Examples of simple PDEs and the phenomena they describe

Evolution equations are of the form

$$\frac{\partial u}{\partial t} = F(u, u_x, u_{xx}, \dots),$$

in other words, the temporal rate of change of some quantity u is determined by its spatial configuration. The x-variable may be in $\mathbb{R}^n$ (and nonlinear).

For many of the linear equations we will study, it is profitable to determine all solutions of the form e^{ax+bt}, especially when a and b are imaginary:

$$u(x,t) = e^{i(kx - \omega t)}$$

k = wavenumber
ω = frequency (circular)

The "phase velocity" of this oscillatory wave is

$$v_{ph} = \frac{\omega}{k}$$

because this is the speed at which a fixed value of the phase ~~wave~~ propagates. Typically ω and k satisfy a relation

$$\omega = W(k).$$

The "group velocity" of a packet of waves ~~near a~~ with wavenumber near k_0 is

$$v_{gr} = \frac{dW}{dk}(k_0)$$

The simplest explanation for this definition is this: Let's superimpose two waves with slightly varying wavenumbers centered at k and frequencies centered at $\omega = W(k)$. If Δk is small, then $\Delta\omega \approx W'(k)\,\Delta k$:

$$e^{i((k+\Delta k)x - (\omega-\Delta\omega)t)} + e^{i((k+\Delta k)x-(\omega+\Delta\omega)t)}$$

$$= \underbrace{e^{i(kx-\omega t)}}_{\text{"fast/phase" varying}} \underbrace{\left[2\cos(\Delta k x - \Delta\omega t)\right]}_{\text{slowly-varying envelope}}$$

Thus the envelope that carries "groups" of waves travels at a speed $\Delta\omega/\Delta k$, or $W'(k)$.

- $\boxed{u_t = -cu_x}$: transport/advection. Solutions are of the form

$$u(x,t) = f(x-ct),$$

and oscillatory solutions, in particular, are

$$u(x,t) = e^{i(\alpha x - \alpha c t)}, \quad \omega = ck$$

The phase velocity is c, as is the group velocity.

- $\boxed{u_t = a(x)u}$: reaction. This is not genuinely a PDE but rather a family of ODEs, one for each x. The solution is $u(x,t) = u(x,0)\,e^{a(x)t}$. This idea is significant in PDEs when it appears in conjunction with other phenomena, as diffusion.

- $\boxed{u_t = \sigma u_{xx}}$: diffusion. $u(x,t) = e^{ax+bt} \Rightarrow b = \sigma a^2$. The relation between a and b is parabolic. If $a = i\alpha$ with α real, then a solution is

$$u(x,t) = e^{i\alpha x - \sigma\alpha^2 t}$$

Thus oscillations are damped — the higher the oscillation, the faster the damping. The equation says that the time derivative is proportional to the concavity, so a configuration $u(x,t_0)$ at some time t_0 will tend to "even out" faster the more oscillatory it is.

- $\boxed{iu_t = \gamma u_{xx}}$: quadratic dispersion - a "Schrödinger" equation. This looks much like the diffusion equation, but it behaves very differently. Oscillatory solutions are of the form

$$u(x,t) = e^{i(kx + \gamma k^2 t)}$$

The relation between frequency and wavenumber is

$$\omega = \gamma k^2$$

and the phase and group velocities are

$$v_{ph} = \gamma k$$
$$v_{gr} = 2\gamma k$$

Since the group velocity is not constant, groups of waves (wave packets) ~~travel~~ at different k travel at different speeds and thus disperse.

Because of this phenomenon, the relation $\omega = W(k)$ is called the dispersion relation.

- $\boxed{u_t = \gamma u_{xxx}}$: cubic dispersion. Oscillatory solutions are

$$u(x,t) = e^{i(kx - \gamma k^3 t)},$$

so the dispersion relation is

$$\omega = \gamma k^3$$

and the phase and group velocities are

$$v_{ph} = \gamma k^2$$
$$v_{gr} = 3\gamma k^2$$

So this equation also exhibits dispersion of wave packets at differing frequencies.

