue Problem Involving Bessel Functions

In this section we determine the eigenvalues of the singular Sturm-Liouville problem (m fixed):

$$\boxed{\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0} \tag{7.7.34}$$

$$\boxed{f(a) = 0} \tag{7.7.35}$$

$$\boxed{|f(0)| < \infty.} \tag{7.7.36}$$

By the change of variables $z = \sqrt{\lambda} r$, (7.7.34) becomes Bessel's differential equation,

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2) f = 0.$$

The general solution is a linear combination of Bessel functions, $f = c_1 J_m(z) + c_2 Y_m(z)$. The scale change implies that in terms of the radial coordinate r,

$$\boxed{f = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r).} \tag{7.7.37}$$

Applying the homogeneous boundary conditions (7.7.35) and (7.7.36) will determine the eigenvalues. $f(0)$ must be finite. However, $Y_m(0)$ is infinite. Thus, $c_2 = 0$, implying that

$$\boxed{f = c_1 J_m(\sqrt{\lambda} r).} \tag{7.7.38}$$

Thus, the condition $f(a) = 0$ determines the eigenvalues:

$$\boxed{J_m(\sqrt{\lambda} a) = 0.} \tag{7.7.39}$$

We see that $\sqrt{\lambda} a$ must be a zero of the Bessel function $J_m(z)$. Later in Sec. 7.8.1, we show that a Bessel function is a decaying oscillation. There is an infinite number of zeros of each Bessel function $J_m(z)$. Let z_{mn} designate the nth zero of $J_m(z)$. Then

$$\sqrt{\lambda} a = z_{mn} \quad \text{or} \quad \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2. \tag{7.7.40}$$

For each m, there is an infinite number of eigenvalues, and (7.7.40) is analogous to $\lambda = (n\pi/L)^2$, where $n\pi$ are the zeros of $\sin x$.

Example. Consider $J_0(z)$, sketched in detail in Fig. 7.7.1. From accurate tables, it is known that the first zero of $J_0(z)$ is $z = 2.4048255577\ldots$. Other zeros are recorded in Fig. 7.7.1. The eigenvalues are $\lambda_{0n} = (z_{0n}/a)^2$. Separate tables of the zeros are available. The *Handbook of Mathematical Functions* (Abramowitz and Stegun [1974]) is one source. Alternatively, over 700 pages are devoted to Bessel functions in *A Treatise on the Theory of Bessel Functions* by Watson [1995].

Eigenfunctions. The eigenfunctions are thus

$$J_m\left(\sqrt{\lambda_{mn}} r\right) = J_m\left(z_{mn}\frac{r}{a}\right), \tag{7.7.41}$$

for $m = 0, 1, 2, \ldots, n = 1, 2, \ldots$. For each m, these are an infinite set of eigenfunctions for the singular Sturm-Liouville problem, (7.7.34)–(7.7.36). *For fixed m* they are orthogonal with weight r [as already discussed, see (7.7.22)]:

$$\int_0^a J_m\left(\sqrt{\lambda_{mp}} r\right) J_m\left(\sqrt{\lambda_{mq}} r\right) r\, dr = 0, \quad p \neq q. \tag{7.7.42}$$

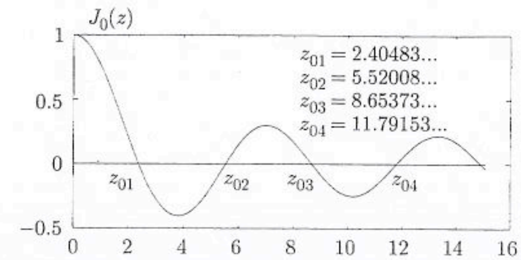

Figure 7.7.1 Sketch of $J_0(z)$ and its zeros.

It is known that this infinite set of eigenfunctions (m fixed) is complete. Thus, any piecewise smooth function of r can be represented by a generalized Fourier series of the eigenfunctions:

$$\alpha(r) = \sum_{n=1}^{\infty} a_n J_m\left(\sqrt{\lambda_{mn}} r\right), \tag{7.7.43}$$

where m is fixed. This is sometimes known as a **Fourier-Bessel series**. The coefficients can be determined by the orthogonality of the Bessel functions (with weight r):

$$a_n = \frac{\int_0^a \alpha(r) J_m\left(\sqrt{\lambda_{mn}} r\right) r\, dr}{\int_0^a J_m^2\left(\sqrt{\lambda_{mn}} r\right) r\, dr} \tag{7.7.44}$$

This illustrates the one-dimensional orthogonality of the Bessel functions. We omit the evaluation of the normalization integrals $\int_0^a J_m^2\left(\sqrt{\lambda_{mn}} r\right) r\, dr$ (e.g., see Churchill [1972] and Berg and McGregor [1966]).

7.7.8 Initial Value Problem for a Vibrating Circular Membrane

The vibrations $u(r, \theta, t)$ of a circular membrane are described by the two-dimensional wave equation, (7.7.1), with u being fixed on the boundary, (7.7.2), subject to the initial conditions (7.7.3). When we apply the method of separation of variables, we obtain four families of product solutions, $u(r, \theta, t) = f(r)g(\theta)h(t)$:

$$J_m\left(\sqrt{\lambda_{mn}} r\right) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} \left\{ \begin{array}{c} \cos c\sqrt{\lambda_{mn}} t \\ \sin c\sqrt{\lambda_{mn}} t \end{array} \right\}. \tag{7.7.45}$$

To simplify the algebra, we will assume that the membrane is initially at rest,

$$\frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta) = 0.$$

Thus, the $\sin c\sqrt{\lambda_{mn}} t$ terms in (7.7.45) will not be necessary. Then according to the principle of superposition, we attempt to satisfy the initial value problem by considering the infinite linear combination of the remaining product solutions:

$$\begin{aligned} u(r,\theta,t) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta \cos c\sqrt{\lambda_{mn}}t \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta \cos c\sqrt{\lambda_{mn}}t. \end{aligned} \tag{7.7.46}$$

The initial position $u(r,\theta,0)=\alpha(r,\theta)$ implies that

$$\begin{aligned} \alpha(r,\theta) &= \sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty} A_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\right)\cos m\theta \\ &+ \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} B_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\right)\sin m\theta. \end{aligned} \tag{7.7.47}$$

By properly arranging the terms in (7.7.47), we see that this is an ordinary Fourier series in θ. Their Fourier coefficients are Fourier-Bessel series (note that m is fixed). Thus, the coefficients may be determined by the orthogonality of $J_m\left(\sqrt{\lambda_{mn}}r\right)$ with weight r [as in (7.7.44)]. As such we can determine the coefficients by repeated application of one-dimensional orthogonality. Two families of coefficients A_{mn} and B_{mn} (including $m=0$) can be determined from one initial condition since the periodicity in θ yielded two eigenfunctions corresponding to each eigenvalue.

However, it is somewhat easier to determine all the coefficients using two-dimensional orthogonality. Recall that for the two-dimensional eigenvalue problem,

$$\nabla^2\phi + \lambda\phi = 0$$

with $\phi = 0$ on the circle of radius a, the two-dimensional eigenfunctions are the doubly infinite families

$$\phi_\lambda(r,\theta) = J_m\left(\sqrt{\lambda_{mn}}r\right)\left\{\begin{array}{c}\cos m\theta \\ \sin m\theta\end{array}\right\}.$$

Thus,

$$\alpha(r,\theta) = \sum_\lambda A_\lambda \phi_\lambda(r,\theta), \tag{7.7.48}$$

where $\sum_\lambda$ stands for a summation over all eigenfunctions [actually two double sums, including both $\sin m\theta$ and $\cos m\theta$ as in (7.7.47)]. These eigenfunctions $\phi_\lambda(r,\theta)$ are orthogonal (in a two-dimensional sense) with weight 1. We then immediately calculate A_λ (representing both A_{mn} and B_{mn}),

$$A_\lambda = \frac{\int\int \alpha(r,\theta)\phi_\lambda(r,\theta)\,dA}{\int\int \phi_\lambda^2(r,\theta)\,dA}. \tag{7.7.49}$$

Here $dA = r\,dr\,d\theta$. In two dimensions the weighting function is constant. However, for geometric reasons $dA = r\,dr\,d\theta$. Thus, **the weight r that appears in the one-dimensional orthogonality of Bessel functions is just a geometric factor**.

7.7.9 Circularly Symmetric Case

In this subsection, as an example, we consider the vibrations of a circular membrane, with $u=0$ on the circular boundary, in the case in which the initial conditions are circularly symmetric (meaning independent of θ). We could consider this as a special case of the general problem, analyzed in Sec. 7.7.8. An alternative method, which yields the same result, is to reformulate the problem. The symmetry of the problem, including the initial conditions suggests that the entire solution should be circularly symmetric; there should be no dependence on the angle θ. Thus,

$$u = u(r,t) \quad \text{and} \quad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad \text{since} \qquad \frac{\partial^2 u}{\partial\theta^2} = 0.$$

The mathematical formulation is thus

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)} \tag{7.7.50}$$

$$\text{BC:} \quad \boxed{u(a,t) = 0} \tag{7.7.51}$$

$$\text{IC:} \quad \boxed{\begin{aligned} u(r,0) &= \alpha(r) \\ \frac{\partial u}{\partial t}(r,0) &= \beta(r). \end{aligned}} \tag{7.7.52}$$

We note that the partial differential equation has two independent variables. We need not study this problem in this chapter, which is reserved for more than two independent variables. We could have analyzed this problem earlier. However, as we will see, Bessel functions are the radially dependent functions, and thus it is more natural to discuss this problem in the present part of this text.

We will apply the method of separation of variables to (7.7.50)–(7.7.52). Looking for product solutions,

$$u(r,t) = \phi(r)h(t), \tag{7.7.53}$$

yields

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda, \tag{7.7.54}$$

where $-\lambda$ is introduced because we suspect that the displacement oscillates in time. The time-dependent equation,

$$\frac{d^2h}{dt^2} = -\lambda c^2 h,$$

has solutions $\sin c\sqrt{\lambda}t$ and $\cos c\sqrt{\lambda}t$ if $\lambda > 0$. The eigenvalue problem for the separation constant is

$$\boxed{\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0} \tag{7.7.55}$$

$$\boxed{\phi(a) = 0} \tag{7.7.56}$$

$$\boxed{|\phi(0)| < \infty.} \tag{7.7.57}$$

Since (7.7.55) is in the form of a Sturm-Liouville problem, we immediately know that eigenfunctions corresponding to distinct eigenvalues are orthogonal with weight r.

From the Rayleigh quotient we could show that $\lambda > 0$. Thus, we may use the transformation

$$z = \sqrt{\lambda}r, \tag{7.7.58}$$

in which case (7.7.55) becomes

$$\frac{d}{dz}\left(z\frac{d\phi}{dz}\right) + z\phi = 0 \quad \text{or} \quad z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + z^2\phi = 0. \tag{7.7.59}$$

We may recall that Bessel's differential equation of order m is

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + \left(z^2 - m^2\right)\phi = 0, \tag{7.7.60}$$

with solutions being Bessel functions of order m, $J_m(z)$ and $Y_m(z)$. A comparison with (7.7.60) shows that (7.7.59) is Bessel's differential equation of order 0. The general solution of (7.7.59) is thus a linear combination of the zeroth-order Bessel functions:

$$\phi = c_1J_0(z) + c_2Y_0(z) = c_1J_0\left(\sqrt{\lambda}r\right) + c_2Y_0\left(\sqrt{\lambda}r\right), \tag{7.7.61}$$

in terms of the radial variable. The singularity condition at the origin (7.7.57) shows that $c_2 = 0$, since $Y_0\left(\sqrt{\lambda}r\right)$ has a logarithmic singularity at $r = 0$:

$$\phi = c_1J_0\left(\sqrt{\lambda}r\right). \tag{7.7.62}$$

Finally, the eigenvalues are determined by the condition at $r = a$, (7.7.56), in which case

$$J_0\left(\sqrt{\lambda}a\right) = 0. \tag{7.7.63}$$

Thus, $\sqrt{\lambda}a$ must be a zero of the zeroth Bessel function. We thus obtain an infinite number of eigenvalues, which we label $\lambda_1, \lambda_2, \ldots$.

