The abstract theory of unbounded ops in Hilbert space. Ref: Reed/Simon I

An unbounded operator $T$ from a Hilbert space $\mathcal{K}$ to a Hilbert space $\mathcal{H}$ is normally defined on a (typically dense) subvector space of $\mathcal{K}$, called the domain of $T$ and denoted by $\mathcal{D}(T)$:

$$ T : \mathcal{D}(T) \subset \mathcal{K} \rightarrow \mathcal{H} $$

The graph norm of $T$ is defined on $\mathcal{D}(T)$ by

$$ \|u\|_T = \|u\|_\mathcal{K} + \|Tu\|_\mathcal{H} $$

Defn $T : \mathcal{D}(T) \subset \mathcal{K} \rightarrow \mathcal{H}$ is closed if $\mathcal{D}(T)$ is complete in the graph norm of $T$. (Assume $T$ is densely defined)

Equivalently, $T$ is closed if either of the following equiv. conditions holds:

- The graph of $T$ in $\mathcal{K} \oplus \mathcal{H}$, $\Gamma(T) = \{(u,v) \in \mathcal{K} \oplus \mathcal{H} : u \in \mathcal{D}(T), v = Tu\}$, is closed in $\mathcal{K} \oplus \mathcal{H}$

- If $\{u_n\}$ is a sequence from $\mathcal{D}(T)$ s.th. $u_n \rightarrow u$ in $\mathcal{K}$ and $Tu_n \rightarrow v$ in $\mathcal{H}$, then $u \in \mathcal{D}(T)$ and $v = Tu$.

Defn The adjoint $T^* : \mathcal{D}(T^*) \subset \mathcal{H} \rightarrow \mathcal{K}$ is defined through

$$ \mathcal{D}(T^*) = \left\{ u \in \mathcal{H} : \exists C \text{ s.t. } |(Tv,u)| \leq C \|u\|_\mathcal{H} + \|v\|_\mathcal{H} \quad \forall v \in \mathcal{D}(T) \right\} $$

For $u \in \mathcal{D}(T^*)$, $T^*$ is the unique element of $\mathcal{K}$ such that

$$ (Tv,u) = (v,T^*u) \quad \forall v \in \mathcal{D}(T). $$

The adjoint is well defined as an operator because $T$ is densely defined (our standing assumption).
Fact: The adjoint $T^*$ of a densely defined operator $T: K \rightarrow \mathcal{H}$ is closed.

Proof. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence from $\mathcal{D}(T^*)$ such that $u_n \rightarrow u$ in $K$ and $T^*u_n \rightarrow w$ in $\mathcal{H}$. By definition of $T^*$, $\forall u \in \mathcal{D}(T)$, we have $(Tu, u_n) = (u, T^*u_n)$, $n = 1, 2, \ldots$. By convergence, we obtain $(Tu, v) = (u, w)$, $\forall u \in \mathcal{D}(T)$. This means that $u \mapsto (Tu, v)$ is a bounded functional and thus $v \in \mathcal{D}(T^*)$ and $w = T^*v$. This proves that $T^*$ is closed.

Defn. An operator $T: \mathcal{D}(T) \subset K \rightarrow \mathcal{H}$ is closeable if the closure $\overline{T(K)}$ of the graph $\mathcal{G}(T)$ of $T$ in $K \oplus \mathcal{H}$ is the graph of an operator. This operator $\overline{T}$ is called the closure of $T$.

Equivalently, $T$ is closeable if either of the two following conditions holds:

- If $\{u_n\}_{n=1}^{\infty}$ is a sequence from $\mathcal{D}(T)$ such that $u_n \rightarrow 0$ and $Tu_n \rightarrow u$, then $u = 0$.
- $T^*$ is densely defined, that is, $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$.

Thus:

- If $T$ is closeable, then $\overline{T} = T$.
- If $T$ and $T^*$ are densely defined, then $T^{**} = \overline{T}$.
Example 1 \( \nabla^*_0 : (C^0(R))^n \to L^2(R^n) \) is the divergence operator on \( C^0 \) functions on \( \mathbb{R}^n \). It is densely defined.

- By definition, \( f \in \mathcal{D}(\nabla^*_0) \) if \( \exists F = \nabla f \in (L^2(R))^n \)

\[
\int_{R^n} f \nabla \cdot \Phi + \int_{R^n} \Phi \cdot \nabla f = 0 \quad \forall \Phi \in \mathcal{D}(\nabla^*_0),
\]

so \( \nabla = - (\nabla^*_0)^* \) and \( \nabla \) is therefore a closed operator.

Its domain \( \mathcal{D}(\nabla) = H^1(R^n) \) is complete in the graph norm of \( \nabla \), which is just the Sobolev norm \( \|
abla \|_{H^1} \).

Point of view: \( H^1(R^n) \) is complete simply because it is defined as the domain of the adjoint of a densely-defined operator in the operator norm.