- $\boxed{u_t = -uu_x}$: nonlinear advection. The speed of propagation of a value of u depends on u itself. In general, we may have $u_t = c(u)u_x$. Higher values may propagate faster than lower values and overtake them, causing shocks, or caustics (optics).

- $\boxed{u_{tt} = c^2 u_{xx}}$: two directional waves. Solutions of the form e^{ax+bt} have $c^2a^2 - b^2 = 0$, so the relation between a and b is hyperbolic. The general solution (in 1D) is

$$u(x,t) = f_1(x-ct) + f_2(x+ct)$$

Oscillatory solutions with wavenumber k are

$$u(x,t) = a_+ e^{i(kx-ckt)} + a_- e^{i(kx+ckt)}$$

The dispersion relation is $\omega^2 = c^2k^2$, or $\omega = \pm ck$. Thus the phase velocity and group velocity are both c and waves do not disperse.

- $\boxed{\Delta u(x_1, x_2) = 0}$: the Laplace equation — the shape of a taut drum membrane. The sum of the second derivatives of u with respect to the spatial variables is zero; so the concavities in orthogonal directions must cancel one another. Solutions $e^{ax_1 + bx_2}$ satisfy the elliptic equation $a^2 + b^2 = 0$. Thus, if one of a and b is real, the other is imaginary, and we have solutions such as

$$u(x_1, x_2) = \sin\beta x_1 e^{\beta x_2}, \cos\beta x_1 \sinh\beta x_2, \text{ etc.}$$

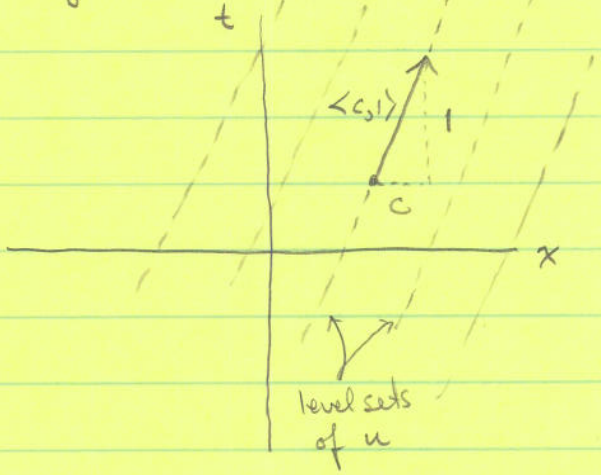
S'more PDEs

- $u_t + u_{xxxx} = 0$: the beam equation (inelastic beam)
- $u_t + \underbrace{\alpha u u_x}_{\text{nonlinear advection}} + \underbrace{u_{xxx}}_{\text{dispersion}} = 0$: The Korteweg-deVries (KdV) equation
- $iu_t + u_{xx} + \gamma |u|^2 u = 0$: The nonlinear Schrödinger (NLS) equation
- $u_{tt} - u_{xx} + k^2 u = 0$: Klein-Gordon equation

  $u_{tt} - u_{xx} + K(u)$: Nonlinear K-G equation
- $u_t + \underbrace{c(u) u_x}_{\text{nonlinear advection}} = \underbrace{\nu u_{xx}}_{\text{diffusion}}$
- $u_t = \mu \Delta u + R(u)$: reaction-diffusion equation
- $|\nabla u| = 1$: eikonal equation for geometric optics
- $\begin{cases} u_t + u \cdot \nabla u - \Delta u = -\nabla p \\ \nabla \cdot u = 0 \end{cases}$ (advection; viscosity $\Rightarrow$ diffusion; forcing by pressure gradient; incompressibility)
- $\nabla \cdot \tau \nabla u + k^2 \varepsilon u = 0$: Helmholtz equation - harmonic solutions of the wave equation
- $\begin{aligned} \frac{\partial E}{\partial t} &= \nabla \times H - J \\ \frac{\partial H}{\partial t} &= -\nabla \times E \end{aligned}$: The Maxwell system for electromagnetic fields in a vacuum.

