Boundary conditions and unique solutions

Let us reconsider the weak-form PDE with its strong-form counterpart:

(0) Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \int_\Omega h u \bar{v} - \lambda \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} \qquad \forall \; v \in H^1_0(\Omega)$$

(0') $-\nabla \cdot \sigma \nabla u - \lambda \varepsilon u = f$ in Ω

Problem (0) does not have a unique solution, but we have identified certain restrictions on u that do give rise to unique solutions, as long as λ is not contained in a set $\{\lambda_i\}_{i=1}^{\infty}$ ($\lambda_i \to \infty$ as $i \to \infty$) of characteristic values for that restriction.

(1) Vanishing <u>Dirichlet</u> boundary values.

Find $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} - \lambda \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} \qquad \forall \; v \in H^1_0(\Omega)$$

(1') $-\nabla \cdot \sigma \nabla u - \lambda \varepsilon u = f$ in Ω

$u = 0$ on $\partial\Omega$

(2) Vanishing <u>Neumann</u> boundary values

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} - \lambda \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} \qquad \forall \; v \in H^1(\Omega)$$

(2') $-\nabla \cdot \sigma \nabla u - \lambda \varepsilon u = f$ in Ω

$\partial_n u = 0$ on $\partial\Omega$

(3) Homogeneous <u>Robin</u> boundary conditions

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \int_\Omega h u \bar{v} - \lambda \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} \qquad \forall \quad v \in H^1(\Omega)$$

(3') $-\nabla \cdot \sigma \nabla u - \lambda \varepsilon u = f$ in Ω

$\sigma \partial_n u + h u = 0$ on $\partial\Omega$

(4) <u>Periodic</u> boundary conditions on $Q_1 = [0,1]^n$

Find $u \in H^1_{per}(Q_1)$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} - \lambda \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} \qquad \forall \quad v \in H^1_{per}(Q_1)$$

(4') $-\nabla \cdot \sigma \nabla u - \lambda \varepsilon u = f$ in Q_1

$u(x_1, \dots, 0, \dots, x_n) = u(x_1, \dots, 1, \dots, x_n)$ for each i and

$\sigma \partial_n u(x_1, \dots, 0, \dots, x_n) = -\sigma \partial_n u(x_1, \dots, 1, \dots, x_n)$ $(x_j)_{j \neq i} \in [0,1]^{n-1}$.

↑ ith place

Some abstract theory of the relation between symmetric forms and operators in Hilbert space will put the <u>strong form</u> of the PDE-BVP into its correct mathematical framework.

First, let us illustrate with the Dirichlet case.

The Dirichlet case

$$\begin{pmatrix} 0 < \sigma_- < \sigma\xi\cdot\xi < \sigma_+ \quad \text{for } |\xi|=1 \\ 0 < \varepsilon_- < \varepsilon < \varepsilon_+ \\ f \in L^2 \end{pmatrix}$$

Define the operator T_0 by

$$\begin{cases} T_0 : H_0^1(\Omega) \subset L^2(\Omega,\varepsilon) \longrightarrow \left(L^2(\Omega,\sigma)\right)^n , \\ T_0 u = \nabla u \quad \text{for } u \in H_0^1(\Omega) . \end{cases}$$

To characterize the adjoint T_0^*, observe that $F \in \mathcal{D}(T_0^*)$ if there exists an element of $L^2(\Omega,\varepsilon)$, denoted by $T_0^* F$, s.th.

$$\int_\Omega \sigma \nabla u \cdot F - \int_\Omega \varepsilon u T_0^* F = 0 \qquad \forall \quad u \in \mathcal{D}(\nabla_0)$$

OR $\displaystyle\int_\Omega \nabla u \cdot (\sigma F) + \int_\Omega u(-\varepsilon T_0^* F) = 0 \qquad \forall \quad u \in \mathcal{D}(\nabla_0)$

(σ & ε positive), so we see that $-\varepsilon T_0^* F = \nabla\cdot\sigma F$. Thus, T_0^* is given by

$$\begin{cases} \mathcal{D}(T_0^*) = \left\{ F \in \left(L^2(\Omega)\right)^n : \sigma F \in \mathcal{D}(\nabla\cdot) \right\} , \\ T_0^* F = -\frac{1}{\varepsilon} \nabla\cdot\sigma F , \quad F \in \mathcal{D}(T_0^*) . \end{cases}$$

Problem (1) is to find $u \in \mathcal{D}(T_0)$ such that

$$(T_0 u, T_0 v)_\sigma - \lambda(u,v)_\varepsilon = (f,v)_\varepsilon \qquad \forall \quad v \in \mathcal{D}(T_0) .$$

Since $|(T_0 u, T_0 v)| \leq \|f+\lambda u\|_\varepsilon \|v\|$, we have $T_0 u \in \mathcal{D}(T_0^*)$, and thus Problem (1) becomes $(T_0^* T_0 u, v)_\sigma - \lambda(u,v)_\varepsilon = (f,v)_\varepsilon \quad \forall\ v \in \mathcal{D}(T_0)$.