We have obtained two infinite families of product solutions

$$J_0\left(\sqrt{\lambda_n}r\right)\sin c\sqrt{\lambda_n}t \quad \text{and} \quad J_0\left(\sqrt{\lambda_n}r\right)\cos c\sqrt{\lambda_n}t.$$

According to the principle of superposition, we seek solutions to our original problem, (7.7.50)–(7.7.52) in the form

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} a_nJ_0\left(\sqrt{\lambda_n}r\right)\cos c\sqrt{\lambda_n}t + \sum_{n=1}^{\infty} b_nJ_0\left(\sqrt{\lambda_n}r\right)\sin c\sqrt{\lambda_n}t.} \tag{7.7.64}$$

As before, we determine the coefficients a_n and b_n from the initial conditions. $u(r,0) = \alpha(r)$ implies that

$$\alpha(r) = \sum_{n=1}^{\infty} a_nJ_0\left(\sqrt{\lambda_n}r\right). \tag{7.7.65}$$

The coefficients a_n are thus the Fourier-Bessel coefficients (of order 0) of $\alpha(r)$. Since $J_0\left(\sqrt{\lambda_n}r\right)$ forms an orthogonal set with weight r, we can easily determine a_n,

$$\boxed{a_n = \frac{\int_0^a \alpha(r)J_0\left(\sqrt{\lambda_n}r\right) r\,dr}{\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right) r\,dr}.} \tag{7.7.66}$$

In a similar manner, the initial condition $\partial/\partial t\; u(r,0) = \beta(r)$ determines b_n.

EXERCISES 7.7

*7.7.1. Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

with $u(a,\theta,t) = 0, u(r,\theta,0) = 0$, and $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$.

7.7.2. Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \quad \text{subject to} \quad \frac{\partial u}{\partial r}(a,\theta,t) = 0$$

with initial conditions

(a) $u(r,\theta,0) = 0$, $\quad \frac{\partial u}{\partial t}(r,\theta,0) = \beta(r)\cos 5\theta$
(b) $u(r,\theta,0) = 0$, $\quad \frac{\partial u}{\partial t}(r,\theta,0) = \beta(r)$
(c) $u(r,\theta,0) = \alpha(r,\theta)$, $\quad \frac{\partial u}{\partial t}(r,\theta,0) = 0$
*(d) $u(r,\theta,0) = 0$, $\quad \frac{\partial u}{\partial t}(r,\theta,0) = \beta(r,\theta)$

7.7.3. Consider a vibrating quarter-circular membrane, $0 < r < a, 0 < \theta < \pi/2$, with $u = 0$ on the entire boundary.

*(a) Determine an expression for the frequencies of vibration.

(b) Solve the initial value problem if

$$u(r,\theta,0) = g(r,\theta), \qquad \frac{\partial u}{\partial t}(r,\theta,0) = 0.$$

7.7.4. Consider the displacement $u(r,\theta,t)$ of a "pie-shaped" membrane of radius a (and angle $\pi/3 = 60°$) that satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

Assume that $\lambda > 0$. Determine the natural frequencies of oscillation if the boundary conditions are

(a) $u(r,0,t) = 0, \quad u(r,\pi/3,t) = 0, \quad \frac{\partial u}{\partial r}(a,\theta,t) = 0$

(b) $u(r,0,t) = 0, \quad u(r,\pi/3,t) = 0, \quad u(a,\theta,t) = 0$

*7.7.5. Consider the displacement $u(r,\theta,t)$ of a membrane whose shape is a 90° sector of an annulus, $a < r < b, 0 < \theta < \pi/2$, with the conditions that $u = 0$ on the entire boundary. Determine the natural frequencies of vibration.

7.7.6. Consider the circular membrane satisfying

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

subject to the boundary condition

$$u(a,\theta,t) = -\frac{\partial u}{\partial r}(a,\theta,t).$$

(a) Show that this membrane only oscillates.

(b) Obtain an expression that determines the natural frequencies.

(c) Solve the initial value problem if

$$u(r,\theta,0) = 0, \quad \frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta.$$

7.7.7. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a circle of radius a with zero temperature around the entire boundary, if initially

$$u(r,\theta,0) = f(r,\theta).$$

Briefly analyze $\lim_{t\to\infty} u(r,\theta,t)$. Compare this to what you expect to occur using physical reasoning as $t \to \infty$.

*7.7.8. Reconsider Exercise 7.7.7, but with the entire boundary insulated.

7.7.9. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a semicircle of radius a and briefly analyze the $\lim_{t\to\infty}$ if the initial conditions are

$$u(r,\theta,0) = f(r,\theta)$$

and the boundary conditions are

(a)	$u(r,0,t) = 0,$	$u(r,\pi,t) = 0,$	$\frac{\partial u}{\partial r}(a,\theta,t) = 0$
* (b)	$\frac{\partial u}{\partial \theta}(r,0,t) = 0,$	$\frac{\partial u}{\partial \theta}(r,\pi,t) = 0,$	$\frac{\partial u}{\partial r}(a,\theta,t) = 0$
(c)	$\frac{\partial u}{\partial \theta}(r,0,t) = 0,$	$\frac{\partial u}{\partial \theta}(r,\pi,t) = 0,$	$u(a,\theta,t) = 0$
(d)	$u(r,0,t) = 0,$	$u(r,\pi,t) = 0,$	$u(a,\theta,t) = 0$

*7.7.10. Solve for $u(r,t)$ if it satisfies the circularly symmetric heat equation

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

subject to the conditions

$$\begin{aligned} u(a,t) &= 0 \\ u(r,0) &= f(r). \end{aligned}$$

Briefly analyze the $\lim_{t\to\infty}$.

7.7.11. Reconsider Exercise 7.7.10 with the boundary condition

$$\frac{\partial u}{\partial r}(a,t) = 0.$$

7.7.12. For the following differential equations, what is the expected approximate behavior of all solutions *near $x = 0$*?

*(a) $x^2\frac{d^2y}{dx^2} + (x-6)y = 0$

(b) $x^2\frac{d^2y}{dx^2} + \left(x^2 + \frac{3}{16}\right)y = 0$

*(c) $x^2\frac{d^2y}{dx^2} + (x+x^2)\frac{dy}{dx} + 4y = 0$

(d) $x^2\frac{d^2y}{dx^2} + (x+x^2)\frac{dy}{dx} - 4y = 0$

*(e) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (6+x^3)y = 0$

(f) $x^2\frac{d^2y}{dx^2} + \left(x + \frac{1}{4}\right)y = 0$

7.7.13. Using the one-dimensional Rayleigh quotient, show that $\lambda > 0$ as defined by (7.7.18)–(7.7.20).

7.8 More on Bessel Functions

7.8.1 Qualitative Properties of Bessel Functions

It is helpful to have some understanding of the sketch of Bessel functions. Let us rewrite Bessel's differential equation as

$$\frac{d^2f}{dz^2} = -\left(1 - \frac{m^2}{z^2}\right) f - \frac{1}{z}\frac{df}{dz}, \tag{7.8.1}$$

in order to compare it with the equation describing the motion of a spring-mass system (unit mass, spring "constant" k and frictional coefficient c):

$$\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}.$$

The equilibrium is $y = 0$. Thus, we might think of Bessel's differential equation as representing a time-varying frictional force ($c = 1/t$) and a time-varying "restoring" force ($k = 1 - m^2/t^2$). The latter force is a variable restoring force only for $t > m (z > m)$. We might expect the solutions of Bessel's differential equation to be similar to a damped oscillator (at least for $z > m$). The larger z gets, the closer the variable spring constant k approaches 1 and the more the frictional force tends to vanish. The solution should oscillate with frequency approximately 1, but should slowly decay. This is similar to an underdamped spring-mass system, but the solutions to Bessel's differential equation should decay more slowly than any exponential since the frictional force is approaching zero. Detailed numerical solutions of Bessel functions are sketched in Fig. 7.8.1, verifying these points. Note that *for small* z,

$$\begin{aligned} J_0(z) &\approx 1 & Y_0(z) &\approx \frac{2}{\pi}\ln z \\ J_1(z) &\approx \tfrac{1}{2}z & Y_1(z) &\approx -\frac{2}{\pi}z^{-1} \\ J_2(z) &\approx \tfrac{1}{8}z^2 & Y_2(z) &\approx -\frac{4}{\pi}z^{-2}. \end{aligned} \tag{7.8.2}$$

These sketches vividly show a property worth memorizing: **Bessel functions of the first and second kind look like decaying oscillations.** In fact, it is known that $J_m(z)$ and $Y_m(z)$ may be accurately approximated for large z by simple algebraically decaying oscillations *for large* z:

$$\boxed{\begin{aligned} J_m(z) &\sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \quad \text{as } z \to \infty \\ Y_m(z) &\sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \quad \text{as } z \to \infty. \end{aligned}} \tag{7.8.3}$$

These are known as asymptotic formulas, meaning that the approximations improve as $z \to \infty$. In Sec. 5.9 we claimed that approximation formulas similar to (7.8.3)

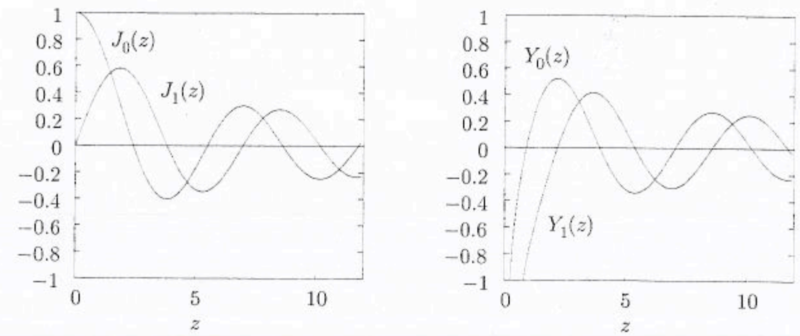

Figure 7.8.1 Sketch of various Bessel functions.

always exist for any Sturm-Liouville problem for the large eigenvalues $\lambda \gg 1$. Here $\lambda \gg 1$ implies that $z \gg 1$ since $z = \sqrt{\lambda} r$ and $0 < r < a$ (as long as r is not too small).

A derivation of (7.8.3) requires facts beyond the scope of this text. However, information such as (7.8.3) is readily available from many sources.[6] We notice from (7.8.3) that the only difference in the approximate behavior for large z of all these Bessel functions is the precise phase shift. We also note that the frequency is approximately 1 (and period 2π) for large z, consistent with the comparison with a spring-mass system with vanishing friction and $k \to 1$. Furthermore, the amplitude of oscillation, $\sqrt{2/\pi z}$, decays more slowly as $z \to \infty$ than the exponential rate of decay associated with an underdamped oscillator, as previously discussed qualitatively.