Transport, or advection

In one spatial dimension (transport along a narrow channel), the simplest transport equation is



$$u_t + cu_x = 0 \tag{1}$$

This can be written as

$$\langle u_x, u_t \rangle \cdot \langle c, 1 \rangle = 0,$$

so $\langle c,1 \rangle$ is the direction in (x,t)-space in which u is invariant. Any function of the form

$$u(x,t) = f(x-ct), \tag{2}$$

whose level sets on the x-axis travel with speed c. Evidently, the linear function $y = x-ct$ characterizes solutions of (1): all functions of y are solutions of (1). To see rigorously that all solutions of (1) are of the form (2), let us rewrite the PDE using the new coordinate system

$$\begin{cases} y = x - ct \\ s = t \end{cases}.$$

The chain rule yields $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = -c\frac{\partial u}{\partial y} + \frac{\partial u}{\partial s} \end{cases}$

Thus the PDE (1) is transformed to the very simple form

$$\frac{\partial u}{\partial s} = 0$$

in the coordinates (y,s). This implies that u is a function of y alone, that is,

$$u = f(y) = f(x-ct).$$

In fact $u(x,t) = y = x-ct$ is itself a solution, and since y serves as a new coordinate in $\mathbb{R}^2$, along with t, we see that all solutions of (1) are functions of this "simplest" one $x-ct$.

Let's extend the transport equation to <u>n spatial dimensions</u>:

(3) $$\frac{\partial u}{\partial t} + c_1 \frac{\partial u}{\partial x_1} + \dots + c_n \frac{\partial u}{\partial x_n} = 0.$$

This can be written as $u_t + c \cdot D_x u = 0$,
where $c = (c_1, \dots, c_n)$ and $D_x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$.
The functions $y_i = x_i - ct$ are constant in the direction of $\langle c, 1 \rangle$, that is, they satisfy the PDE (3).
This suggests that we use the change of variables

$$\begin{cases} y_i = x_i - c_i t, & i = 1, \dots, n \\ s = t \end{cases}$$

which yields

$$\begin{cases} \dfrac{\partial u}{\partial x_i} = \dfrac{\partial u}{\partial y_i}, & i = 1, \dots, n \\ \dfrac{\partial u}{\partial t} = \displaystyle\sum_{j=1}^{n} -c_j \frac{\partial u}{\partial y_j} + \frac{\partial u}{\partial s} \end{cases}$$

The PDE (3) is transformed to the simple equation

$$\frac{\partial u}{\partial s} = 0 ,$$

whose solutions are simply functions of $y_1, \dots, y_n$, $u = f(y_1, \dots, y_n)$, or

$$u(x,t) = f(x_1 - c_1 t, \dots, x_n - c_n t) . \tag{4}$$

The <u>initial-value problem</u>. Given the solution (4), it is easy to solve the problem

$$\begin{cases} u_t + c \cdot D_x u = 0 \\ u(x,0) = f(x) , \quad x \in \mathbb{R}^n \end{cases}$$

because the solution is just (4) itself — the initial values (density of a substance, for example) are transported in time with velocity c.

<u>Nonconstant velocity</u>. We can solve the transport equation, at least theoretically, if the transport is not spatially homogeneous $c = c(x)$, and even in the nonautonomous case in which the transport velocity varies spatially and temporally:

$$\begin{cases} u_t + c(x,t) \cdot D_x u = 0 \\ u(x,0) = f(x) , \quad x \in \mathbb{R}^n . \end{cases}$$

To do this, we should find functions $y_1, \dots, y_n$ (new coord.s) that are constant along the paths in (x,t)-space that are tangent to the vectors $\langle c(x,t), 1 \rangle$; moreover, at $t=0$, x and y should coincide. This is accomplished by allowing $\mathbb{R}^n \times \{0\}$ (points in $\mathbb{R}^n$ at time 0) to parameterize these paths. Such a path $(x(s), s)$ is characterized by the system of ordinary differential equations (ODEs):

$$\frac{dx}{ds} = c(x,t) \quad , \quad x(0) = y$$

$$\frac{dt}{ds} = 1 \quad , \quad t(0) = 0$$

[Notice that, if $c(x,t) = c \in \mathbb{R}$, then the solution is $x = y + cs$, $t = s$, which yields the coordinates $y = x - ct$ we used before.] A solution exists if $c(x,t)$ is Lipschitz continuous; the solution is a function of the initial value $y \in \mathbb{R}^n$ and s:

$$\begin{cases} x = \xi(y,s) \\ t = s \end{cases} \quad , \quad \xi(y,0) = y$$

The map $(y,s) \longmapsto (x,t)$ is locally a homeomorphism, and it is differentiable if $c(x,t)$ is. As long as the Jacobian matrix does not vanish, the map has a differentiable inverse:

$$\begin{cases} y = \eta(x,t) \\ s = t \end{cases} \quad , \quad \eta(x,0) = x \qquad (5)$$

These facts fall under the topic of the dependence of solutions to ODEs on initial values. See, for example, Ch. 1, Thm 7.1–7.2 in Coddington/Levinson.

Let us take (s) as a change of variable and transform the transport equation with velocity $c(x,t)$:

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x_j} \to 0$$

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial t} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial t} \to 1$$

Using the PDE $u_t + c(x,t)u_x = 0$ and the ODE defining the transformation, $dx/ds = c(x,t)$, we obtain

$$0 = u_t + \frac{\partial x}{\partial s} u_x = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \sum_{i=1}^{n} \left(\frac{\partial y_j}{\partial x_i} \frac{\partial x_i}{\partial s} + \frac{\partial y_j}{\partial t} \frac{dt}{ds} \right) + \frac{\partial u}{\partial s}$$

Since the expression in parentheses is equal to $\frac{\partial y_j}{\partial s} = 0$, we find that the transport equation simplifies again to

$$\frac{\partial u}{\partial s} = 0$$

Therefore, $u = f(y_1, \ldots, y_n)$, or

$$u(x,t) = f(\eta_1(x,t), \ldots, \eta_n(x,t))$$
$$u(x,0) = f(x_1, \ldots, x_n).$$

Physical interpretation: If a chemical is dissolved in a fluid that moves at a velocity of $c(x,t)$, which may depend on position and time, the chemical is transported, or advected, by the fluid. Its concentration does not change along the curves $x = \xi(y,t)$, but remains what it was at the point y at the initial time $t=0$.

Diffusion

Let's begin with the physical idea behind the diffusion equation. Let the scalar field u denote the concentration of some stuff in space, say a chemical, so that u may carry units of mol/vol. Let the vector field F denote the flux density, which carries units of mol/time/area. The meanings of u and F are expressed as follows: The total amount of stuff Q in a volume Ω is

$$Q = \int_\Omega u \, dV,$$

and the rate at which stuff passes from one side of a surface S to the other is

$$\text{Flux thru } S = \int_S F \cdot n \, dS.$$

Now, given a volume Ω, the rate at which the amount Q of stuff in Ω is changing can be written in terms of u or F:

$$\frac{dQ}{dt} = \int_\Omega \frac{\partial u}{\partial t} dV = -\int_{\partial\Omega} F \cdot n \, dS. \qquad (n \text{ pointing outward})$$

By the divergence theorem, $\int_{\partial\Omega} F \cdot n \, dS = \int_\Omega \operatorname{div} F \, dV$, we obtain

$$\int_\Omega \frac{\partial u}{\partial t} dV = -\int_\Omega \operatorname{div} F \, dV,$$

and since this holds for all regions Ω, we conclude that

$$\frac{\partial u}{\partial t} = -\operatorname{div} F.$$

Now we impose the constitutive law

$$F = -\sigma \nabla u,$$

where σ is a tensor, and arrive at the diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \sigma \nabla u = \nabla \cdot \sigma \nabla u. \tag{6}$$

If $\sigma = 1$, this is called "the" diffusion equation, or, more commonly, "the" heat equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot \nabla u = \nabla^2 u = \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{7}$$

Some boundary-value problems.

If σ is constant, we have the equation

$$\frac{\partial u}{\partial t} = \sigma \Delta u, \tag{8}$$

which can be transformed into the heat equation by using the new time variable σt. What the heat equation is saying is that

"the rate of change of u is proportional to its convexity."

