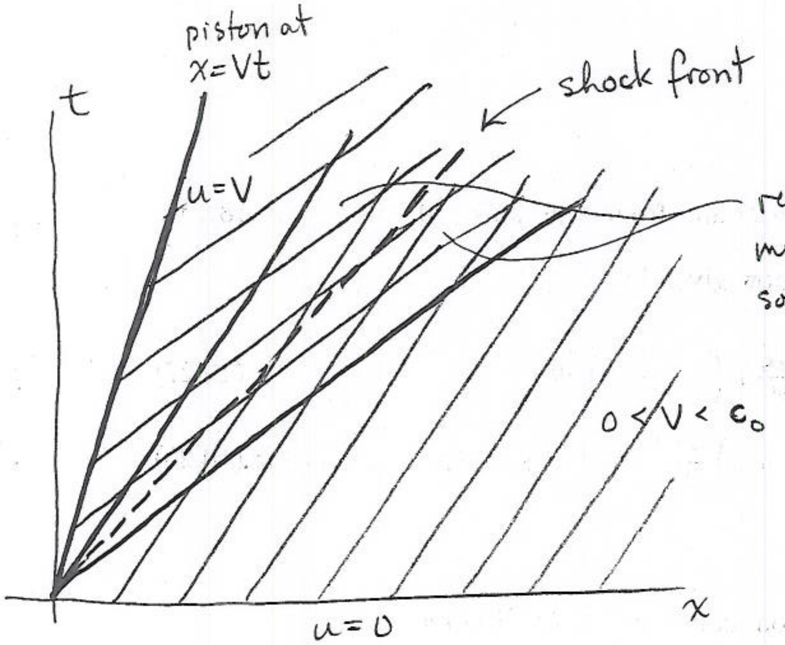
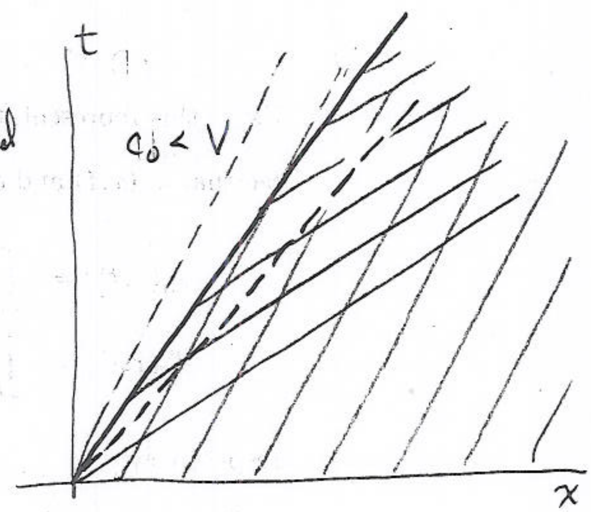




The placement of the shock is determined by the rule of equal areas, as in the simple example ① on page 43.

The viscid Burgers equation: tempering shocks by diffusion.

$$\text{(B)} \qquad u_t + \underbrace{u u_x}_{\substack{\text{advection}\\ \text{of velocity}}} = \underbrace{\nu u_{xx}}_{\substack{\text{diffusion due}\\ \text{to viscocity}}} \quad ; \quad u(x,0) = g(x).$$

The transformation of dependent variable

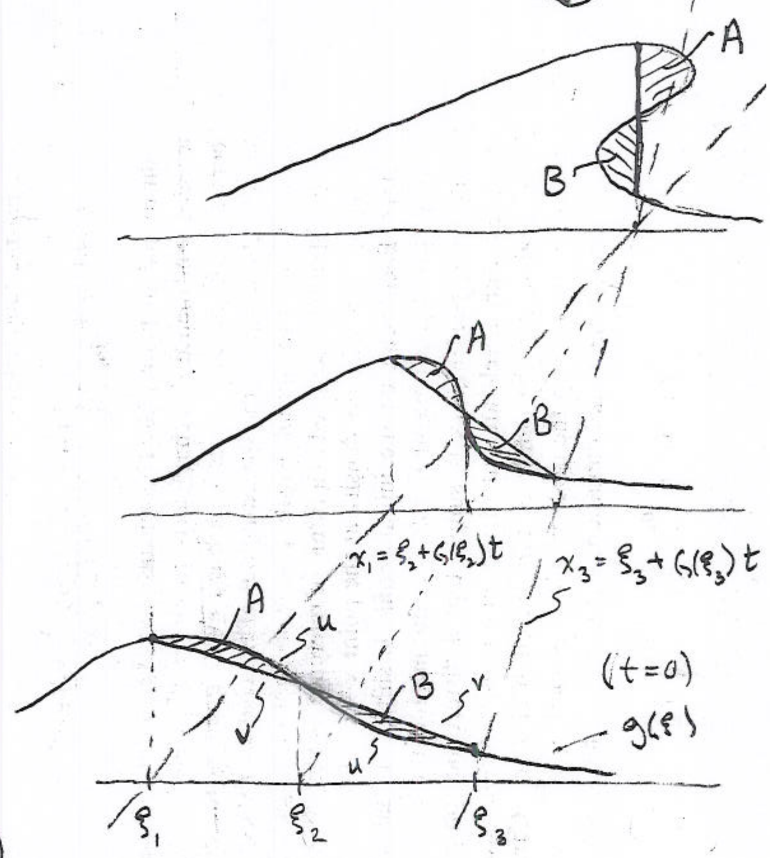
$$u(x,t) = -2\nu (\log \varphi)_x$$
$$\varphi(x,t) = \exp\left(-\frac{1}{2\nu}\int_0^x u(y,t)\,dy\right)$$