7.8.2 Asymptotic Formulas for the Eigenvalues

Approximate values of the zeros of the eigenfunctions $J_m(z)$ may be obtained using these asymptotic formulas, (7.8.3). For example, for $m = 0$, for large z

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right).$$

The zeros approximately occur when $z - \pi/4 = -\pi/2 + s\pi$, but s must be large (in order for z to be large). Thus, the large zeros are given approximately by

$$z \sim \pi\left(s - \frac{1}{4}\right), \tag{7.8.4}$$

for large integral s. We claim that formula (7.8.4) becomes more and more accurate as n increases. In fact, since the formula is reasonably accurate already for $n = 2$ or 3

[6] A personal favorite, highly recommended to students with a serious interest in the applications of mathematics to science and engineering, is *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, originally published *inexpensively* by the National Bureau of Standards in 1964 and in 1974 reprinted by Dover in paperback.

Table 7.8.1: Zeros of $J_0(z)$

n	z_{0n}	Exact	Large z formula (7.8.4)	Error	Percentage error	$z_{0n} - z_{0(n-1)}$
1	z_{01}	2.40483 ...	2.35619	0.04864	2.0	—
2	z_{02}	5.52008 ...	5.49779	0.02229	0.4	3.11525
3	z_{03}	8.65373 ...	8.63938	0.01435	0.2	3.13365
4	z_{04}	11.79153 ...	11.78097	0.01156	0.1	3.13780

(see Table 7.8.1), it may be unnecessary to compute the zero to a greater accuracy than is given by (7.8.4). A further indication of the accuracy of the asymptotic formula is that we see that the differences of the first few eigenvalues are already nearly π (as predicted for the large eigenvalues).

7.8.3 Zeros of Bessel Functions and Nodal Curves

We have shown that the eigenfunctions are $J_m\left(\sqrt{\lambda_{mn}}r\right)$ where $\lambda_{mn} = (z_{mn}/a)^2$, z_{mn} being the nth zero of $J_m(z)$. Thus, the eigenfunctions are

$$J_m\left(z_{mn}\frac{r}{a}\right).$$

For example, for $m = 0$, the eigenfunctions are $J_0(z_{0n}r/a)$, where the sketch of $J_0(z)$ is reproduced in Fig. 7.8.2 (and the zeros are marked). As r ranges from 0 to a, the argument of the eigenfunction $J_0(z_{0n}r/a)$ ranges from 0 to the nth zero, z_{0n}. At $r = a$, $z = z_{0n}$, the nth zero. Thus, the nth eigenfunction has $n-1$ zeros in the interior. Although originally stated for regular Sturm-Liouville problems, it is also valid for singular problems (if eigenfunctions exist).

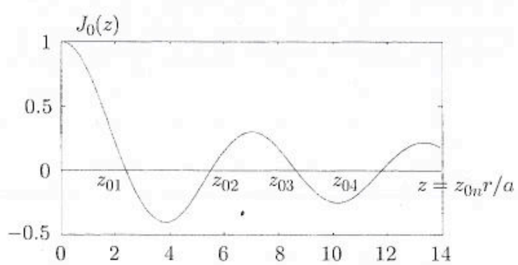

Figure 7.8.2 Sketch of $J_0(z)$ and its zeros.

The separation of variables solution of the wave equation is

$$u(r, \theta, t) = f(r)g(\theta)h(t),$$

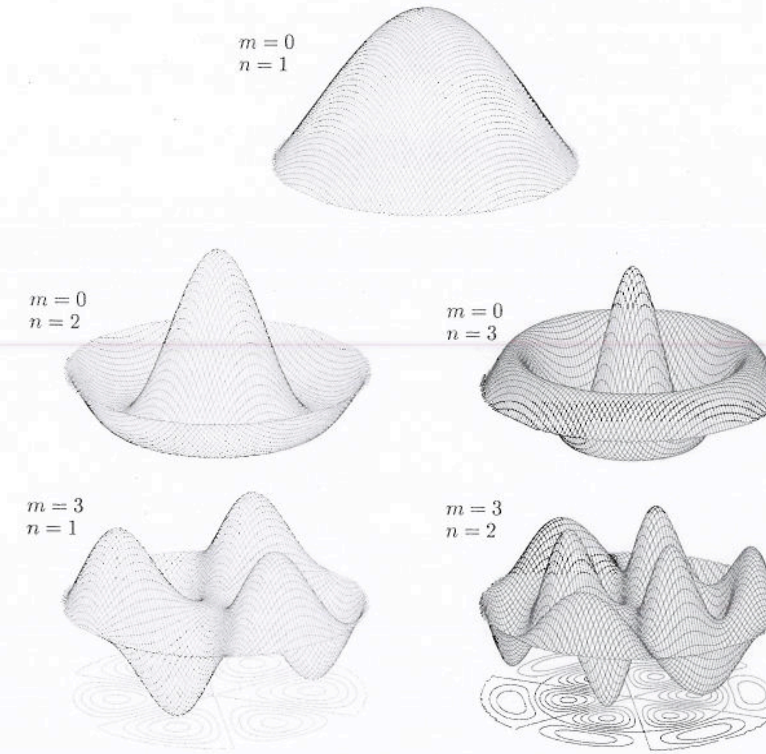

Figure 7.8.3 Normal nodes and nodal curves for a vibrating circular membrane.

where

$$u(r, \theta, t) = J_m\left(z_{mn}\frac{r}{a}\right)\left\{\begin{array}{c}\sin m\theta \\ \cos m\theta\end{array}\right\}\left\{\begin{array}{c}\sin c\sqrt{\lambda_{mn}}t \\ \cos c\sqrt{\lambda_{mn}}t\end{array}\right\}, \tag{7.8.5}$$

known as a normal mode of oscillation and is graphed for fixed t in Fig. 7.8.3. For each $m \neq 0$ there are four families of solutions (for $m = 0$ there are two families). Each mode oscillates with a characteristic natural frequency, $c\sqrt{\lambda_{mn}}$. At certain positions along the membrane, known as **nodal curves**, the membrane will be unperturbed for all time (for vibrating strings we called these positions nodes). The nodal curve for the $\sin m\theta$ mode is determined by

$$J_m\left(z_{mn}\frac{r}{a}\right)\sin m\theta = 0. \tag{7.8.6}$$

The nodal curve consists of all points where $\sin m\theta = 0$ or $J_m(z_{mn}r/a) = 0$; $\sin m\theta$ is zero along $2m$ distinct rays, $\theta = s\pi/m, s = 1, 2, \ldots, 2m$. In order for there to be a zero of $J_m(z_{mn}r/a)$ for $0 < r < a$, $z_{mn}r/a$ must equal an earlier zero of

$J_m(z)$, $z_{mn}r/a = z_{mp}, p = 1, 2, \ldots, n-1$. There are thus $n-1$ circles along which $J_m(z_{mn}r/a) = 0$ besides $r = a$. We illustrate this for $m = 3$, $n = 2$ in Fig. 7.8.3, where the nodal circles are determined from a table.

7.8.4 Series Representation of Bessel Functions

The usual method of discussing Bessel functions relies on series solution methods for differential equations. We will obtain little useful information by pursuing this topic. However, some may find it helpful to refer to the formulas that follow.

First we review some additional results concerning series solutions around $z = 0$ for second-order linear differential equations:

$$\frac{d^2 f}{dz^2} + a(z)\frac{df}{dz} + b(z)f = 0. \tag{7.8.7}$$

Recall that $z = 0$ is an ordinary point if both $a(z)$ and $b(z)$ have Taylor series around $z = 0$. In this case we are guaranteed that all solutions may be represented by a convergent Taylor series,

$$f = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots,$$

at least in some neighborhood of $z = 0$.

If $z = 0$ is not an ordinary point, then we call it a singular point (e.g., $z = 0$ is a singular point of Bessel's differential equation). If $z = 0$ is a singular point, we cannot state that all solutions have Taylor series around $z = 0$. However, if $a(z) = R(z)/z$ and $b(z) = S(z)/z^2$ with $R(z)$ and $S(z)$ having Taylor series, then we can say more about solutions of the differential equation near $z = 0$. For this case known as a **regular singular point**, the coefficients $a(z)$ and $b(z)$ can have *at worst* a simple pole and double pole, respectively. It is possible for the coefficients $a(z)$ and $b(z)$ not to be that singular. For example, if $a(z) = 1 + z$ and $b(z) = (1 - z^3)/z^2$, then $z = 0$ is a regular singular point. Bessel's differential equation in the form (7.8.7) is

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + \frac{z^2 - m^2}{z^2} f = 0.$$

Here $R(z) = 1$ and $S(z) = z^2 - m^2$; both have Taylor series around $z = 0$. Therefore, $z = 0$ is a regular singular point for Bessel's differential equation.

For a regular singular point at $z = 0$, it is known by the **method of Frobenius** that at least one solution of the differential equation is in the form

$$f = z^p \sum_{n=0}^{\infty} a_n z^n, \tag{7.8.8}$$

that is, z^p times a Taylor series, where p is one of the solutions of the quadratic indicial equation. One method to obtain the indicial equation is to substitute $f = z^p$

into the corresponding equidimensional equation that results by replacing $R(z)$ by $R(0)$ and $S(z)$ by $S(0)$. Thus

$$p(p-1) + R(0)p + S(0) = 0$$

is the indicial equation. If the two values of p (the roots of the indicial equation) differ by a noninteger, then two independent solutions exist in the form (7.8.8). If the two roots of the indicial equation are identical, then only one solution is in the form (7.8.8) and the other solution is more complicated but always involves logarithms. If the roots differ by an integer, then sometimes both solutions exist in the form (7.8.8), while other times form (7.8.8) only exists corresponding to the larger root p and a series beginning with the smaller root p must be modified by the introduction of logarithms. Details of the method of Frobenius are presented in most elementary differential equations texts.

For Bessel's differential equation, we have shown that the indicial equation is

$$p(p-1) + p - m^2 = 0,$$

since $R(0) = 1$ and $S(0) = -m^2$. Its roots are $\pm m$. If $m = 0$, the roots are identical. Form (7.8.8) is valid for one solution, while logarithms must enter the second solution. For $m \neq 0$ the roots of the indicial equation differ by an integer. Detailed calculations also show that logarithms must enter. The following infinite series can be verified by substitution and are often considered as definitions of $J_m(z)$ and $Y_m(z)$:

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+m}}{k!(k+m)!} \tag{7.8.9}$$

$$\begin{aligned} Y_m(z) &= \frac{2}{\pi}\left[\left(\log\frac{z}{2} + \gamma\right) J_m(z) - \frac{1}{2}\sum_{k=0}^{m-1} \frac{(m-k-1)!(z/2)^{2k-m}}{k!}\right. \\ &+ \left.\frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\left[\varphi(k) + \varphi(k+m)\right]\frac{(z/2)^{2k+m}}{k!(m+k)!}\right], \end{aligned} \tag{7.8.10}$$

where

(i) $\varphi(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/k$, $\phi(0) = 0$

(ii) $\gamma = \lim_{k\to\infty}[\varphi(k) - \ln k] = 0.5772157\ldots$, known as Euler's constant.

(iii) If $m = 0$, $\sum_{k=0}^{m-1} \equiv 0$.