Since $\mathcal{D}(T_0)$ is dense in L^2, we obtain finally

$$T_0^* T_0 u - \lambda u = \varepsilon^{-1} f$$

where $\mathcal{D}(T_0^* T_0) = \left\{ u \in \mathcal{D}(T_0) : T_0 u \in \mathcal{D}(T_0^*) \right\}$.

The operator $T_0^* T_0$ is more concretely characterized by

$$\mathcal{D}(T_0^* T_0) = \{ u \in H_0^1(\Omega) : \sigma \nabla u \in \mathcal{D}(\nabla \cdot) \},$$

$$T_0^* T_0 u = -\tfrac{1}{\varepsilon} \nabla \cdot \sigma \nabla u \quad \text{for} \quad u \in \mathcal{D}(T_0^* T_0).$$

In view of the more general (distributional) definition of the divergence, the condition $F \in \mathcal{D}(\nabla \cdot)$ is often expressed as $\nabla \cdot F \in L^2(\Omega)$.

In summary, problem (1) is equivalent to the problem

$$(1'') \quad \begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that } \sigma \nabla u \in \mathcal{D}(\nabla \cdot) \text{ and} \\ -\nabla \cdot \sigma \nabla u - \lambda \varepsilon u = f, \end{cases}$$

or, more casually, find $u \in L^2(\Omega)$ such that $\nabla u \in L^2$ and

$$\begin{cases} -\nabla \cdot \sigma \nabla u - \lambda \varepsilon u = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

We now understand this to mean (1″) in the rigorous sense and see that (1″) is equivalent to (1′).

|| <u>Notice</u> that the <u>operator</u> $T_0^* T_0$ <u>subsumes</u> <u>the</u> homogeneous <u>boundary conditions</u> – that is, <u>homogeneous</u> bdy conditions $u|_{\partial\Omega} = 0$

<u>Fact</u> : $T_0^* T_0$ is self-adjoint in $L^2(\Omega, \varepsilon)$, that is,

$$\mathcal{D}((T_0^* T_0)^*) = \mathcal{D}(T_0^* T_0) \text{ and } (T_0^* T_0)^* u = T_0^* T_0 u \;\; \forall \; u \in \mathcal{D}(T_0^* T_0).$$

The problems (1) – (4) fit into the following abstract framework.

Let T be a closed unbounded operator $T : \mathcal{D}(T) \subset \mathcal{H} \to K$ with dense domain, and consider the problem of finding $u \in \mathcal{D}(T)$ such that

$$(*) \qquad (Tu, Tv)_K - \lambda (u,v)_{\mathcal{H}} = (f,v) \quad \forall \quad v \in \mathcal{D}(T),$$

where f is an arbitrary element of $\mathcal{H}$. Any solution is necessarily in $\mathcal{D}(T^*T)$ because $Tu \in \mathcal{D}(T^*)$, that is,

$$|(Tu, Tv)| \leq \|\lambda u + f\| \|v\| .$$

So $(*)$ is equivalent to solving

$$T^*Tu - \lambda u = f$$

We will show that T^*T is self-adjoint, but first, we abstract further and, in place of (Tu, Tv), consider unbounded closed symmetric quadratic forms $a(u,v)$ in $\mathcal{H}$. Such a form is defined on $Q(a) \oplus Q(a)$, where the form domain $Q(a)$ is dense in $\mathcal{H}$. It is linear in one argument (say, the first) and conjugate linear in the second, and <u>symmetry</u> means $a(u,v) = \overline{a(v,u)}$ for all $u, v \in Q(a)$. The quadratic form a is <u>positive</u> if $a(u,u) \geq 0$ for all $u \in Q(a)$.

If a is positive, then there is a natural form norm $\|\cdot\|_a$ on $Q(a)$:

$$\|u\|_a^2 = a(u,u) + (u,u),$$

and one says that a is <u>closed</u> if $Q(a)$ is complete with respect to this norm. If a is symmetric (this is implied by positivity for complex Hilbert spaces), there is an associated inner product

$$(u,v)_a := a(u,v) + (u,v)$$

which makes $Q(a)$ a Hilbert space.

Let us maintain the assumption that a is symmetric, closed, and positive with dense domain.

For $u \in Q(a)$, if $a(u,\cdot)$ is bounded in the norm $\|\cdot\|_{\mathcal{H}}$, that is, $\exists C>0$ s.th. $|a(u,v)| < C\|v\|$ $\forall$ $v \in Q(a)$, then by the density of $Q(a)$, there is a unique element $\hat{u} \in \mathcal{H}$ s.th. $a(u,v) = (\hat{u},v)$ $\forall$ $v \in Q(a)$.