converts (B) into the diffusion equation

$$\text{(D)} \qquad \varphi_t = \nu \varphi_{xx}, \qquad \varphi(x,0) = \exp\left(-\frac{1}{2\nu}\int_0^x g(y)\,dy\right).$$

Digression on shocks:

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = g(x) \end{cases} \qquad \left(c(u) = u; \; F(u) = \tfrac{1}{2} u^2 \right)$$



A rule of constant areas:

Fix ξ_1 and ξ_2, and set $x_1(t)$ and $x_2(t)$ equal to the characteristic lines emanating from $\{t=0\}$:

$$x_1 = \xi_1 + G(\xi_1)t, \qquad \left[G(\xi) = g(c(\xi)) = g(\xi) \right]$$
$$x_2 = \xi_2 + G(\xi_2)t.$$

The straight line connecting $(\xi_1, g(\xi_1))$ and $(\xi_2, g(\xi_2))$ evolves by the equation $u_t + u u_x = 0$ into the line connecting $(x_1, g(\xi_1))$ and $(x_2, g(\xi_2))$ because the characteristic curves are straight lines. Call this solution v. We have

$$\frac{d}{dt}\int_{x_1}^{x_2} v(x,t)\,dx = \frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = F(u(x_1)) - F(u(x_2)) + u(x_2)\dot{x}_2 - u(x_1)\dot{x}_1 .$$

In other words, since these derivatives depend on u or v only through their values at the endpoints x_1, x_2, the area A is constant in time. Likewise, the area B is constant in time. This argument is easily extended to multivalued solutions, so the areas A and B are constant in the picture above.

Given that a shock occurs at $x_1 = x_3$, or when the characteristics emanating from ξ_1 and ξ_3 cross, we know from the equal-areas rule that $A = B$. Thus the condition for a shock [for $c(u) = u$] is expressed in terms of the initial condition as the statement that

$$\int_{\xi_1}^{\xi_2} g(\xi)\, d\xi = \frac{1}{2}\left(g(\xi_1) + g(\xi_2)\right)(\xi_2 - \xi_1) .$$

Return to the viscid Burgers equation

The solution of the diffusion equation (D) on p. 46 is obtained from the heat kernel, with the observation that, with $s = \nu t$, the equation $\varphi_t = \nu \varphi_{xx}$ becomes $\varphi_s = \varphi_{xx}$:

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \varphi(y,0) \exp\left(-\frac{|y-x|^2}{4\nu t}\right) dy$$

$$= \frac{1}{\sqrt{4\pi\nu t}} \int \exp\left[-\frac{1}{2\nu}\underbrace{\left(\int_0^y g(z)\,dz + \frac{|y-x|^2}{2t}\right)}_{G(y;x,t)}\right] dy$$

$$\Rightarrow \quad u(x,t) = -2\nu(\log\varphi)_x = \frac{\int \frac{x-y}{t} \exp\left(-\frac{1}{2\nu} G(y;x,t)\right) dy}{\int \exp\left(-\frac{1}{2\nu} G(y;x,t)\right) dy}$$

The numerator and denominator are "Laplace integrals", and their asymptotic behavior can be obtained (see chapter 6 in Bender / Orszag, Advanced Math. Meth. for Sci. and Engr., for example):

If $\mathcal{G}$ attains its minimum at a single point ξ, we must have

$$(*) \qquad \frac{\partial \mathcal{G}}{\partial y}(\xi; x, t) = g(\xi) - \frac{(x-\xi)}{t} = 0 ,$$

and we have

$$\int \frac{x-y}{t} \exp\left(-\frac{1}{2\nu}\mathcal{G}(y; x, t)\right) dy \sim \frac{x-\xi}{t}\sqrt{\frac{4\pi\nu}{G''(\xi)}}\, e^{-G(\xi)/2\nu}$$

$$\int \exp\left(-\frac{1}{2\nu}\mathcal{G}(y; x, t)\right) dy \sim \sqrt{\frac{4\pi\nu}{G''(\xi)}}\, e^{-G(\xi)/2\nu}$$

$$\Rightarrow \quad u(x,t) \sim \frac{x-\xi}{t} =: u_a(x,t) \quad \text{[the asymptotic soln.]}$$

From $(*)$, we obtain

$$u_a(x,t) = g(\xi)$$
$$x = \xi + g(\xi)t ,$$

which is a branch of a solution of $u_t + uu_x = 0$.