We have obtained these from the previously mentioned handbook edited by Abramowitz and Stegun.

EXERCISES 7.8

7.8.1. The boundary value problem for a vibrating annular membrane $1 < r < 2$ (fixed at the inner and outer radii) is

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0$$

with $f(1) = 0$ and $f(2) = 0$, where $m = 0, 1, 2, \ldots$.

(a) Show that $\lambda > 0$.

*(b) Obtain an expression that determines the eigenvalues.

(c) For what value of m does the smallest eigenvalue occur?

*(d) Obtain an upper and lower bound for the smallest eigenvalue.

(e) Using a trial function, obtain an upper bound for the lowest eigenvalue.

(f) Compute approximately the lowest eigenvalue from part (b) using tables of Bessel functions. Compare to parts (d) and (e).

7.8.2. Consider the temperature $u(r, \theta, t)$ in a quarter-circle of radius a satisfying

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

subject to the conditions

$$\begin{aligned} u(r,0,t) &= 0 \quad u(a,\theta,t) = 0 \\ u(r,\pi/2,t) &= 0 \quad u(r,\theta,0) = G(r,\theta). \end{aligned}$$

(a) Show that the boundary value problem is

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{\mu}{r}\right) f = 0$$

with $f(a) = 0$ and $f(0)$ bounded.

(b) Show that $\lambda > 0$ if $\mu \geq 0$.

(c) Show that for each μ, the eigenfunction corresponding to the smallest eigenvalue has no zeros for $0 < r < a$.

*(d) Solve the initial value problem.

7.8.3. Reconsider Exercise 7.8.2 with the boundary conditions

$$\frac{\partial u}{\partial \theta}(r,0,t) = 0, \quad \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{2}, t\right) = 0, \quad u(a,\theta,t) = 0.$$

7.8.4. Consider the boundary value problem

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0$$

with $f(a) = 0$ and $f(0)$ bounded. For each integral m, show that the nth eigenfunction has $n - 1$ zeros for $0 < r < a$.

7.8.5. Using the known asymptotic behavior as $z \to 0$ and as $z \to \infty$, roughly sketch for all $z > 0$

(a) $J_4(z)$	(b) $Y_1(z)$	(c) $Y_0(z)$
(d) $J_0(z)$	(e) $Y_5(z)$	(f) $J_2(z)$

7.8.6. Determine approximately the *large* frequencies of vibration of a circular membrane.

7.8.7. Consider Bessel's differential equation

$$z^2\frac{d^2 f}{dz^2} + z\frac{df}{dz} + \left(z^2 - m^2\right) f = 0.$$

Let $f = y/z^{1/2}$. Derive that

$$\frac{d^2 y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0.$$

*7.8.8. Using Exercise 7.8.7, determine exact expressions for $J_{1/2}(z)$ and $Y_{1/2}(z)$. Use and verify (7.8.3) and (7.7.33) in this case.

7.8.9. In this exercise use the result of Exercise 7.8.7. If z is large, verify as much as possible concerning (7.8.3).

7.8.10. In this exercise use the result of Exercise 7.8.7 in order to improve on (7.8.3):

(a) Substitute $y = e^{iz}w(z)$ and show that

$$\frac{d^2 w}{dz^2} + 2i\frac{dw}{dz} + \frac{\gamma}{z^2}w = 0, \text{ where } \gamma = \frac{1}{4} - m^2.$$

(b) Substitute $w = \sum_{n=0}^{\infty} \beta_n z^{-n}$. Determine the first few terms β_n (assuming that $\beta_0 = 1$).

(c) Use part (b) to obtain an improved asymptotic solution of Bessel's differential equation. For real solutions, take real and imaginary parts.

(d) Find a recurrence formula for β_n. Show that the series diverges. (Nonetheless, a finite series is very useful.)

7.8.11. In order to "understand" the behavior of Bessel's differential equation as $z \to \infty$, let $x = 1/z$. Show that $x = 0$ is a singular point, but an irregular singular point. [The asymptotic solution of a differential equation in the neighborhood of an irregular singular point is analyzed in an unmotivated way in Exercise 7.8.10. For a more systematic presentation, see advanced texts on asymptotic or perturbation methods (such as Bender and Orszag [1999].)]

7.8.12. The lowest eigenvalue for (7.7.34)–(7.7.36) for $m = 0$ is $\lambda = (z_{01}/a)^2$. Determine a reasonably accurate upper bound by using the Rayleigh quotient with a trial function. Compare to the exact answer.

7.8.13. Explain why the nodal circles in Fig. 7.8.3 are nearly equally spaced.

7.9 Laplace's Equation in a Circular Cylinder

7.9.1 Introduction

Laplace's equation,

$$\nabla^2 u = 0, \tag{7.9.1}$$

represents the steady-state heat equation (without sources). We have solved Laplace's equation in a rectangle (Sec. 2.5.1) and Laplace's equation in a circle (Sec. 2.5.2). In both cases, when variables were separated, oscillations occur in one direction, but not in the other. Laplace's equation in a rectangular box can also be solved by the method of separation of variables. As shown in some exercises in Chapter 7, the three independent variables yield two eigenvalue problems that have oscillatory solutions and solutions in one direction that are not oscillatory.

A more interesting problem is to consider Laplace's equation in a circular cylinder of radius a and height H. Using circular cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z, \end{aligned}$$

Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{7.9.2}$$

We prescribe u (perhaps temperature) on the entire boundary of the cylinder:

$$\begin{array}{lrcl} \text{top:} & u(r,\theta,H) & = & \beta(r,\theta) \\ \text{bottom:} & u(r,\theta,0) & = & \alpha(r,\theta) \\ \text{lateral side:} & u(a,\theta,z) & = & \gamma(\theta,z). \end{array}$$

There are three nonhomogeneous boundary conditions. One approach is to break the problem up into the sum of three simpler problems, each solving Laplace's equation,

$$\nabla^2 u_i = 0, \quad i = 1,2,3,$$

where $u = u_1 + u_2 + u_3$. This is illustrated in Fig. 7.9.1. In this way each problem satisfies two homogeneous boundary conditions, but the sum satisfies the desired nonhomogeneous conditions. We separate variables once, for all three cases, and then proceed to solve each problem individually.

7.9.2 Separation of Variables

We begin by looking for product solutions,

$$u(r,\theta,z) = f(r)g(\theta)h(z), \tag{7.9.3}$$

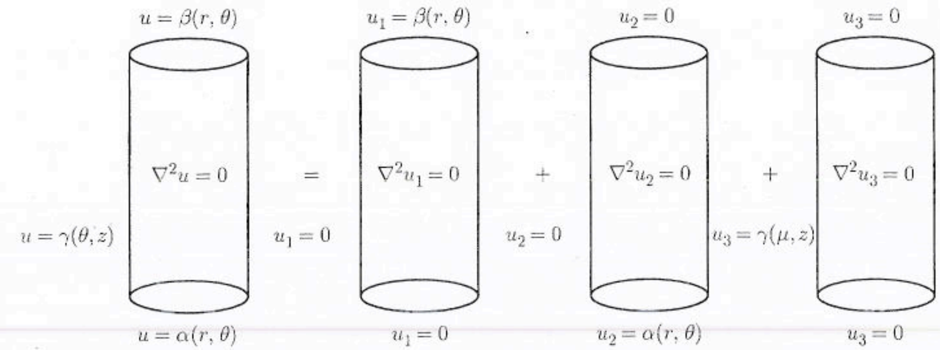

Figure 7.9.1 Laplace's equation in a circular cylinder.

for Laplace's equation. Substituting (7.9.3) into (7.9.2) and dividing by $f(r)g(\theta)h(z)$ yields

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2}\frac{1}{g}\frac{d^2 g}{d\theta^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = 0. \tag{7.9.4}$$

We immediately can separate the z-dependence and hence

$$\frac{1}{h}\frac{d^2 h}{dz^2} = \lambda. \tag{7.9.5}$$

Do we expect oscillations in z? From Fig. 7.9.1 we see that oscillations in z should be expected for the u_3-problem but not necessarily for the u_1- or u_2-problem. Perhaps $\lambda < 0$ for the u_3-problem but not for the u_1- and u_2-problems. Thus, we do not specify λ at this time. The r and θ parts also can be separated if (7.9.4) is multiplied by r^2 [and (7.9.5) is utilized]:

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \lambda r^2 = -\frac{1}{g}\frac{d^2 g}{d\theta^2} = \mu. \tag{7.9.6}$$

A second separation constant μ is introduced, with the anticipation that $\mu > 0$ because of the expected oscillations in θ for all three problems. In fact, the implied periodic boundary conditions in θ dictate that

$$\mu = m^2, \tag{7.9.7}$$

and that $g(\theta)$ can be either $\sin m\theta$ or $\cos m\theta$, where m is a nonnegative integer, $m = 0,1,2,\ldots$. A Fourier series in θ will be appropriate for all these problems.

In summary, the θ-dependence is $\sin m\theta$ and $\cos m\theta$, and the remaining two differential equations are

$$\frac{d^2 h}{dz^2} = \lambda h \tag{7.9.8}$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r^2 - m^2\right)f = 0. \tag{7.9.9}$$

These two differential equations contain only one unspecified parameter λ. Only one will become an eigenvalue problem. The eigenvalue problem needs two homogeneous boundary conditions. Different results occur for the various problems, u_1, u_2, and u_3. For the u_3-problem, there are two homogeneous boundary conditions in z, and thus (7.9.8) will become an eigenvalue problem [and (7.9.9) will have nonoscillatory solutions]. However, for the u_1- and u_2-problems there do not exist two homogeneous boundary conditions in z. Instead, there should be two homogeneous conditions in r. One of these is at $r = a$. The other must be a singularity condition at $r = 0$, which occurs due to the singular nature of polar (or circular cylindrical) coordinates at $r = 0$ and the singular nature of (7.9.9) at $r = 0$:

$$|f(0)| < \infty. \tag{7.9.10}$$

Thus, we will find that for the u_1- and u_2-problems, (7.9.9) will be the eigenvalue problem. The solution of (7.9.9) will oscillate, whereas the solution of (7.9.8) will not oscillate. We next describe the details of all three problems.

7.9.3 Zero Temperature on the Lateral Sides and on the Bottom or Top

The mathematical problem for u_1 is

$$\nabla^2 u_1 = 0 \tag{7.9.11}$$

$$u_1(r, \theta, 0) = 0 \tag{7.9.12}$$

$$u_1(r, \theta, H) = \beta(r, \theta) \tag{7.9.13}$$

$$u_1(a, \theta, z) = 0. \tag{7.9.14}$$

The temperature is zero on the bottom. By separation of variables in which the nonhomogeneous condition (7.9.13) is momentarily ignored, $u_1 = f(r)g(\theta)h(z)$. The θ-part is known to equal $\sin m\theta$ and $\cos m\theta$ (for integral $m \geq 0$). The z-dependent equation, (7.9.8), satisfies only one homogeneous condition, $h(0) = 0$. The r-dependent equation will become a boundary value problem determining the separation constant λ. The two homogeneous boundary conditions are

$$f(a) = 0 \tag{7.9.15}$$

$$|f(0)| < \infty. \tag{7.9.16}$$

The eigenvalue problem, (7.9.9) with (7.9.15) and (7.9.16), is one that was analyzed in Sec. 7.8. There we showed that $\lambda > 0$ (by directly using the Rayleigh quotient).