Define the linear operator A in $\mathcal{H}$ by

$$\begin{cases} \mathcal{D}(A) = \{u \in Q(a) : \exists\, C>0 \text{ s.th. } |a(u,v)| < C\|v\| \ \forall\, v \in Q(a)\}, \\ (Au,v) = a(u,v) \quad \forall\, v \in Q(a). \end{cases}$$

<u>Theorem</u> A is self-adjoint.

<u>Proof</u> First, A is symmetric because, if $u,v \in \mathcal{D}(A) \subset Q(a)$,

$$(Au,v) = a(u,v) = \overline{a(v,u)} = \overline{(Av,u)} = (u,Av).$$

Next we prove that $A+I : \mathcal{D}(A) \to \mathcal{H}$ is surjective, where I is the identity operator.

Given $f \in \mathcal{H}$,

$$|(f,v)| \le \|f\|\,\|v\| \le \|f\|\,\|v\|_a \quad \forall \quad v \in Q(a),$$

so by the Riesz Lemma, there exists a unique $u \in Q(a)$ such that $a(u,v) + (u,v) = (u,v)_a = (f,v) \quad \forall \quad v \in Q(a)$.

Since $|a(u,v)| \le \|f-u\|\,\|v\| \quad \forall \; v \in Q(a)$, we have $u \in \mathcal{D}(A)$ and $(Au,v) + (u,v) = (f,v) \quad \forall \quad v \in Q(a)$. Thus

$$(A+I)u = f .$$

This demonstrates the surjectivity of $A+I$.

Finally, we show that $\mathcal{D}(A^*) = \mathcal{D}(A)$, which, together with symmetry of A will prove its self-adjointness.

Let $u \in \mathcal{D}(A^*) \supset \mathcal{D}(A)$. By the surjectivity of $A+I$, there exists $v \in \mathcal{D}(A)$ such that $(A+I)v = (A^*+I)u$. Since $Av = A^*v$ we have $(A^*+I)(v-u) = 0$. Let $w \in \mathcal{H}$ be given. There exists $z \in \mathcal{D}(A)$ s.th. $(A+I)z = w$, so

$$(w, v-u) = ((A+I)z, v-u) = (z, (A^*+I)(v-u)) = 0 .$$

Thus $u = v \in \mathcal{D}(A)$, and we have demonstrated that $\mathcal{D}(A^*) = \mathcal{D}(A)$.

Recall that all self-adjoint operators are densely defined and closed: An operator that is not densely defined does not have an adjoint operator, and all adjoint operators are closed.

<u>Theorem</u> The form a is determined by its associated self-adjoint operator A. That is, if A is the self-adjoint operator associated to the forms a and b, then $a = b$.

<u>Proof</u> The key is to show that $\mathcal{D}(A)$ is dense in $Q(a)$ in the norm $\|\cdot\|_a$. Let $u \in Q(a)$ be such that $(u, w)_a = 0$ for all $w \in \mathcal{D}(A)$. This means that

$$(u, Aw) + (u, w) = a(u, w) + (u, w) = 0 \quad \forall \ w \in \mathcal{D}(A).$$

As we have seen, $A + I$ is surjective, so given $f \in \mathcal{H}$, $\exists \ w \in \mathcal{D}(A)$ such that $Aw + w = f$, and thus

$$(u, f) = (u, Aw) + (u, w) = 0.$$

Since this holds for all $f \in \mathcal{H}$, we obtain $u = 0$. It follows that $\mathcal{D}(A)$ is in fact dense in $Q(a)$ in the norm $\|\cdot\|_a$. The Hilbert space $Q(a), (\cdot,\cdot)_a$ is therefore fully determined by A and thus so is the form a.

The operator A is <u>positive</u> because a is. This means that

$$(Au, u) \geq 0 \quad \text{for all} \quad u \in \mathcal{D}(A).$$

__Theorem__ If A is a positive self-adjoint operator in $\mathcal{H}$, then there exists a unique closed positive symmetric quadratic form in $\mathcal{H}$ such that A is its associated self-adjoint operator.

The proof (see Reed/Simon Vol. I, § VIII.6, Example 2) makes use of the spectral theorem for self-adjoint operators.

There is therefore a one-to-one correspondence between positive symmetric closed quadratic forms in $\mathcal{H}$ and positive self-adjoint operators in $\mathcal{H}$. This correspondence can be extended to semibounded forms, for which

$$a(u,u) \geq -M(u,u) \quad \forall\, u \in Q(a),$$

where M is some real number.

The following simple example illustrates the way in which a positive self-adjoint operator is associated to a quadratic form.