If there are two local minima of $\mathcal{G}(y; x, t)$ (in y), say at ξ_1 and ξ_2, then both satisfy $(*)$, and we have

$$x = \xi_1 + g(\xi_1)t = \xi_2 + g(\xi_2)t ,$$

indicating a crossing of characteristics for $u_t + uu_x = 0$.

In this case, the asymptotic solution at (x,t) is either $u_a = g(\xi_1)$ or $u = g(\xi_2)$, depending on whether $\mathcal{G}(\xi_1) \gtrless \mathcal{G}(\xi_2)$.

The switch (or shock as $\nu \to 0$) takes place when $\mathcal{G}(\xi_1) = \mathcal{G}(\xi_2)$;

$$\mathcal{G}(\xi_1; x, t) = \mathcal{G}(\xi_2; x, t)$$

$$\Leftrightarrow \quad \int_0^{\xi_1} g(z)\,dz + \frac{(\xi_1 - x)^2}{2t} = \int_0^{\xi_2} g(z)\,dz + \frac{(\xi_2 - x)^2}{2t}$$

$$\left(\text{Now use } \quad g(\xi_i) - \frac{(x-\xi_i)}{t} = 0 \right)$$

$$\Leftrightarrow \quad \int_{\xi_1}^{\xi_2} g(z)\,dz = \frac{t}{2}\left[\left(\frac{\xi_1 - x}{t}\right)^2 - \left(\frac{\xi_2 - x}{t}\right)^2 \right]$$

$$= \frac{t}{2}\left[g(\xi_1)^2 - g(\xi_2)^2 \right]$$

$$= \frac{1}{2}\left(g(\xi_1) + g(\xi_2) \right) \cdot t \left(g(\xi_1) - g(\xi_2) \right)$$

$$= \frac{1}{2}\left(g(\xi_1) + g(\xi_2) \right) \left(\xi_2 - \xi_1 \right),$$

which is exactly the shock condition on p. 48. !!

Asymptotic Laplace integrals

$$\int_{-\infty}^{\infty} f(t) e^{x\phi(t)}\,dt = \underbrace{E^1_{exp}}_{\text{exponentially small compared to LHS.}} + \int_{-\varepsilon}^{\varepsilon} f(t) e^{x\phi(t)}\,dt$$

$$\int_{-\varepsilon}^{\varepsilon} f(t) e^{x\phi(t)}\,dt = \int_{-\varepsilon}^{\varepsilon} \left(f(0) + t f_1(t)\right) e^{x\left(\phi(0) + \phi''(0)t^2/2 + t^3\phi_3(t)\right)}\,dt \qquad (f(0) \neq 0)$$

$$= f(0) e^{x\phi(0)} \int_{-\varepsilon}^{\varepsilon} \left(1 + t g_1(t)\right) e^{x\phi''(0)t^2/2} \left(1 + x t^3 \psi_3(t)\right) dt$$

$$= f(0) e^{x\phi(0)} \int_{-\varepsilon}^{\varepsilon} e^{x\phi''(0)t^2/2}\,dt + E = F + E$$

$$E = f(0) e^{x\phi(0)} \int_{-\varepsilon}^{\varepsilon} e^{x\phi''(0)t^2/2} \left(t g_1(t) + x t^3 \psi_3(t) + x t^4 g_1(t)\psi_3(t)\right) dt$$

$$|E| \le |f(0)| e^{x\phi(0)} \int_{-\infty}^{\infty} e^{x\phi''(0)t^2/2} \left(|t| M_1 + x|t^3| M_3 + x t^4 M_1 M_3\right) dt$$

$$= |f(0)| e^{x\phi(0)} \int_{-\infty}^{\infty} e^{-s^2} \left(\frac{1}{A\sqrt{x}}|s| M_1 + \frac{1}{A^3\sqrt{x}}|s^3| M_3 + \frac{1}{A^4 x} s^4 M_1 M_3\right) \frac{ds}{A\sqrt{x}}$$

$$= O\left(\tfrac{1}{x}\right)$$

$$s = \sqrt{x} A t \qquad A = \left(-\frac{\phi''(0)}{2}\right)^{1/2}$$

$$\frac{1}{A\sqrt{x}} s = t$$

$$\frac{1}{A^3\sqrt{x}} |s^3| = |t^3| x$$

$$\frac{1}{A^4 x} s^4 = t^4 x$$

This can be strengthened w/ a bit more smoothness that allows the odd powers of t to give 0 integral.

$$F = E^2_{exp} + \underbrace{f(0) e^{x\phi(0)} \int_{-\infty}^{\infty} e^{x\phi''(0)t^2/2}\,dt}$$

$$= f(0) e^{x\phi(0)} \sqrt{\frac{2}{-x\phi''(0)}} \int_{-\infty}^{\infty} e^{-s^2}\,ds = \underbrace{f(0) e^{x\phi(0)} \sqrt{\frac{2\pi}{-x\phi''(0)}}}_{\text{Leading order}} .$$

$$\boxed{\text{So} \int_{-\infty}^{\infty} f(t) e^{x\phi(t)}\,dt \sim f(0) e^{x\phi(0)} \sqrt{\frac{2\pi}{-x\phi''(0)}} .}$$