Furthermore, we showed that the general solution of (7.9.9) is a linear combination of Bessel functions of order m with argument $\sqrt{\lambda}r$:

$$f(r) = c_1 J_m\left(\sqrt{\lambda}r\right) + c_2 Y_m\left(\sqrt{\lambda}r\right) = c_1 J_m\left(\sqrt{\lambda}r\right), \tag{7.9.17}$$

which has been simplified using the singularity condition, (7.9.16).Then the homogeneous condition, (7.9.15), determines λ:

$$J_m\left(\sqrt{\lambda}a\right) = 0. \tag{7.9.18}$$

Again $\sqrt{\lambda}a$ must be a zero of the mth Bessel function, and the notation λ_{mn} is used to indicate the infinite number of eigenvalues for each m. The eigenfunction $J_m\left(\sqrt{\lambda_{mn}}r\right)$ oscillates in r.

Since $\lambda > 0$, the solution of (7.9.8) that satisfies $h(0) = 0$ is proportional to

$$h(z) = \sinh\sqrt{\lambda}z. \tag{7.9.19}$$

No oscillations occur in the z-direction. There are thus two doubly infinite families of product solutions:

$$\sinh\sqrt{\lambda_{mn}}z\; J_m\left(\sqrt{\lambda_{mn}}r\right)\begin{Bmatrix}\sin m\theta\\ \cos m\theta\end{Bmatrix}, \tag{7.9.20}$$

oscillatory in r and θ, but nonoscillatory in z. The principle of superposition implies that we should consider

$$\begin{aligned} u(r,\theta,z) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sinh\sqrt{\lambda_{mn}}z\; J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sinh\sqrt{\lambda_{mn}}z\; J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta. \end{aligned} \tag{7.9.21}$$

The nonhomogeneous boundary condition, (7.9.13), $u_1(r, \theta, H) = \beta(r, \theta)$, will determine the coefficients A_{mn} and B_{mn}. It will involve a Fourier series in θ and a Fourier-Bessel series in r. Thus we can solve A_{mn} and B_{mn} using the two one-dimensional orthogonality formulas. Alternatively, the coefficients are more easily calculated using the two-dimensional orthogonality of $J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta$ and $J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta$ (see Sec. 7.8). We omit the details.

In a similar manner, one can obtain u_2. We leave as an exercise the solution of this problem.

7.9.4 Zero Temperature on the Top and Bottom

A somewhat different mathematical problem arises if we consider the situation in which the top and bottom are held at zero temperature. The problem for u_3 is

$$\nabla^2 u_3 = 0 \tag{7.9.22}$$

$$u_3(r,\theta,0) = 0 \tag{7.9.23}$$

$$u_3(r,\theta,H) = 0 \tag{7.9.24}$$

$$u_3(a,\theta,z) = \gamma(\theta,z). \tag{7.9.25}$$

We may again use the results of the method of separation of variables. The periodicity again implies that the θ-part will relate to a Fourier series (i.e., $\sin m\theta$ and $\cos m\theta$). However, unlike what occurred in Sec. 7.9.3, the z-equation has two homogeneous boundary conditions:

$$\frac{d^2h}{dz^2} = \lambda h \tag{7.9.26}$$

$$h(0) = 0 \tag{7.9.27}$$

$$h(H) = 0. \tag{7.9.28}$$

This is the simplest Sturm-Liouville eigenvalue problem (in a somewhat different form). In order for $h(z)$ to oscillate and satisfy (7.9.27) and (7.9.28), the separation constant λ must be negative. In fact, we should recognize that

$$\lambda = -\left(\frac{n\pi}{H}\right)^2 \quad n = 1,2,\ldots \tag{7.9.29}$$

$$h(z) = \sin\frac{n\pi z}{H}. \tag{7.9.30}$$

The boundary conditions at top and bottom imply that we will be using an ordinary Fourier sine series in z.

We have oscillations in z and θ. The r-dependent solution should not be oscillatory; they satisfy (7.9.9), which using (7.9.29) becomes

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(-\left(\frac{n\pi}{H}\right)^2 r^2 - m^2\right) f = 0. \tag{7.9.31}$$

A homogeneous condition, in the form of a singularity condition, exists at $r = 0$,

$$|f(0)| < \infty, \tag{7.9.32}$$

but there is no homogeneous condition at $r = a$.

Equation (7.9.31) looks similar to Bessel's differential equation but has the wrong sign in front of the r^2 term. It cannot be changed into Bessel's differential equation using a real transformation. If we let

$$s = i\left(\frac{n\pi}{H}\right) r \tag{7.9.33}$$

where $i = \sqrt{-1}$, then (7.9.31) becomes

$$s\frac{d}{ds}\left(s\frac{df}{ds}\right) + \left(s^2 - m^2\right) f = 0 \quad \text{or} \quad s^2\frac{d^2f}{ds^2} + s\frac{df}{ds} + \left(s^2 - m^2\right) f = 0.$$

We recognize this as exactly Bessel's differential equation, and thus

$$f = c_1 J_m(s) + c_2 Y(s) \quad \text{or} \quad f = c_1 J_m\left(i\frac{n\pi}{H}r\right) + c_2 Y_m\left(i\frac{n\pi}{H}r\right). \tag{7.9.34}$$

Therefore, the solution of (7.9.31) can be represented in terms of Bessel functions of an imaginary argument. This is not very useful since Bessel functions are not usually tabulated in this form.

Instead, we introduce a real transformation that eliminates the dependence on $n\pi/H$ of the differential equation:

$$w = \frac{n\pi}{H} r.$$

Then (7.9.31) becomes

$$w^2\frac{d^2f}{dw^2} + w\frac{df}{dw} + \left(-w^2 - m^2\right) f = 0. \tag{7.9.35}$$

Again the wrong sign appears for this to be Bessel's differential equation. Equation (7.9.35) is a modification of Bessel's differential equation, and its solutions, *which have been well tabulated*, are known as **modified Bessel functions**.

Equation (7.9.35) has the same kind of singularity at $w = 0$ as Bessel's differential equation. As such, the singular behavior could be determined by the method of Frobenius.[7] Thus, we can specify *one solution to be well defined at $w = 0$, called the* **modified Bessel function of order m of the first kind**, denoted $I_m(w)$. Another independent solution, which is *singular at the origin, is called the* **modified Bessel function of order m of the second kind**, denoted $K_m(w)$. Both $I_m(w)$ and $K_m(w)$ are well-tabulated functions. We will need very little knowledge concerning $I_m(w)$ and $K_m(w)$. The general solution of (7.9.31) is thus

$$f = c_1 K_m\left(\frac{n\pi}{H}r\right) + c_2 I_m\left(\frac{n\pi}{H}r\right). \tag{7.9.36}$$

Since K_m is singular at $r = 0$ and I_m is not, it follows that $c_1 = 0$ and $f(r)$ is proportional to $I_m(n\pi r/H)$. We simply note that both $I_m(w)$ and $K_m(w)$ are nonoscillatory and are not zero for $w > 0$. A discussion of this and further properties are given in Sec. 7.9.5.

There are thus two doubly infinite families of product solutions:

$$I_m\left(\frac{n\pi}{H}r\right)\sin\frac{n\pi z}{H}\cos m\theta \quad \text{and} \quad I_m\left(\frac{n\pi}{H}r\right)\sin\frac{n\pi z}{H}\sin m\theta. \tag{7.9.37}$$

These solutions are oscillatory in z and θ, but nonoscillatory in r. The principle of superposition, equivalent to a Fourier sine series in z and a Fourier series in θ, implies that

$$\begin{aligned} u_3(r,\theta,z) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} E_{mn} I_m\left(\frac{n\pi}{H}r\right)\sin\frac{n\pi z}{H}\cos m\theta \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\, I_m\left(\frac{n\pi}{H}r\right)\sin\frac{n\pi z}{H}\sin m\theta. \end{aligned} \tag{7.9.38}$$

The coefficients E_{mn} and F_{mn} can be determined [if $I_m(n\pi a/H) \neq 0$] from the nonhomogeneous equation (7.9.25) either by two iterated one-dimensional orthogonality results or by one application of two-dimensional orthogonality. In the next section we will discuss further properties of $I_m(n\pi a/H)$, including the fact that it has no positive zeros.

In this way the solution for Laplace's equation inside a circular cylinder has been determined given any temperature distribution along the entire boundary.

7.9.5 Modified Bessel Functions

The differential equation that defines the modified Bessel functions is

$$w^2\frac{d^2f}{dw^2} + w\frac{df}{dw} + \left(-w^2 - m^2\right)f = 0. \tag{7.9.39}$$

Two independent solutions are denoted $K_m(w)$ and $I_m(w)$. The behavior in the neighborhood of the singular point $w = 0$ is determined by the roots of the indicial

[7] Here it is easier to use the complex transformation (7.9.33). Then the infinite series representation for Bessel functions is valid for complex arguments, avoiding additional calculations.

equation, $\pm m$, corresponding to approximate solutions near $w = 0$ of the forms $w^{\pm m}$ (for $m \neq 0$) and w^0 and $w^0 \ln w$ (for $m = 0$). We can choose the two independent solutions such that one is well behaved at $w = 0$ and the other singular.

A good understanding of these functions comes from also analyzing their behavior as $w \to \infty$. Roughly speaking, for large w (7.9.39) can be rewritten as

$$\frac{d^2f}{dw^2} \approx -\frac{1}{w}\frac{df}{dw} + f. \tag{7.9.40}$$

Thinking of this as Newton's law for a particle with certain forces, the $-1/w\, df/dw$ term is a weak damping force tending to vanish as $w \to \infty$. We might expect as $w \to \infty$ that

$$\frac{d^2f}{dw^2} \approx f,$$

which suggests that the solution should be a linear combination of an exponentially growing e^w and exponentially decaying e^{-w} term. In fact, the weak damping has its effects (just as it did for ordinary Bessel functions). We state (but do not prove) a more advanced result, namely that the asymptotic behavior for large w of solutions of (7.9.39) are approximately $e^{\pm w}/w^{1/2}$. Thus, both $I_m(w)$ and $K_m(w)$ are linear combinations of these two, one exponentially growing and the other decaying.

There is only one independent linear combination that decays as $w \to \infty$. There are many combinations that grow as $w \to \infty$. We *define* $K_m(w)$ to be a solution that decays as $w \to \infty$. It must be proportional to $e^{-w}/w^{1/2}$ and it is defined *uniquely* by

$$K_m(w) \sim \sqrt{\frac{\pi}{2}}\frac{e^{-w}}{w^{1/2}}, \tag{7.9.41}$$

as $w \to \infty$. As $w \to 0$ the behavior of $K_m(w)$ will be some linear combination of the two different behaviors (e.g., w^m and w^{-m} for $m \neq 0$). In general, it will be composed of both and hence will be singular at $w = 0$. In more advanced treatments it is shown that

$$K_m(w) \sim \begin{cases} \ln w & m = 0 \\ \frac{1}{2}(m-1)!\left(\frac{1}{2}w\right)^{-m} & m \neq 0, \end{cases} \tag{7.9.42}$$

as $w \to 0$. The most important facts about this function is that the $K_m(w)$ **exponentially decays as $w \to \infty$ but is singular at $w = 0$.**

Since $K_m(w)$ is singular at $w = 0$, we would like to define a second solution $I_m(w)$ not singular at $w = 0$. $I_m(w)$ is *defined* uniquely such that

$$I_m(w) \sim \frac{1}{m!}\left(\frac{1}{2}w\right)^m, \tag{7.9.43}$$

as $w \to 0$. As $w \to \infty$, the behavior of $I_m(w)$ will be some linear combination of the two different asymptotic behaviors ($e^{\pm w}/w^{1/2}$). In general, it will be composed of both and hence is expected to exponentially grow as $w \to \infty$. In more advanced works, it is shown that

$$I_m(w) \sim \sqrt{\frac{1}{2\pi w}} e^w, \tag{7.9.44}$$

as $w \to \infty$. The most important facts about this function is that **$I_m(w)$ is well behaved at $w = 0$ but grows exponentially as $w \to \infty$.**

Some modified Bessel functions are sketched in Fig. 7.9.2. Although we have not proved it, note that both $I_m(w)$ and $K_m(w)$ are not zero for $w > 0$.