__Example__ Define $A : \mathcal{D}(A) \subset L^2(0,1) \longrightarrow L^2(0,1)$ by

$$\begin{cases} \mathcal{D}(A) = \{ f \in L^2(0,1) : \; x f(x) \in L^2(0,1) \}, \\ (Af)(x) = x f(x) \quad \text{for } f \in \mathcal{D}(A). \end{cases}$$

A is self-adjoint, and its associated form a is given by

$$\begin{cases} Q(a) = \{ f \in L^2(0,1) : \sqrt{x}\, f(x) \in L^2(0,1) \} \\ a(f,g) = \int_0^\infty x f(x) \overline{g(x)}\, dx \quad \text{for } f, g \in Q(a) \end{cases}$$

One can verify that a and A are indeed associated by the one-to-one correspondence discussed above.

Let $T : \mathcal{D}(T) \subset \mathcal{H} \longrightarrow K$ be a closed operator, and define a quadratic form a in $\mathcal{H}$ by

$$\begin{cases} Q(a) = \mathcal{D}(T) , \\ a(u,u) = (Tu, Tu) , \end{cases}$$

This form is closed because T is and it is evidently symmetric and positive. The self-adjoint operator A associated with a is given by

$$\begin{cases} \mathcal{D}(A) = \{ u \in Q(a) : (Tu,Tv) < C\|v\| \ \forall\ v \in Q(a) \text{ and some } C>0 \}, \\ (Au,v) = (Tu,Tv) \quad \forall\ v \in Q(a), \text{ for } u \in \mathcal{D}(A) . \end{cases}$$

The operator T^*T is naturally defined by

$$\begin{cases} \mathcal{D}(T^*T) = \{ u \in \mathcal{D}(T) : Tu \in \mathcal{D}(T^*) \} , \\ (T^*Tu, v) = (Tu, Tv) \ \forall\ v \in \mathcal{D}(T), \text{ for } u \in \mathcal{D}(T^*T) . \end{cases}$$

It is evident that $\mathcal{D}(A) = \mathcal{D}(T^*T)$ and that the actions of the two operators coincide on this common domain.

If $f \in \mathcal{H}$, then the following problems are equivalent:

(P) Find $u \in \mathcal{D}(T)$ such that

$$(Tu, Tv) - \lambda(u,v) = (f,v) \quad \text{for all } v \in \mathcal{D}(T)$$

(P') Find $u \in \mathcal{D}(T^*T)$ such that

$$(T^*T - \lambda) u = f$$

OR, more generally,

(P) Find $u \in Q(a)$ s.th.
$a(u,v) - \lambda(u,v) = (f,v) \ \forall\ v \in Q(v)$

(P') Find $u \in \mathcal{D}(A)$ s.th.
$(A - \lambda) u = f$

Let us put problems (1,1') – (4,4') into the framework of (P, P').

In each of them, we take $\mathcal{H} = L^2(\Omega, \varepsilon)$ and $K = (L^2(\Omega,\sigma))^n$

and replace f in (1)-(4) with $\varepsilon^{-1} f$ in (P).

<u>Problem 1</u> $\quad \mathcal{D}(T_0) = H_0^1(\Omega)$, $\quad T_0 u = \nabla u$ for $u \in H_0^1(\Omega)$

Weak form: Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_K - \lambda (u,v)_{\mathcal{H}} = (\varepsilon^{-1} f, v)_{\mathcal{H}} \quad \forall \quad v \in H_0^1(\Omega)$$

Operator form: $\mathcal{D}(T_0^* T_0) = \{ u \in H_0^1(\Omega) : \nabla u \in \mathcal{D}(T_0^*) \}$

$$= \{ u : u \in L^2,\ \nabla u \in L^2,\ \underbrace{u|_{\partial\Omega} = 0}_{\text{(Dirichlet)}},\ -\varepsilon^{-1} \nabla \cdot \sigma \nabla u \in L^2 \},$$

$$T_0^* T_0 u = -\varepsilon^{-1} \nabla \cdot \sigma \nabla u \quad \text{for } u \in \mathcal{D}(T_0^* T_0)$$

Find $u \in \mathcal{D}(T_0^* T_0)$ s.th. $\quad -\varepsilon^{-1} \nabla \cdot \sigma \nabla u - \lambda u = \varepsilon^{-1} f$

<u>Problem 2</u> $\quad \mathcal{D}(T) = H^1(\Omega)$, $\quad Tu = \nabla u$ for $u \in H^1(\Omega)$

Weak form: Find $u \in H^1(\Omega)$ such that

$$(\nabla u, \nabla v)_K - \lambda (u,v)_{\mathcal{H}} = (\varepsilon^{-1} f, v)_{\mathcal{H}} \quad \forall \quad v \in H^1(\Omega)$$

Operator form: $\mathcal{D}(T^* T) = \{ u \in H^1(\Omega) : \nabla u \in \mathcal{D}(T^*) \}$