- Since \( C^0(R^n) \subset H^1(R^n) \) and \( C^0(R^n) \) is dense in \( L^2(R^n) \), \( \nabla \) is densely defined. Thus \( -\nabla^* = \overline{\nabla^*} \), that is, the adjoint of \( -\nabla \) is the closure of the operator \( \nabla^*_0 \). We can characterize \( \overline{\nabla^*_0} \) by recalling the relation

\[
\int_{R^n} f \nabla \cdot \Phi + \int_{R^n} \Phi \cdot \nabla f = \int_{R^n} f \Phi \cdot n \quad \forall \Phi \in \mathcal{D}(\nabla), \quad f \in H^1(R^n), \quad \Phi \cdot n \in H^{-\frac{1}{2}}(\partial R^n)
\]

The criterion for \( \Phi \in \mathcal{D}(\overline{\nabla^*_0}) = \mathcal{D}(\nabla^*) \) is that the LHS be equal to zero for all \( f \in H^1(R^n) = \mathcal{D}(\nabla) \). This amounts to the condition that \( \Phi \cdot n = 0 \). Thus we call this operator

\[
\nabla^*_0 = \overline{\nabla^*_0} = -\nabla^* \\
\text{with domain } \mathcal{D}(\nabla^*_0) = \{ \Phi \in \mathcal{D}(\nabla) : \Phi \cdot n = 0 \}.
\]
The operator $\nabla$ with domain $H^1(\Omega)$ is the gradient operator on its maximal domain, and the operator $\nabla^*$ is the divergence operator on its minimal domain (assuming it is closed).

--- in the $L^2$-sense.

**Example 2**

* Begin with $\nabla_c : C_c^\infty(\Omega) \subset L^2(\Omega) \to (L^2(\Omega))^n$, which is densely defined.

* $\nabla^*_c$ is the closed operator s.t.: $F \in \mathcal{D}(\nabla^*_c)$ if $\exists \ f = \nabla^*_c F = -\nabla F \in L^2(\Omega)$ such that

$$\int_{\Omega} F \cdot \nabla \phi + \int_{\Omega} (\nabla F) \cdot \phi = 0 \quad \forall \ \phi \in C_c^\infty(\Omega) = \mathcal{D}(\nabla_c),$$

that is, $\nabla^*_c = -\nabla_c$. Again, $\nabla^*_c$ is a closed operator, and it is densely defined because $(C_c^\infty(\Omega))^n \subset \mathcal{D}(\nabla_c)$ and $(C_c^\infty(\Omega))^n$ is dense in $(L^2(\Omega))^n$.

* $\nabla^*_c = -\nabla_c$ since $\nabla_c$ and $\nabla^*_c = -\nabla^*_c$ are densely defined.

Recall again the relation

$$\int_{\Omega} F \cdot \nabla \phi + \int_{\Omega} (\nabla F) \cdot \phi = \int_{\partial \Omega} (F \cdot n) \phi \quad \forall \ F \in \mathcal{D}(\nabla), \ \phi \in H^1(\Omega).$$

The criterion for $\phi \in \mathcal{D}(\nabla_c) = \mathcal{D}(-\nabla^*_c)$ is that the LHS of (1) vanish for all $F \in \mathcal{D}(\nabla)$. This means that $\phi|_{\partial \Omega} = 0$, or $\phi \in H^1_0(\Omega)$.

Thus, $\nabla_c$ should be denoted by $\nabla$; it is a closed operator with domain $H^1_0(\Omega)$ — the minimal domain of the gradient (assuming a closed operator) — $\mathcal{D}(\nabla)$ is the maximal domain of the divergence — in the $L^2$-sense.
Example 3

$H^1_{\text{per}}(\mathbb{R}) = \{ u \in H^1(\mathbb{R}) : u(x,0) = u(x,1) \forall x \in (0,1) \}
\text{ and } u(0,y) = u(1,y) \forall y \in (0,1) \}^*$

[Of course, $u(x,0), \text{etc.}$ are understood in the sense of $H^{1/2}$]

$H^1_{\text{per}}(\mathbb{R})$ is a closed subspace of $H^1(\mathbb{R})$.

* Define $\nabla_{\text{per}} : H^1_{\text{per}}(\mathbb{R}) \to (L^2(\mathbb{R}))^n$ to be the restriction of $\nabla$ to $H^1_{\text{per}}(\mathbb{R})$. $\nabla_{\text{per}}$ is closed, because $H^1_{\text{per}}(\mathbb{R})$ is closed in the graph norm of $\nabla$, and is densely defined. $\nabla_{\text{per}}$ is the periodic gradient operator.

* To characterize $\nabla^*_\text{per}$, we use the relation

$\int_{\mathbb{R}} F \cdot \phi + \int_{\mathbb{R}} (\nabla \cdot F) \phi = -\int_{\mathbb{R}} (Fw) \phi, \quad F \in \mathcal{D}(\nabla), \phi \in H^1(\mathbb{R}).$ (1)

Since $\mathcal{D}(\nabla_{\text{per}}) = \mathcal{D}(\nabla_0)$, we have $\mathcal{D}(\nabla^*_\text{per}) \subseteq \mathcal{D}(\nabla^*_0) = \mathcal{D}(\nabla)$. The additional condition is that the LHS of (1) must vanish for all $\phi \in H^1_{\text{per}}(\mathbb{R})$ but for all $\phi \in H^1_{\text{per}}(\mathbb{R})$. One verifies that

$\mathcal{D}(\nabla^*_\text{per}) = \{ F \in \mathcal{D}(\nabla) : (Fw)(x,0) = -Fw(x,1) \text{ and } (Fw)(x,1) = -Fw(x,0) \}^*$

where equality of normal components is in the sense of $H^{1/2}$.

Thus, we denote $-\nabla^*_\text{per}$ by $\nabla^*_{\text{per}}$ - the periodic divergence operator.

* Since $\nabla_{\text{per}}$ is closed, it is the adjoint of $-\nabla^*_{\text{per}}$.

$\nabla_{\text{per}}$ and $-\nabla^*_{\text{per}}$ are mutually adjoint.