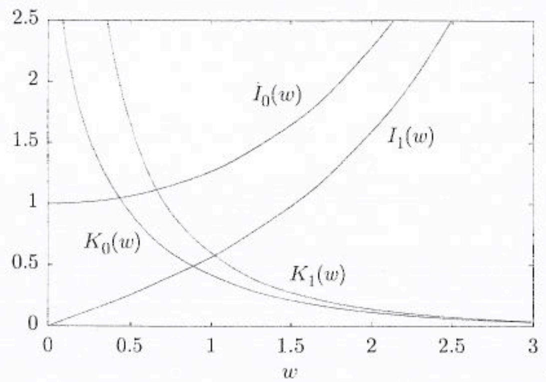

Figure 7.9.2 Various modified Bessel functions (from Abramowitz and Stegun [1974]).

EXERCISES 7.9

7.9.1. Solve Laplace's equation inside a circular cylinder subject to the boundary conditions

(a) $u(r,\theta,0) = \alpha(r,\theta)$, $u(r,\theta,H) = 0$, $u(a,\theta,z) = 0$

*(b) $u(r,\theta,0) = \alpha(r)\sin 7\theta$, $u(r,\theta,H) = 0$, $u(a,\theta,z) = 0$

(c) $u(r,\theta,0) = 0$, $u(r,\theta,H) = \beta(r)\cos 3\theta$, $\frac{\partial u}{\partial r}(a,\theta,z) = 0$

(d) $\frac{\partial u}{\partial z}(r,\theta,0) = \alpha(r)\sin 3\theta$, $\frac{\partial u}{\partial z}(r,\theta,H) = 0$, $\frac{\partial u}{\partial r}(a,\theta,z) = 0$

(e) $\frac{\partial u}{\partial z}(r,\theta,0) = \alpha(r,\theta)$, $\frac{\partial u}{\partial z}(r,\theta,H) = 0$, $\frac{\partial u}{\partial r}(a,\theta,z) = 0$

For (e) only, under what condition does a solution exist?

7.9.2. Solve Laplace's equation inside a semicircular cylinder, subject to the boundary conditions

(a) $u(r,\theta,0) = 0$, $u(r,\theta,H) = \alpha(r,\theta)$, $u(r,0,z) = 0$,
$u(r,\pi,z) = 0$, $u(a,\theta,z) = 0$

*(b) $u(r,\theta,0) = 0$, $\frac{\partial u}{\partial z}(r,\theta,H) = 0$, $u(r,0,z) = 0$,
$u(r,\pi,z) = 0$, $u(a,\theta,z) = \beta(\theta,z)$

(c) $\frac{\partial}{\partial z}u(r,\theta,0) = 0$, $\frac{\partial}{\partial z}u(r,\theta,H) = 0$, $\frac{\partial u}{\partial \theta}(r,0,z) = 0$,
$\frac{\partial u}{\partial \theta}(r,\pi,z) = 0$, $\frac{\partial u}{\partial r}(a,\theta,z) = \beta(\theta,z)$

For (c) only, under what condition does a solution exist?

(d) $u(r,\theta,0) = 0$, $u(r,0,z) = 0$, $u(a,\theta,z) = 0$,
$u(r,\theta,H) = 0$, $\frac{\partial u}{\partial \theta}(r,\pi,z) = \alpha(r,z)$

7.9.3. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a quarter-circular cylinder ($0 < \theta < \pi/2$ with radius a and height H) subject to the initial condition

$$u(r,\theta,z,0) = f(r,\theta,z)$$

Briefly explain what temperature distribution you expect to be approached as $t \to \infty$. Consider the following boundary conditions

(a) $u(r,\theta,0) = 0$, $u(r,\theta,H) = 0$, $u(r,0,z) = 0$,
$u(r,\pi/2,z) = 0$, $u(a,\theta,z) = 0$

*(b) $\frac{\partial u}{\partial z}(r,\theta,0) = 0$, $\frac{\partial u}{\partial z}(r,\theta,H) = 0$, $\frac{\partial u}{\partial \theta}(r,0,z) = 0$,
$\frac{\partial u}{\partial \theta}(r,\pi/2,z) = 0$, $\frac{\partial u}{\partial r}(a,\theta,z) = 0$

(c) $u(r,\theta,0) = 0$, $u(r,\theta,H) = 0$, $\frac{\partial u}{\partial \theta}(r,0,z) = 0$,
$u(r,\pi/2,z) = 0$, $\frac{\partial u}{\partial r}(a,\theta,z) = 0$

7.9.4. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a cylinder (of radius a and height H) subject to the initial condition,

$$u(r,\theta,z,0) = f(r,z),$$

independent of θ, if the boundary conditions are

*(a) $u(r,\theta,0,t)=0,$ $u(r,\theta,H,t)=0,$ $u(a,\theta,z,t)=0$

(b) $\frac{\partial u}{\partial z}(r,\theta,0,t)=0,$ $\frac{\partial u}{\partial z}(r,\theta,H,t)=0,$ $\frac{\partial u}{\partial r}(a,\theta,z,t)=0$

(c) $u(r,\theta,0,t)=0,$ $u(r,\theta,H,t)=0,$ $\frac{\partial u}{\partial r}(a,\theta,z,t)=0$

7.9.5. Determine the three ordinary differential equations obtained by separation of variables for Laplace's equation in spherical coordinates

$$0=\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}.$$

7.10 Spherical Problems and Legendre Polynomials

7.10.1 Introduction

Problems in a spherical geometry are of great interest in many applications. In the exercises, we consider the three-dimensional heat equation inside the spherical earth. Here, we consider the three-dimensional wave equation which describes the vibrations of the earth:

$$\frac{\partial^2 u}{\partial t^2}=c^2\nabla^2 u, \tag{7.10.1}$$

where u is a local displacement. In geophysics, the response of the real earth to point sources is of particular interest due to earthquakes and nuclear testing. Solid vibrations of the real earth are more complicated than (7.10.1). Compressional waves called P for primary are smaller than shear waves called S for secondary, arriving later because they propagate at a smaller velocity. There are also long (L) period surface waves, which are the most destructive in severe earthquakes because their energy is confined to a thin region near the surface. Real seismograms are more complicated because of scattering of waves due to the interior of the earth not being uniform. Measuring the vibrations is frequently used to determine the interior structure of the earth needed not only in seismology but also in mineral exploration, such as petroleum engineering. All displacements solve wave equations. Simple mathematical models are most valid for the destructive long waves, since the variations in the earth are averaged out for long waves. For more details, see Aki and Richards [1980], *Quantitative Seismology*. We use spherical coordinates (ρ, θ, ϕ), where ϕ is the angle from the pole and θ is the usual cylindrical angle. The

boundary condition we assume is $u(a,\theta,\phi,t)=0$, and the initial displacement and velocity distribution is given throughout the entire solid:

$$u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi) \tag{7.10.2}$$

$$\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho,\theta,\phi). \tag{7.10.3}$$

Problems with nonhomogeneous boundary conditions are treated in Chapter 8.

7.10.2 Separation of Variables and One-Dimensional Eigenvalue Problems

We use the method of separation of variables. As before, we first introduce product solutions of space and time:

$$u(\rho,\theta,\phi,t)=w(\rho,\theta,\phi)h(t). \tag{7.10.4}$$

We have already separated space and time, so that we know

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \tag{7.10.5}$$

$$\nabla^2 w+\lambda w = 0, \tag{7.10.6}$$

where the first separation constant λ satisfies the multidimensional eigenvalue problem (7.10.6) subject to being zero on the boundary of the sphere. The frequencies of vibration of the solid sphere are given by $c\sqrt{\lambda}$.

Using the equation for the Laplacian in spherical coordinates (reference from Chapter 1), we have

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial w}{\partial\rho})+\frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial w}{\partial\phi})+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 w}{\partial\theta^2}+\lambda w=0. \tag{7.10.7}$$

We seek product solutions of the form

$$w(\rho,\theta,\phi)=f(\rho)q(\theta)g(\phi). \tag{7.10.8}$$

To save some algebra, since the coefficients in (7.10.7) do not depend on θ, we note that it is clear that the eigenfunctions in θ are $\cos m\theta$ and $\sin m\theta$, corresponding to the periodic boundary conditions associated with the usual Fourier series in θ on the interval $-\pi\le\theta\le\pi$. In this case the term $\frac{\partial^2 w}{\partial\theta^2}$ in (7.10.7) may be replaced by $-m^2 w$. We substitute (7.10.8) into (7.10.7), multiply by ρ^2, divide by $f(\rho)g(\phi)$, and introduce the third (counting $-m^2$as number two) separation constant μ:

$$\frac{1}{f}\frac{d}{d\rho}(\rho^2\frac{df}{d\rho})+\lambda\rho^2=-\frac{1}{g\sin\phi}\frac{d}{d\phi}(\sin\phi\frac{dg}{d\phi})-\frac{m^2}{\sin^2\phi}=\mu. \tag{7.10.9}$$

The two ordinary differential equations that are the fundamental part of the eigenvalue problems in ϕ and ρ are

$$\boxed{\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+(\lambda\rho^2-\mu)f = 0} \tag{7.10.10}$$

$$\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu\sin\phi-\frac{m^2}{\sin\phi}\right)g = 0. \tag{7.10.11}$$

The homogeneous boundary conditions associated with (7.10.10) and (7.10.11) will be discussed shortly. We will solve (7.10.11) first because it does not depend on the eigenvalues λ of (7.10.10).

Equation (7.10.11) is a Sturm-Liouville differential equation (for each m) in the angular coordinate ϕ with eigenvalue μ and nonnegative weight $\sin\phi$. Equation (7.10.11) is defined from $\phi = 0$ (North Pole) to $\phi = \pi$ (South Pole). However, (7.10.11) is not a regular Sturm-Liouville problem since $p = \sin\phi$ must be > 0 and $\sin\phi = 0$ at both ends. There is no physical boundary condition at the singular endpoints. Instead we will insist the solution is bounded at each endpoint: $|g(0)| < \infty$ and $|g(\pi)| < \infty$. We claim that the usual properties of eigenvalues and eigenfunctions are valid. In particular, there is an infinite set of eigenfunctions (for each fixed m) corresponding to different eigenvalues μ_{nm} , and these eigenfunctions will be an orthogonal set with weight $\sin\phi$.