Now, $F \in \mathcal{D}(T^*)$ if $\sigma F \in \mathcal{D}(\nabla \cdot)$ and

$$0 = \int_\Omega \sigma F \cdot \nabla \bar{v} + \int_\Omega \varepsilon (\varepsilon^{-1} \nabla \cdot \sigma F) \bar{v} = \int_{\partial\Omega} (\sigma F \cdot n) \bar{v} \quad \forall \ v \in H^1(\Omega),$$

— that is, $\sigma F \cdot n = 0$ on $\partial\Omega$.

so $\mathcal{D}(T^* T) = \{ u : u \in L^2,\ \nabla u \in L^2,\ -\varepsilon^{-1} \nabla \cdot \sigma \nabla u \in L^2,\ \underbrace{\sigma \nabla u \cdot n = 0}_{\text{(Neumann)}} \}$

Find $u \in \mathcal{D}(T^* T)$ s.th. $\quad \underbrace{-\varepsilon^{-1} \nabla \cdot \sigma \nabla u}_{T^* T u} - \lambda u = \varepsilon^{-1} f$

<u>Problem 3</u> $Q(a) = H^1(\Omega)$, $a(u,v) = \int_\Omega \sigma \nabla u, \nabla \bar{v} + \int_{\partial\Omega} h u \bar{v}$, in $\mathcal{H}$.

Suppose $u \in \mathcal{D}(A)$. Then Au is the unique L^2 function such that

(*) $$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \int_{\partial\Omega} h u \bar{v} - \int_\Omega \varepsilon (Au) \bar{v} = 0 \quad \forall \; v \in H^1(\Omega).$$

Weak form: Find $u \in Q(a)$ s.th. $a(u,v) - \lambda(u,v) = (\varepsilon^{-1} f, v)$ $\forall \; v \in Q(a)$

In particular, this holds for all $v \in H^1_0(\Omega)$, that is,

$$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} - \int_\Omega \varepsilon (Au) \bar{v} = 0 \quad \forall \; v \in H^1_0(\Omega).$$

This means that $Au = T_0^* u = -\varepsilon^{-1} \nabla \cdot \sigma \nabla u$. Now, for each $v \in H^1(\Omega)$, we must have

$$\int_{\partial\Omega} (\sigma \nabla u \cdot n) \bar{v} + \int_{\partial\Omega} h u \bar{v} = \int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \int_{\partial\Omega} h u \bar{v} + \int_\Omega (\nabla \cdot \sigma \nabla u) \bar{v} = 0.$$

Since the trace map from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is surjective, we have

$$\int_{\partial\Omega} (\sigma \nabla u \cdot n - h u) \phi = 0 \quad \forall \; \phi \in H^{1/2}(\partial\Omega),$$

and thus $\sigma \nabla u \cdot n - h u = 0$ in $H^{-1/2}(\partial\Omega)$. Thus,

$$\mathcal{D}(A) = \{ u \in Q(a) \; ; \; Q(a) \to \mathbb{C} :: v \mapsto a(u,v) \text{ is bdd in } L^2 \}$$
$$= \{ u : u \in L^2, \nabla u \in L^2, -\varepsilon^{-1} \nabla \cdot \sigma \nabla u \in L^2, \underbrace{\boxed{\sigma \nabla u \cdot n = h u}}_{\text{(Robin)}} \}$$

So the operator form of the <u>weak-form PDE (P)</u> is (P'):
in terms of a p. 54

$$\begin{cases} \text{Find } u \in \mathcal{D}(A) \text{ s.th.} \\ -\varepsilon^{-1} \nabla \cdot \sigma \nabla u - \lambda u = \varepsilon^{-1} f \end{cases}$$

Problem 4 $\quad \mathcal{D}(T_{per}) = H^1_{per}(\Omega)$, $\quad T_{per}u = \nabla u$ for $u \in H^1_{per}(\Omega)$

Weak form: Find $u \in H^1_{per}(\Omega)$ such that

$$(\nabla u, \nabla \bar{v})_K - \lambda (u,v)_{\mathcal{H}} = (\varepsilon^{-1} f, v)_{\mathcal{H}} \quad \forall \quad v \in H^1_{per}(\Omega).$$

Operator form: $\mathcal{D}(T^*_{per}T_{per}) = \{ u \in H^1_{per}(\Omega) : \nabla u \in \mathcal{D}(T^*_{per}) \}$

Now, $F \in \mathcal{D}(T^*_{per})$ if $\sigma F \in \mathcal{D}(\nabla\cdot)$ and

$$0 = \int_\Omega \sigma F \cdot \nabla \bar{v} + \int_\Omega \varepsilon (\varepsilon^{-1} \nabla \cdot \sigma F) \bar{v} = \int_{\partial\Omega} (\sigma F \cdot n) \bar{v} \quad \forall \ v \in H^1_{per}(\Omega),$$

so $\mathcal{D}(T^*_{per}T_{per}) = \{ u : u \in L^2, \nabla u \in L^2, -\varepsilon^{-1} \nabla \cdot \sigma \nabla u \in L^2$, and $\sigma \nabla u \cdot n$ satisfies the periodic condition $\}$

