The abstract theory of unbdd ops in Hilbert space. Ref: Reed/Simon I

An unbounded linear operator T from a Hilbert space $\mathcal{K}$ to a Hilbert space $\mathcal{H}$ is normally defined on a (typically dense) subvector space of $\mathcal{K}$, called the domain of T and denoted by $\mathcal{D}(T)$:

$$T : \mathcal{D}(T) \subset \mathcal{K} \longrightarrow \mathcal{H}.$$

The graph norm of T is defined on $\mathcal{D}(T)$ by

$$\|u\|_T = \|u\|_K + \|Tu\|_{\mathcal{H}}$$

Defn $T: \mathcal{D}(T) \subset \mathcal{K} \to \mathcal{H}$ is closed if $\mathcal{D}(T)$ is complete in the graph norm of T. (Assume T is densely defined)

Equivalently, T is closed if either of the fllg equiv. conds holds:

- The graph of T in $K \oplus \mathcal{H}$, $\Gamma(T) = \{(u,v) \in K \oplus \mathcal{H} : u \in \mathcal{D}(T), v = T(u)\}$ is closed in $K \oplus \mathcal{H}$
- If $\{u_n\}$ is a sequence from $\mathcal{D}(T)$ s.th. $u_n \to u$ in K and $Tu_n \to v$ in $\mathcal{H}$, then $u \in \mathcal{D}(T)$ and $v = Tu$.

Defn The adjoint $T^*: \mathcal{D}(T^*) \subset \mathcal{H} \to \mathcal{K}$ is defined through

$$\mathcal{D}(T^*) = \{u \in \mathcal{H} : \exists\, C \text{ s.th. } |(Tv,u)| < C\|v\| \ \forall\, v \in \mathcal{D}(T)\};$$

for $u \in \mathcal{D}(T^*)$, T^*u is the unique element of $\mathcal{K}$ such that $(Tv,u) = (v,T^*u) \ \forall\ v \in \mathcal{D}(T)$.

The adjoint is well defined as an operator because T is densely defined (our standing assumption).

<u>Fact</u> The <u>adjoint T^*</u> of a densely defined operator $T: \mathcal{K} \to \mathcal{H}$ <u>is closed</u>.

<u>Proof</u> Suppose that $\{v_n\}_{n=1}^{\infty}$ is a sequence from $\mathcal{D}(T^*)$ s.th. $v_n \to v$ in $\mathcal{K}$ and $T^* v \to w$ in $\mathcal{H}$. By defn. of T^*, $\forall\ u \in \mathcal{D}(T)$, we have $(Tu, v_n) = (u, T^* v_n)$, $n = 1, 2, \ldots$. By convergence, we obtain $(Tu, v) = (u, w)$ $\forall\ u \in \mathcal{D}(T)$. This means that $u \mapsto (Tu, v)$ is a bounded functional and thus $v \in \mathcal{D}(T^*)$ and $w = T^* v$. This proves that T^* is closed.

<u>Defn</u>. An operator $T: \mathcal{D}(T) \subset K \to \mathcal{H}$ is <u>closeable</u> if the closure $\overline{\Gamma(T)}$ of the graph $\Gamma(T)$ of T in $K \oplus \mathcal{H}$ is the graph of an operator. This operator $\overline{T}$ is called the <u>closure</u> of T.

Equivalently, T is closeable if either of the two flly conditions holds:

- If $\{u_i\}_{i=1}^{\infty}$ is a sequence from $\mathcal{D}(T)$ s.th. $u_i \to 0$ and $Tu_i \to u$, then $u = 0$.
- T^* is densely defined, that is, $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$.

<u>Thm</u>
- If T is closeable, then $\overline{\overline{T}} = \overline{T}$.
- If T and T^* are densely defined, then $T^{**} = \overline{T}$.

Example 1

* $\nabla_c\cdot : (C_c^\infty(\Omega))^n \longrightarrow L^2(\Omega)$ is the divergence operator on C_c^∞ functions on Ω. It is densely defined.
$\subset (L^2(\Omega))^n$

* By definition, $f \in \mathcal{D}(\nabla)$ if $\exists\ F = \nabla f \in (L^2(\Omega))^n$

$$\int_\Omega f\,\nabla\cdot\Phi + \int_\Omega \nabla f\cdot\Phi = 0 \quad \forall\ \Phi \in \mathcal{D}(\nabla_c\cdot),$$

so $\nabla = -(\nabla_c\cdot)^*$ and ∇ is therefore a closed operator:
Its domain $\mathcal{D}(\nabla) = H^1(\Omega)$ is complete in the graph norm of ∇, which is just the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$.

Point of view: $H^1(\Omega)$ is complete simply because it is defined as the domain of the adjoint of a densely-defined operator in the operator norm.

* Since $C_c^\infty(\Omega) \subset H^1(\Omega)$ and $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, ∇ is densely defined. Thus $-\nabla^* = \overline{\nabla_c\cdot}$, that is, the adjoint of $-\nabla$ is the closure of the operator $\nabla_c\cdot$. We can characterize $\overline{\nabla_c\cdot}$ be recalling the relation

$$(*)\qquad \int_\Omega f\,\nabla\cdot\Phi + \int_\Omega \nabla f\cdot\Phi = \int_{\partial\Omega} f\,\Phi\cdot n, \qquad f \in H^1(\Omega),\ \Phi \in \mathcal{D}(\nabla\cdot),\ \Phi\cdot n \in H^{-1/2}(\partial\Omega)$$

The criterion for $\Phi \in \mathcal{D}(\overline{\nabla_c\cdot}) = \mathcal{D}(-\nabla^*)$ is that the LHS be equal to zero for all $f \in H^1(\Omega) = \mathcal{D}(-\nabla)$. This amounts to the condition that $\Phi\cdot n = 0$. Thus we call this operator

$$\nabla_0\cdot = \overline{\nabla_c\cdot} = -\nabla^*$$

with domain $\mathcal{D}(\nabla_0\cdot) = \{\Phi \in \mathcal{D}(\nabla\cdot) : \Phi\cdot n = 0\}$.