Some physical problems help illustrate this principle

Temperature in a rod that is insulated along its length but not at the ends, which are in ice water, and therefore fixed at 0°. Suppose that the rod lies on the interval $[0, \pi]$.

One can verify that the functions

$$u_n(x,t) = \sin nx\, e^{-\sigma n^2 t} \qquad n = 1, 2, 3, \ldots$$

satisfy (8) and the boundary values $u(0,t) = u(\pi,t) = 0 \;\forall\, t > 0$. Since (8) is linear, combinations of these also satisfy the boundary-value problem:

$$u(x,t) = \sum_{n=1}^{N} a_n \sin nx\, e^{-\sigma n^2 t}.$$

If N is allowed to be ∞, then one must consider carefully the question of convergence of the series and its Laplacian Δu. We will not examine this issue yet. But notice for now that

$$u(x,0) = \sum_{n=1}^{N} a_n \sin nx, \tag{9}$$

that is, the initial temperature is a combination of $\sin nx$ shapes, each of which decays at a different rate, with the higher oscillations decaying faster [more variation $\Rightarrow$ faster diffusion]. The long-time behavior is that the temperature approaches an equilibrium of 0° along the length of the rod.

Insulated endpoints In this case, the functions

$$u_n(x,t) = \cos nx\, e^{-\sigma n^2 t}, \qquad n = 0, 1, 2, \ldots$$

satisfy (8) with the conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$ $\forall\, t > 0$, which correspond to zero heat flux through the endpoints. Again, one can take linear combinations:

$$u(x,t) = \sum_{n=0}^{N} a_n \cos nx\, e^{-\sigma n^2 t} \longrightarrow a_0 \text{ as } t \to \infty$$

Mixed boundary values. If the left end of the rod is maintained at 0° and the right end is insulated, then the general solution is

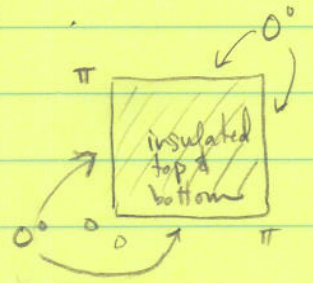
$$u(x,t) = \sum_{n=0}^{\infty} a_n \sin\left(n+\tfrac{1}{2}\right)x \, e^{-\sigma n^2 t} \qquad x \in [0,\pi], \quad t>0$$

For an insulated circular wire, we have

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) e^{-\sigma n^2 t},$$

$$\theta \in [0, 2\pi], \quad t>0.$$

An insulated square plate. If the edges are maintained at 0°, the general solution is



$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin nx \sin my \, e^{-\sigma(n^2+m^2)t}$$

For a disk of radius 1 with 0° on the boundary, we have

$$u(r,\theta,t) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} a_{nm} e^{in\theta} J_n(\gamma_{nm} r) \, e^{-\sigma^2 \gamma_{nm}^2 t}$$

where J_n is the Bessel function of order n that is smooth at $r=0$ and γ_{nm} is its m^{th} nonzero root.

The term "general solution" is used loosely here. We will make this rigorous in the following sense. The "basis" functions we have used in the spatial domain form an orthonormal Hilbert-space basis for L^2 of $[0,\pi]$, $[0,\pi]^2$, or the disk. If $\{a_n\}$ or $\{a_{nm}\}$ is in the sense of ℓ^2, then these series make sense as generalized solutions to the

initial boundary-value problem, where the initial condition is given explicitly as a Fourier series (or Fourier-Bessel series...), as in equation (4). What is interesting is that the boundary conditions determine the type of Fourier decomposition that is necessary, and this affects the time evolution.

The guiding principle is this: The sequences $\{\sin nx\}_{n=1}^{\infty}$, $\{\cos nx\}_{n=0}^{\infty}$, $\{\sin nx \sin mx\}_{n,m=1}^{\infty}$, etc. are the eigenfunctions of the Laplace operator Δ, subject to certain homogeneous boundary conditions. The boundary condition must make Δ a <u>self-adjoint</u> operator in order for the associated eigenfunctions to give a basis of L^2. The theory is very delicate, and we postpone it to later.