The periodic condition on $G \cdot n$ is that

$$\int_{\partial\Omega} (G \cdot n) \bar{v} = 0 \quad \forall \quad v \in H^1_{per}(\Omega)$$

This means that

$$G \cdot n(x_1, \ldots, 0, \ldots, x_n) = -G \cdot n(x_1, \ldots, 1, \ldots, x_n)$$

for each $i = 1, \ldots, n$ and $(x_j)_{j \neq i} \in [0,1]^{n-1}$.

Thus $\mathcal{D}(T^*_{per} T_{per})$ is the space of functions satisfying the periodic boundary conditions in (4'), but now we understand this rigorously.

Let us investigate the "smallest" and "largest" domains of the operator $-\varepsilon^{-1}\nabla\cdot\sigma\nabla$. We retain the spaces

$\mathcal{H} = L^2(\Omega, \varepsilon)$ and $K = (L^2(\Omega,\sigma))^n$.

Let T_0 be defined as in Problem 1 and T as in Problem 2 (p. 55):

$$\mathcal{D}(T_0) = H_0^1(\Omega) \subset \mathcal{H}, \quad T_0 u = \nabla u$$

$$\mathcal{D}(T) = H^1(\Omega) \subset \mathcal{H}, \quad Tu = \nabla u$$

Since $\mathcal{D}(T_0) \subset \mathcal{D}(T)$, $\mathcal{D}(T_0^*) \supset \mathcal{D}(T^*)$.

The domains of all of the operators $T_0^* T_0$, T^*T, and $T_{per}^* T_{per}$ contain the minimal domain

$$\mathcal{D}(T^* T_0) = \{ u \in H_0^1(\Omega) : \sigma\nabla u \in \mathcal{D}(\nabla_0\cdot) \}$$

$$= \{ u : u \in L^2, \nabla u \in L^2, \nabla\cdot\sigma\nabla u \in L^2, u|_{\partial\Omega} = 0, \partial_n \sigma\nabla u|_{\partial\Omega} = 0 \}$$

and are contained in the maximal domain

$$\mathcal{D}(T_0^* T) = \{ u \in H^1(\Omega) : \sigma\nabla u \in \mathcal{D}(\nabla\cdot) \}$$

$$= \{ u : u \in L^2, \nabla u \in L^2, \nabla\cdot\sigma\nabla u \in L^2 \}.$$

In fact,

- $T^* T_0$ and $T_0^* T$ are adjoints of one another;
- $T^* T_0$ is symmetric;
- $\mathcal{D}(T^* T_0) \subset \mathcal{D}(T_0^* T)$, and $T^* T_0 u = T_0^* T u \quad \forall\, u \in \mathcal{D}(T^* T_0)$;
- $T_0^* T u = -\varepsilon^{-1}\nabla\cdot\sigma\nabla \quad \forall\, u \in \mathcal{D}(T_0^* T)$.

If A is equal to $T_0^* T_0$, $T^* T$, or $T_{per}^* T_{per}$,
and if we set $A_{min} = T^* T_0$ and $A_{max} = T_0^* T$,
we have

$$\mathcal{D}(A_{min}) \subsetneq \mathcal{D}(A) \subsetneq \mathcal{D}(A_{max}) .$$

The operator A is a <u>self-adjoint extension of the symmetric operator A_{min}</u>.

Given that A is positive, we have already seen that $A+I : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is bijective. Let us prove that

$\boxed{(A+I)^{-1} : \mathcal{H} \to \mathcal{H} \text{ is compact.}}$

Consider the chain of maps

$$\begin{array}{ccccccc} L^2(\Omega,\varepsilon) & \longrightarrow & Q(a)^* & \longrightarrow & Q(a) & \hookrightarrow & L^2(\Omega,\varepsilon) \\ f & \underset{\text{bdd}}{\longmapsto} & (f,\cdot) & \underset{\text{bdd}}{\longmapsto} & u & \underset{\text{compact}}{\longmapsto} & u \end{array}$$

where u is such that $a(u,v) + (u,v) = (f,v) \;\; \forall \; v \in Q(a)$, that is, $(A+I)u = f$. The inclusion $Q(a) \hookrightarrow L^2(\Omega,\varepsilon)$ is compact by the Rellich-Kondrachov Theorem because $Q(a) \subset H^1(\Omega)$ and $\|\cdot\|_a$ is equivalent to $\|\cdot\|_{H^1}$ in $Q(a)$.

This composite map is $A+I$ as an operator from L^2 to L^2, and since it is the composition of bounded operators and a compact operator, it is compact.