Equation (7.10.10) is a Sturm-Liouville differential equation (for each m and n) in the radial coordinate ρ with eigenvalue λ and weight ρ^2. One homogeneous boundary condition is $f(a) = 0$. Equation (7.10.10) is a singular Sturm-Liouville problem because of the zero at $\rho = 0$ in the coefficient in front of $df/d\rho$. Spherical coordinates are singular at $\rho = 0$, and solutions of the Sturm-Liouville differential equation must be bounded there: $|f(0)| < \infty$. We claim that this singular problem still has an infinite set of eigenfunctions (for each fixed m and n) corresponding to different eigenvalues λ_{knm} , and these eigenfunctions will form an orthogonal set with weight ρ^2.

7.10.3 Associated Legendre Functions and Legendre Polynomials

A (not obvious) change of variables has turned out to simplify the analysis of the differential equation that defines the orthogonal eigenfunctions in the angle ϕ:

$$x = \cos\phi. \tag{7.10.12}$$

As ϕ goes from 0 to π, this is a one-to-one transformation in which x goes from 1 to −1. We will show that both endpoints remain singular points. Derivatives are transformed by the chain rule, $\frac{d}{d\phi} = \frac{dx}{d\phi}\frac{d}{dx} = -\sin\phi\frac{d}{dx}$. In this way (7.10.11) becomes after dividing by $\sin\phi$ and recognizing that $\sin^2\phi = 1-\cos^2\phi = 1-x^2$:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dg}{dx}\right]+\left(\mu-\frac{m^2}{1-x^2}\right)g = 0. \tag{7.10.13}$$

This is also a Sturm-Liouville equation, and eigenfunctions will be orthogonal in x with weight 1. This corresponds to the weight $\sin\phi$ with respect to ϕ since $dx = -\sin\phi\, d\phi$. Equation (7.10.13) has singular points at $x = \pm 1$, which we will show are regular singular points (see Sec. 7.8.4). It is helpful to understand the local behavior near each singular point using a corresponding elementary equidimensional (Euler) equation. We analyze (7.10.13) near $x = 1$ (and claim due to symmetry that the local behavior near $x = -1$ can be the same). The troublesome coefficients $1-x^2 = (1-x)(1+x)$ can be approximated by $-2(x-1)$ near $x = 1$ Thus (7.10.13) may be approximated near $x = 1$ by

$$-2\frac{d}{dx}[(x-1)\frac{dg}{dx}]+\frac{m^2}{2(x-1)}g \approx 0 \tag{7.10.14}$$

since only the singular term that multiplies g is significant. Equation (7.10.14) is an equidimensional (Euler) differential equation whose exact solutions is easy to obtain by substituting $g = (x-1)^p$, from which we obtain $p^2 = m^2/4$ or $p = \pm m/2$. If $m \neq 0$, we conclude that one independent solution is bounded near $x = 1$ [and approximated by $(x-1)^{m/2}$] and the second independent solution is unbounded [and approximated by $(x-1)^{-m/2}$].

Since we want our solution to be bounded at $x = 1$, we can only use the one solution that is bounded at $x = 1$. When we compute this solution (perhaps numerically) at $x = -1$, its behavior must be a linear combination of the two local behaviors near $x = -1$. Usually the solution that is bounded at $x = 1$ will be unbounded at $x = -1$. Only for certain very special values of μ_{mn} (which we have called the eigenvalues) will the solution of (7.10.13) be bounded at both $x = \pm 1$. To simplify significantly the presentation, we will not explain the mysterious but elegant result that the only values of μ for which the solution is bounded at $x = \pm 1$

$$\mu = n(n+1), \tag{7.10.15}$$

where n is an integer with some restrictions we will mention. It is quite remarkable that the eigenvalues do not depend on the important parameter m. Equation (7.10.13) is a linear differential equation whose two independent solutions are called **associated Legendre functions (spherical harmonics)** of the first $P_n^m(x)$ and second kind $Q_n^m(x)$. The first kind is bounded at both $x = \pm 1$ for integer n, so that the eigenfunctions are given by $g(x) = P_n^m(x)$.

If $m = 0$: Legendre polynomials. $m = 0$ corresponds to solutions of the partial differential equation with no dependence on the cylindrical angle θ. In this case the differential equation (7.10.13) becomes

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dg}{dx}\right]+n(n+1)g = 0, \tag{7.10.16}$$

given that it can be shown that the eigenvalues satisfy (7.10.15). By series methods it can be shown that there are elementary Taylor series solutions around $x = 0$

that terminate (are finite series) only when $\mu = n(n+1)$, and hence are bounded at $x = \pm 1$ when $\mu = n(n+1)$. It can be shown (not easy) that if $\mu \neq n(n+1)$, then the solution to the differential equation is not bounded at either ± 1. These important bounded solutions are called Legendre polynomials and are not difficult to compute:

$$\begin{aligned} n &= 0: P_0(x) = 1 \\ n &= 1: P_1(x) = x = \cos\phi \\ n &= 2: P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) = \frac{1}{4}\left(3\cos 2\phi + 1\right). \end{aligned} \tag{7.10.17}$$

These have been chosen such that they equal 1 at $x = 1$ ($\phi = 0$, North Pole). It can be shown that the Legendre polynomials satisfy **Rodrigues' formula**:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n. \tag{7.10.18}$$

Since Legendre polynomials are orthogonal with weight 1, they can be obtained using the Gram-Schmidt procedure (see appendix of Section 7.5). We graph (see Fig. 7.10.1) in x and ϕ the first few eigenfunctions (Legendre polynomials). It can be shown that the Legendre polynomials are a complete set of polynomials, and therefore there are no other eigenvalues besides $\mu = n(n+1)$.

If $m > 0$: the associated Legendre functions. Remarkably, the eigenvalues when $m > 0$ are basically the same as when $m = 0$ given by (7.10.15). Even more remarkable is that the eigenfunctions when $m > 0$ (which we have called **associated Legendre functions**) can be related to the eigenfunctions when $m = 0$ (Legendre polynomials):

$$g(x) = P_n^m(x) = \left(x^2 - 1\right)^{m/2}\frac{d^m}{dx^m}P_n(x). \tag{7.10.19}$$

We note that $P_n(x)$ is the nth-degree Legendre polynomial. The mth derivative will be zero if $n < m$. Thus, the eigenfunctions exist only for $n \geq m$, and the eigenvalues do depend (weakly on m). The infinite number of eigenvalues are

$$\mu = n(n+1), \tag{7.10.20}$$

with the restriction that $n \geq m$. These formulas are also valid when $m = 0$; the associated Legendre functions when $m = 0$ are the Legendre polynomials, $P_n^0(x) = P_n(x)$.

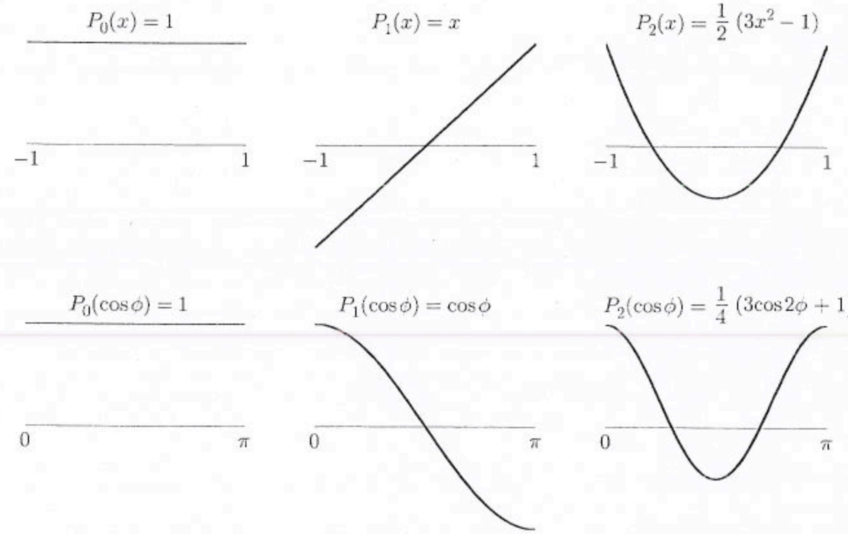

Figure 7.10.1 Legendre polynomials.

7.10.4 Radial Eigenvalue Problems

The radial Sturm-Liouville differential equation, (7.10.10), with $\mu = n(n+1)$,

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \left(\lambda\rho^2 - n(n+1)\right)f = 0, \tag{7.10.21}$$

has the restriction $n \geq m$ for fixed m. The boundary conditions are $f(a) = 0$, and the solution should be bounded at $\rho = 0$. Equation (7.10.21) is nearly Bessel's differential equation. The parameter λ can be eliminated by instead considering $\sqrt{\lambda}\rho$ as the independent variable. However, the result is not quite Bessel's differential equation. It is easy to show (see the Exercises) that if $Z_p(x)$ solves Bessel's differential equation (7.7.25) of order p, then $f(\rho) = \rho^{-1/2}Z_{n+\frac{1}{2}}(\sqrt{\lambda}\rho)$, called **spherical Bessel functions**, satisfy (7.10.21). Since the radial eigenfunctions must be bounded at $\rho = 0$, we have

$$f(\rho) = \rho^{-1/2}J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho), \tag{7.10.22}$$

for $n \geq m$. [If we recall the behavior of the Bessel functions at the origin (7.7.33), we can verify that these solutions are bounded at the origin. In fact, they are zero at the origin except for $n = 0$.] The eigenvalues λ are determined by applying the homogeneous boundary condition at $\rho = a$:

$$J_{n+\frac{1}{2}}(\sqrt{\lambda}a) = 0. \tag{7.10.23}$$

The eigenvalues are determined by the zeroes of the Bessel functions of order $n + \frac{1}{2}$. There is an infinite number of eigenvalues for each n and m. Note that the frequencies of vibration are the same for all values of $m \leq n$.

The spherical Bessel functions can be related to trigonometric functions:

$$x^{-1/2}J_{n+\frac{1}{2}}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right). \tag{7.10.24}$$

7.10.5 Product Solutions, Modes of Vibration, and the Initial Value Problem

Product solutions for the wave equation in three dimensions are

$$u(\rho,\theta,\phi,t) = \cos c\sqrt{\lambda}t \sin c\sqrt{\lambda}t \rho^{-1/2}J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho) \cos m\theta \sin m\theta P_n^m(\cos\phi),$$

where the frequencies of vibration are determined from (7.10.23). The angular parts $Y_n^m \equiv \cos m\theta \sin m\theta P_n^m(\cos\phi)$ are called **surface harmonics** of the first kind. Initial value problems are solved by using superposition of these infinite modes, summing over m, n, and the infinite radial eigenfunctions characterized by the zeros of the Bessel functions. The weights of the three one-dimensional orthogonality give rise to $d\theta, \sin\phi\, d\phi, \rho^2\, d\rho$, which is equivalent to orthogonality in three dimensions with weight 1, since differential volume in spherical coordinates is $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. This can be checked using the Jacobian J of the original transformation since $dx\, dy\, dx = J\, d\rho\, d\theta\, d\phi$ and

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2 \sin\phi.$$

Normalization integrals for associated Legendre functions can be found in reference books such as Abramowitz and Stegun:

$$\int_{-1}^{1} [P_n^m(x)]^2\, dx = (n+\frac{1}{2})^{-1}(n+m)!/(n-m)! \tag{7.10.25}$$

Example. For the purely radial mode $n = 0$ ($m = 0$ only), using (7.10.24) the frequencies of vibration satisfy $\sin(\sqrt{\lambda}a) = 0$, so that

$$\text{circular frequency} = c\sqrt{\lambda} = \frac{j\pi c}{a},$$

where a is the radius of the earth, for example. The fundamental mode $j = 1$ has circular frequency $\frac{\pi c}{a}$ hertz (cycles per second) or a frequency of $\frac{c}{2a}$ per second or a period of $\frac{2a}{c}$ seconds. For the earth we can take $a = 6000$ km and $c = 5$ km/s, giving a period of $\frac{12000}{5} = 2400$ seconds or 40 minutes.