The solution on the line of the heat equation, in Fourier form, is

$$u(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{ikx} e^{-\sigma k^2 t}\, dk .$$

The heat kernel.

The function on $\mathbb{R}^n \times (0,\infty)$ given by

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

(with $\sigma = 1$) is a solution of the heat equation for $t > 0$, as one can easily verify. The function

$$\Phi(x,1) = \phi(x) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4}}$$

has the property

$$\int_{\mathbb{R}^n} \phi(z)\, dV(z) = 1$$

(see Folland, Ch. 0). By a change of coordinate $z = \frac{x-y}{\varepsilon}$, where $y \in \mathbb{R}^n$ and $\varepsilon > 0$ are fixed, we obtain

$$\int_{\mathbb{R}^n} \tfrac{1}{\varepsilon^n} \phi\left(\tfrac{x-y}{\varepsilon}\right) dV(x) = 1$$

In addition, for each function ψ that is C^∞ and has compact support — we say $\psi \in C_c^\infty$ — we have

$$\left| \int_{\mathbb{R}^n} \tfrac{1}{\varepsilon^n} \phi\left(\tfrac{x-y}{\varepsilon}\right) \psi(x)\, dV \right| < \sup_{x \in \mathbb{R}^n} |\psi(x)| \quad \forall\ \varepsilon > 0$$

(10)

$$\int_{\mathbb{R}^n} \tfrac{1}{\varepsilon^n} \phi\left(\tfrac{x-y}{\varepsilon}\right) \psi(x)\, dV \longrightarrow \psi(y) \quad \text{as } \varepsilon \to 0$$

This can be proved by using the change of variable $z = \frac{x-y}{\varepsilon}$ and the Lebesgue Dominated-Convergence Theorem.

With $\varepsilon = t^{1/2}$, we have

(10.5) $$\Phi(x-y,t) = \frac{1}{\varepsilon^n}\phi\left(\frac{x-y}{\varepsilon}\right),$$

and this is called the "heat kernel". The reason is that it serves as the "integral kernel" for the solution of the initial-value problem for the heat equation on $\mathbb{R}^n$. The function $\Phi(x-y,t)$ represents the solution to the heat equation with an initial distribution that is localized (ideally) at one point y.

Theorem If $g \in L^1(\mathbb{R}^n)$, the function

(10.7) $$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dV(y)$$

solves the heat equation for $t>0$, $x \in \mathbb{R}^n$ and converges to g in the distributional sense as $t \to 0$, that is, for each $\psi \in C_c^\infty(\mathbb{R}^n)$,

(11) $$\int_{\mathbb{R}^n} u(x,t)\,\psi(x)\, dV(x) \longrightarrow \int_{\mathbb{R}^n} g(x)\,\psi(x)\, dV(x)$$

as $t \to 0^+$.

Proof That $u(x,t)$ solves the heat equation follows from commutation of integration in y with differentiation in x and t. We will skip this part.

To prove (11), let $\psi \in C_c^\infty$ be given.

$$\int_{\mathbb{R}^n} u(x,t)\,\psi(x)\,dV(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(x-y,t)\,g(y)\,\psi(x)\,dV(y)\,dV(x)$$

Fubini's theorem

$$= \int_{\mathbb{R}^n} g(y) \int_{\mathbb{R}^n} \Phi(x-y,t)\,\psi(x)\,dV(x)\,dV(y)$$

By (10)$^{\S(10.5)}$, the inner integral tends to the function $\psi(y)$ pointwise as $t \to 0$ and is bounded in magnitude by $\sup_{x\in\mathbb{R}^n} |\psi(x)| < \infty$.

By the Lebesgue Dominated-Convergence Theorem, this integral tends to $\int_{\mathbb{R}^n} g(y)\psi(y)\,dV(y)$. ∎

One can prove that $u(x,t) \to g(x)$ at points of continuity of g (see Evans §2.3 Thm 1) and that $g(x)$ need not be L^1 (see Ch. 4 of Folland, for example)

Formula (10.7) demonstrates the infinite propagation speed of initial data for the heat equation.