Eigenvalues

Since $(A+I)^{-1}$ is compact, it has a sequence of eigenvalues $\{\mu_i\}_{i=1}^{\infty}$, repeated according to multiplicity

$$\mu_j \to 0 \quad \text{as} \quad j \to \infty$$
$$\mu_j \neq 0$$
$$\text{mult.}(\mu_j) < \infty$$

The corresponding eigenfunctions $\psi_j \in L^2(\Omega, \varepsilon)$ satisfy

$$(A+I)^{-1}\psi_j = \mu_j \psi_j ,$$

or, equivalently,

$$A\psi_j = (\mu_j^{-1} - 1)\psi_j$$

Set $\lambda_j = \mu_j^{-1} - 1$. Since A is positive, we have $\lambda_j \geq 0$

$$\left[\ \lambda_j(\psi_j, \psi_j) = (\lambda_j\psi_j, \psi_j) = (A\psi_j, \psi_j) \geq 0\ \right], \quad \text{so}$$

$$0 < \mu_j \leq 1 \quad \forall \quad j = 1, 2, \ldots$$

Theorem If $\lambda \neq \lambda_j \ \ \forall \ j = 1, 2, \ldots$, then $A - \lambda I : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is invertible and $(A - \lambda I)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact.

Proof The equation $(A - \lambda)u = f \quad (u \in \mathcal{D}(A),\ f \in \mathcal{H})$ is equivalent to $\quad (A+1)u - (1+\lambda)u = f \iff u - (1+\lambda)(A+1)^{-1}u = (A+1)^{-1}f$
$\iff ((A+1)^{-1} - (1+\lambda)^{-1})u = -(1+\lambda)^{-1}(A+1)^{-1}f$. Since $(A+1)^{-1}$ is compact, $((A+1)^{-1} - (1+\lambda)^{-1})$ has a bounded inverse whenever $(1+\lambda)^{-1} \neq \mu_j$, or whenever $\lambda \neq \mu_j^{-1} - 1 = \lambda_j$ for $j = 1, 2, \ldots$.

Thus, for $\lambda \neq \lambda_j \; \forall j$, $(A-\lambda)u = f$ is equivalent to

$u = -(1+\lambda)^{-1}\left[(A+1)^{-1} - (1+\lambda)^{-1}\right]^{-1}(A+1)^{-1}f$, so we obtain

$$(A-\lambda)^{-1} = -\left[(1+\lambda)(A+1)^{-1} - 1\right]^{-1}(A+1)^{-1},$$

which expresses $(A-\lambda)^{-1}$ as the product of a bounded operator and a compact operator.

Spectral representation

Let use normalize the eigenfunctions so that $\|\psi_j\|_\varepsilon = \int_\Omega \varepsilon |\psi_j|^2 = 1$ and $\int_\Omega \varepsilon \psi_i \bar{\psi}_j = 0$ if $\mu_i = \mu_j$.

The spectral theorem for self-adjoint operators provides the following:

- $\{\psi_j\}_{j=1}^\infty$ is an orthonormal Hilbert-space basis for $\mathcal{H} = L^2(\Omega,\varepsilon)$, that is, $\int_\Omega \varepsilon \psi_i \bar{\psi}_j = \delta_{ij}$, $i,j \in \mathbb{Z}^+$, and, given $f \in \mathcal{H}$, $f = \sum_{j=1}^\infty c_j \psi_j$, where $c_j = \int_\Omega \varepsilon u \bar{\psi}_j$, where the series converges in $L^2(\Omega,\varepsilon)$.
- If $u = \sum_{j=1}^\infty b_j \psi_j \in L^2(\Omega,\varepsilon)$, then $u \in \mathcal{D}(A)$ if and only if $\{\lambda_j b_j\} \in \ell^2$, and, for $u \in \mathcal{D}(A)$,
$$Au = \sum_{j=1}^\infty \lambda_j b_j \psi_j .$$
- For $f = \sum_{j=1}^\infty c_j \psi_j \in L^2(\Omega,\varepsilon)$ and for $\lambda \neq \lambda_j$, $j \in \mathbb{Z}^+$,
$$(A-\lambda)^{-1} f = \sum_{j=1}^\infty \frac{c_j}{\lambda_j - \lambda}\psi_j .$$

Problem 1 in spectral representation, with $\sigma = 1$, $\varepsilon = 1$.

$$\begin{cases} \Delta u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

As we have seen, the rigorous formulation of this problem is

$$\Delta_D u + \lambda u = f,$$

in which the "Dirichlet Laplacian" Δ_D is defined by

$\Delta_D = \nabla \cdot \nabla_0 = -T_0^* T_0$, where $T_0 = \nabla_0$, the gradient operator in $L^2(\Omega)$ with domain $H_0^1(\Omega)$. The domain of Δ_D is

$$\mathcal{D}(\Delta_D) = \left\{ u : u \in L^2, \nabla u \in L^2, \nabla \cdot \nabla u \in L^2, u|_{\partial\Omega} = 0 \right\}$$

(each of these restrictions has a precise definition, as we have discussed).