7.10.6 Laplace's Equation Inside a Spherical Cavity

In electrostatics, it is of interest to solve Laplace's equation inside a sphere with the potential u specified on the boundary $\rho = a$

$$\nabla^2 u = 0 \tag{7.10.26}$$

$$u(a,\theta,\phi) = F(\theta,\phi). \tag{7.10.27}$$

This corresponds to determining the electric potential given the distribution of the potential along the spherical conductor. We can use the previous computations, where we solved by separation of variables. The θ and ϕ equations and their solutions will be the same, a Fourier series in θ involving $\cos m\theta$ and $\sin m\theta$ and a generalized Fourier series in ϕ involving the associated Legendre functions $P_n^m(\cos\phi)$. However, we need to insist that $\lambda = 0$, so that the radial equation (7.10.21) will be different and will not be an eigenvalue problem:

$$\frac{d}{d\rho}(\rho^2\frac{df}{d\rho}) - n(n+1))f = 0. \tag{7.10.28}$$

Here (7.10.28) is an equidimensional equation and can be solved exactly by substituting $f = \rho^r$. By substitution we have $r(r+1) - n(n+1) = 0$, which is a quadratic equation with two different roots $r = n$ and $r = -n-1$ since n is an integer. Since the potential must be bounded at the center $\rho = 0$, we reject the unbounded solution ρ^{-n-1}. Product solutions for Laplace's equation are

$$\rho^n \cos m\theta \sin m\theta P_n^m(\cos\phi), \tag{7.10.29}$$

so that the solution of Laplace's equation is in the form

$$u(\rho,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \rho^n[A_{mn}\cos m\theta + B_{mn}\sin m\theta]P_n^m(\cos\phi). \tag{7.10.30}$$

The nonhomogeneous boundary condition implies that

$$F(\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} a^n[A_{mn}\cos m\theta + B_{mn}\sin m\theta]P_n^m(\cos\phi). \tag{7.10.31}$$

By orthogonality, for example,

$$a^n B_{mn} = \frac{\iint F(\theta,\phi)\sin m\theta P_n^m(\cos\phi)\sin\phi\, d\phi\, d\theta}{\iint \sin^2 m\theta\,[P_n^m(\cos\phi)]^2 \sin\phi\, d\phi\, d\theta}. \tag{7.10.32}$$

A similar expression exists for A_{mn}.

Example. In electrostatics, it is of interest to determine the electric potential inside a conducting sphere if the hemispheres are at different constant potentials. This can be done experimentally by separating two hemispheres by a negligibly small insulated ring. For convenience, we assume the upper hemisphere is at potential $+V$ and the lower hemisphere at potential $-V$. The boundary condition at $\rho = a$ is cylindrically (azimuthally) symmetric; there is no dependence on the angle θ. We solve Laplace's equation under this simplifying circumstance, or we can use the general solution obtained previously. We follow the later procedure. Since there is no dependence on θ, all terms for the Fourier series in θ will be zero in (7.10.30) except for the $m = 0$ term. Thus, the solution of Laplace's equation with cylindrical symmetry can be written as a series involving the Legendre polynomials:

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n \rho^n P_n(\cos\phi). \tag{7.10.33}$$

The boundary condition becomes

$$\left.\begin{array}{l} V \text{ for } 0 < \phi < \pi/2\ (0 < x < 1) \\ -V \text{ for } \pi/2 < \phi < \pi (-1 < x < 0) \end{array}\right\} = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\phi). \tag{7.10.34}$$

Thus, using orthogonality (in $x = \cos\phi$) with weight 1,

$$A_n a^n = \frac{\int_{-1}^{0} -V P_n(x)\,dx + \int_0^1 V P_n(x)\,dx}{\int_{-1}^{1}[P_n(x)]^2\,dx} = \begin{cases} 0 \text{ for } n \text{ even} \\ 2\int_0^1 V P_n(x)\,dx \Big/ \int_{-1}^{1}[P_n(x)]^2\,dx \text{ for } n \text{ odd}, \end{cases} \tag{7.10.35}$$

since $P_n(x)$ is even for n even and $P_n(x)$ is odd for n odd and the potential on the surface of the sphere is an odd function of x. Using the normalization integral (7.10.25) for the denominator and Rodrigues formula for Legendre polynomials, (7.10.18), for the numerator, it can be shown that

$$u(r,\phi) = V[\frac{3}{2}\frac{\rho}{a}P_1(\cos\phi) - \frac{7}{8}(\frac{\rho}{a})^3 P_3(\cos\phi) + \frac{11}{16}(\frac{\rho}{a})^5 P_5(\cos\phi) + \ldots] \tag{7.10.36}$$

For a more detailed discussion of this, see Jackson [1998], *Classical Electrodynamics*.

EXERCISES 7.10

7.10.1. Solve the initial value problem for the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $u(a,\theta,\phi,t) = 0$ and the initial conditions

(a) $u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi)$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(b) $u(\rho,\theta,\phi,0) = 0$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho,\theta,\phi)$

(c) $u(\rho,\theta,\phi,0) = F(\rho,\phi)$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(d) $u(\rho,\theta,\phi,0) = 0$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho,\phi)$

(e) $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos 3\theta$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(f) $u(\rho,\theta,\phi,0) = F(\rho)\sin 2\theta$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(g) $u(\rho,\theta,\phi,0) = F(\rho)$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(h) $u(\rho,\theta,\phi,0) = 0$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho)$

7.10.2. Solve the initial value problem for the heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $u(a,\theta,\phi,t) = 0$ and the initial conditions

(a) $u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi)$

(b) $u(\rho,\theta,\phi,0) = F(\rho,\phi)$

(c) $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos\theta$

(d) $u(\rho,\theta,\phi,0) = F(\rho)$

7.10.3. Solve the initial value problem for the heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $\frac{\partial u}{\partial \rho}(a,\theta,\phi,t) = 0$ and the initial conditions

(a) $u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi)$

(b) $u(\rho,\theta,\phi,0) = F(\rho,\phi)$

(c) $u(\rho,\theta,\phi,0) = F(\rho,\phi)\sin 3\theta$

(d) $u(\rho,\theta,\phi,0) = F(\rho)$

7.10.4. Using the one-dimensional Rayleigh quotient, show that $\mu \geq 0$ (if $m \geq 0$) as defined by (7.10.11). Under what conditions does $\mu = 0$?

7.10.5. Using the one-dimensional Rayleigh quotient, show that $\mu \geq 0$ (if $m \geq 0$) as defined by (7.10.13). Under what conditions does $\mu = 0$?

7.10.6. Using the one-dimensional Rayleigh quotient, show that $\lambda \geq 0$ (if $n \geq 0$) as defined by (7.10.6) with the boundary condition $f(a) = 0$. Can $\lambda = 0$?

7.10.7. Using the three-dimensional Rayleigh quotient, show that $\lambda \geq 0$ as defined by (7.10.11) with $u(a,\theta,\phi,t) = 0$. Can $\lambda = 0$?

7.10.8. Differential equations related to Bessel's differential equation. Use this to show that

$$x^2\frac{d^2f}{dx^2}+x(1-2a-2bx)\frac{df}{dx}+[a^2-p^2+(2a-1)bx+(d^2+b^2)x^2]f=0 \quad (7.10.37)$$

has solutions $x^a e^{bx} Z_p(dx)$, where $Z_p(x)$ satisfies Bessel's differential equation (7.7.25). By comparing (7.10.21) and (7.10.37), we have $a=-\frac{1}{2}, b=0, \frac{1}{4}-p^2=-n(n+1)$, and $d^2=\lambda$. We find that $p=(n+\frac{1}{2})$.

7.10.9. Solve Laplace's equation inside a sphere $\rho < a$ subject to the following boundary conditions on the sphere:

(a) $u(a,\theta,\phi)=F(\phi)\cos 4\theta$

(b) $u(a,\theta,\phi)=F(\phi)$

(c) $\frac{\partial u}{\partial \rho}(a,\theta,\phi)=F(\phi)\cos 4\theta$

(d) $\frac{\partial u}{\partial \rho}(a,\theta,\phi)=F(\phi)$ with $\int_0^\pi F(\phi)\sin\phi\, d\phi=0$

(e) $\frac{\partial u}{\partial \rho}(a,\theta,\phi)=F(\theta,\phi)$ with $\int_0^\pi\int_0^{2\pi} F(\theta,\phi)\sin\phi\, d\theta\, d\phi=0$

7.10.10. Solve Laplace's equation outside a sphere $\rho > a$ subject to the potential given on the sphere:

(a) $u(a,\theta,\phi)=F(\theta,\phi)$

(b) $u(a,\theta,\phi)=F(\phi)$, with cylindrical (azimuthal) symmetry

(c) $u(a,\theta,\phi)=V$ in the upper hemisphere, $-V$ in the lower hemisphere (do not simplify; do not evaluate definite integrals)

7.10.11. Solve Laplace's equation inside a sector of a sphere $\rho < a$ with $0<\theta<\frac{\pi}{2}$ subject to $u(\rho,0,\phi)=0$ and $u(\rho,\frac{\pi}{2},\phi)=0$ and the potential given on the sphere: $u(a,\theta,\phi)=F(\theta,\phi)$.

7.10.12. Solve Laplace's equation inside a hemisphere $\rho < a$ with $z>0$ subject to $u=0$ at $z=0$ and the potential given on the hemisphere: $u(a,\theta,\phi)=F(\theta,\phi)$ [*Hint*: Use symmetry and solve a different problem, a sphere with the antisymmetric potential on the lower hemisphere.]

7.10.13. Show that Rodrigues' formula agrees with the given Legendre polynomials for $n=0$, $n=1$, and $n=2$.

7.10.14. Show that Rodrigues' formula satisfies the differential equation for Legendre polynomials.

7.10.15. Derive (7.10.36) using (7.10.35), (7.10.18), and (7.10.25).

Chapter 8

Nonhomogeneous Problems

8.1 Introduction

In the previous chapters we have developed only one method to solve partial differential equations: the method of separation of variables. In order to apply the method of separation of variables, the partial differential equation (with n independent variables) must be linear and homogeneous. In addition, we must be able to formulate a problem with linear and homogeneous boundary conditions for $n-1$ variables. However, some of the most fundamental physical problems do not have homogeneous conditions.

8.2 Heat Flow with Sources and Nonhomogeneous Boundary Conditions

Time-independent boundary conditions. As an elementary example of a nonhomogeneous problem, consider the heat flow (without sources) in a uniform rod of length L with the temperature fixed at the left end at $A°$ and the right at $B°$. If the initial condition is prescribed, the mathematical problem for the temperature $u(x,t)$ is

$$\text{PDE:} \quad \boxed{\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}} \quad (8.2.1)$$

$$\text{BC1:} \quad \boxed{u(0,t)=A} \quad (8.2.2)$$

$$\text{BC2:} \quad \boxed{u(L,t)=B} \quad (8.2.3)$$