The operator ∇ with domain $H^1(\Omega)$ is the gradient operator on its maximal domain, and the operator $\nabla_0\cdot$ is the divergence operator on its minimal domain (assuming it is closed).
— in the L^2-sense.

Example 2

* Begin with $\nabla_c : C_c^\infty(\Omega) \subset L^2(\Omega) \longrightarrow (L^2(\Omega))^n$, which is densely defined.
* ∇_c^* is the closed operator s.th: $F \in \mathcal{D}(\nabla_c^*)$ if $\exists\ f = \nabla_c^* F = -\nabla\cdot F \in L^2(\Omega)$ such that

$$\int_\Omega F\cdot\nabla_c\phi + \int_\Omega (\nabla\cdot F)\,\phi = 0 \quad \forall\ \phi \in C_c^\infty(\Omega) = \mathcal{D}(\nabla_c),$$

that is, $\nabla_c^* = -\nabla\cdot$. Again, $\nabla\cdot$ is a closed operator, and it is densely defined because $(C_c^\infty(\Omega))^n \subset \mathcal{D}(\nabla\cdot)$ and $(C_c^\infty(\Omega))^n$ is dense in $(L^2(\Omega))^n$.
* $\nabla\cdot^* = -\overline{\nabla_c}$ since ∇_c and $\nabla\cdot = -\nabla_c^*$ are densely defined.

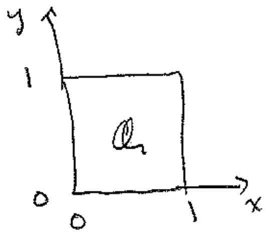
Recall again the relation

$$(\dagger)\qquad \int_\Omega F\cdot\nabla\phi + \int_\Omega (\nabla\cdot F)\phi = \int_{\partial\Omega} (F\cdot n)\phi \quad \forall\ F \in \mathcal{D}(\nabla\cdot),\ \phi \in H^1(\Omega).$$

The criterion for $\phi \in \mathcal{D}(\overline{\nabla_c}) = \mathcal{D}(-\nabla\cdot^*)$ is that the LHS of $(\dagger)$ vanish for all $F \in \mathcal{D}(\nabla\cdot)$. This means that $\phi|_{\partial\Omega} = 0$, or $\phi \in H_0^1(\Omega)$.

Thus, $\overline{\nabla_c}$ should be denoted by ∇_0; it is a closed operator with domain $H_0^1(\Omega)$ — the minimal domain of the gradient (assuming a closed operator). $\mathcal{D}(\nabla\cdot)$ is the maximal domain of the divergence — in the L^2-sense.

Example 3



$$H^1_{per}(\Omega) = \{ u \in H^1(\Omega) : u(x,0) = u(x,1) \ \forall x \in (0,1) \text{ and } u(0,y) = u(1,y) \ \forall y \in (0,1) \}$$

[Of course, $u(x,0)$, etc. are understood in the sense of $H^{1/2}$.]

$H^1_{per}(\Omega)$ is a closed subspace of $H^1(\Omega)$.

* Define $\nabla_{per} : H^1_{per}(\Omega) \to (L^2(\Omega))^n$ to be the restriction of ∇ to $H^1_{per}(\Omega)$. ∇_{per} is closed, because $H^1_{per}(\Omega)$ is closed in the graph norm of ∇, and is densely defined. ∇_{per} is the <u>periodic gradient operator</u>.

* To characterize ∇^*_{per}, we use the relation

$$(\mathrm{H}) \qquad \int_\Omega F \cdot \nabla\phi + \int_\Omega (\nabla \cdot F)\phi = \int_{\partial\Omega} (F \cdot n)\phi \quad , \ F \in \mathcal{D}(\nabla\cdot),\ \phi \in H^1(\Omega).$$

Since $\mathcal{D}(\nabla_{per}) \supset \mathcal{D}(\nabla_0)$, we have $\mathcal{D}(\nabla^*_{per}) \subset \mathcal{D}(\nabla^*_0) = \mathcal{D}(\nabla\cdot)$.

The additional condition is that the LHS of (H) must vanish not for all $\phi \in H^1_0(\Omega)$ but <u>for all $\phi \in H^1_{per}(\Omega)$</u>. One verifies that

$$\mathcal{D}(\nabla^*_{per}) = \{ F \in \mathcal{D}(\nabla\cdot) : (F\cdot n)(x,0) = -F\cdot n(x,1) \text{ and } (F\cdot n)(0,y) = -F\cdot n(1,y) \},$$

where equality of normal components is in the sense of $H^{-1/2}$.

Thus, we denote $-\nabla^*_{per}$ by $\nabla^{\bullet}_{per}$ ~ the <u>periodic divergence operator</u>.

* Since ∇_{per} is closed, it is the adjoint of $-\nabla^{\bullet}_{per}$:

$$\underline{\nabla_{per} \text{ and } -\nabla^{\bullet}_{per} \text{ are mutually adjoint}}$$