Let $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\psi_j\}_{j=1}^{\infty}$ be the eigenvalues and corresponding eigenfunctions of $-\Delta_D$; they are called the Dirichlet eigenvalues and eigenfunctions of the Laplacian, and $\lambda_j > 0$, $\lambda_j \to \infty$ as $j \to \infty$. Let f be given in its spectral expansion,

$$f = \sum_{j=1}^{\infty} b_j \psi_j$$

As long as $\lambda \neq \lambda_j \;\forall j$, we obtain

$$u = (\Delta_D + \lambda)^{-1} f = \sum_{j=1}^{\infty} \frac{b_j}{\lambda - \lambda_j} \psi_j .$$

This solution is, in fact, valid for $\lambda = \lambda_j$ as long as $b_i = 0$ for all i such that $\lambda_i = \lambda_j$, that is, if

$$\int_\Omega \varepsilon \bar{\psi} = 0 \quad \text{for all eigenfunctions } \psi \text{ for eval } \lambda_j .$$

This is the solvability condition for $\Delta_D u + \lambda_j u = f$.

Inhomogeneous Dirichlet boundary condition

$$\begin{cases} \Delta u + \lambda u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Any solution $u \in H^1(\Omega)$ can be decomposed according to the Hilbert space decomposition $H^1(\Omega) = H^1_0(\Omega) \oplus H^1_0(\Omega)^\perp$:

$$u = u_0 + \tilde{u},$$

where $u_0 \in H^1_0(\Omega)$ and $\tilde{u} \in H^1_0(\Omega)^\perp$. As we have seen, this means that

$$\begin{cases} \Delta \tilde{u} - \tilde{u} = 0 & \text{in } \Omega \\ \tilde{u} = g & \text{on } \partial\Omega \end{cases}, \quad u_0 = 0 \quad \text{on } \partial\Omega$$

The BVP for u_0 becomes

$$\begin{cases} \Delta u_0 + \lambda u_0 = -(1+\lambda)\tilde{u} + f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega . \end{cases}$$

Thus $u_0 = (\Delta_D + \lambda)^{-1}\left(f - (1+\lambda)\tilde{u}\right)$.

Variational calculus for the inhomogeneous Dirichlet problem.

The solution of the foregoing problem can be viewed as a the minimizer of the "energy" $\mathcal{E}$ subject to $u|_{\partial\Omega} = g$:

$$\mathcal{E}(u) = \int_\Omega \sigma |\nabla u|^2 - \lambda \int_\Omega \varepsilon |u|^2 - \int_\Omega f u ,$$

where we set $\sigma = 1$ and $\varepsilon = 1$. In terms of the L^2 inner product, we have

$$\mathcal{E}(u) = (\nabla u, \nabla u) - \lambda (u,u) - (f,u) .$$

To find critical functions of $\mathcal{E}$ subject to $u|_{\partial\Omega} = g$, we take a variation hv, with $v \in H_0^1(\Omega)$, $h \in \mathbb{R}$, so that $(u+hv)|_{\partial\Omega} = g$; then we take $h \to 0$ in a difference quotient of $\mathcal{E}$:

$$\frac{1}{h}\left(\mathcal{E}(u+hv) - \mathcal{E}(u)\right) = \frac{1}{2}\left\{(\nabla u, \nabla v) + (\nabla v, \nabla u) - \lambda\left[(u,v)+(v,u)\right]\right\} - (f,v)$$
$$+ \; h\left\{(\nabla v, \nabla v) - \lambda (v,v)\right\}$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{h}\left(\mathcal{E}(u+hv) - \mathcal{E}(u)\right) = \operatorname{Re}\left[(\nabla u, \nabla v) - \lambda(u,v)\right] - (f,v) = \frac{\delta \mathcal{E}}{\delta u}(u,v).$$

This is the variational, or Fréchet, derivative of $\mathcal{E}$ with respect to u at u in the direction of v. Setting this equal to 0 $\forall\, v \in H_0^1(\Omega)$ gives the Euler-Lagrange equations for $\mathcal{E}$:

$$(\nabla u, \nabla v) - \lambda (u,v) = (f,v) \quad \forall\; v \in H_0^1(\Omega), \quad u|_{\partial\Omega} = g$$

which we recognize as the problem on p. 63.

Because of the positivity of $(\nabla u, \nabla u)$ and the convexity of the space $H_0^1(\Omega)$ and the quadratic functional $\mathcal{E}$, the soln. of this problem is the unique minimizer of $\mathcal{E}$ in $H^1(\Omega)$ with $u|_{\partial\Omega} = g$.