Second-order linear PDEs

Let Ω be a bounded open set in $\mathbb{R}^n$.

For illustration, let us consider the physical problem of heat conduction in Ω:

$u(x)$ = temperature

$\rho(u(x))$ = heat density

$F(x, u, \nabla u)$ = heat flux

$f(x)$ = heat sources in the bulk, $x \in \Omega$

$g(x)$ = heat source at the boundary, $x \in \partial\Omega$

$h(x)u(x)$ = flux of heat on boundary from "radiation"

The conservation law:

$$\frac{d}{dt}\int_R \rho(u) = -\int_{\partial R} F(u, \nabla u)\cdot n + \int_R f \quad , \quad \forall\, R \subset\subset \Omega$$

Boundary conditions:

$\rho(u) = g$ on $\partial\Omega$ [or $u = g$ on $\partial\Omega$]

(fix the temperature on the boundary)

OR $F(x, u, \nabla u)\cdot n = g - hu$ on $\partial\Omega$

(radiation and external source of heat)

The <u>PDE formulation</u> of the conservation law is

(1′) $$\frac{\partial}{\partial t}\rho(u) + \nabla\cdot F(x,u,\nabla u) = f \quad \text{in } \Omega$$

For acoustic, elastic, or electromagnetic problems, where <u>forces</u> are involved, there is a second time derivative:

(1″) $$\frac{\partial^2}{\partial t^2}\rho(u) + \nabla\cdot F(x,u,\nabla u) = f \quad \text{in } \Omega$$

Space-time-<u>separable solutions</u>:

In (1′) put $u(x,t) = u(x)e^{-\lambda t}$, or in (1″) put $u(x,t) = u(x)e^{-i\omega t}$,

to obtain

(2) $$-\lambda\rho(u) + \nabla\cdot F(x,u,\nabla u) = f \quad \text{in } \Omega \qquad \left(\lambda = \omega^2 \text{ in } (1'')\right)$$

with either of the following boundary conditions:

(2′) $$u = g \quad \text{on } \partial\Omega$$ (<u>Dirichlet</u> bdy cond.)

(2″) $$F(x,u,\nabla u)\cdot n = g - hu \quad \text{on } \partial\Omega$$ ($h=0$, $F = \nabla u$ $\Rightarrow$ <u>Neumann</u> bdy cond.)

Define $C_0^\infty(\bar\Omega) := \{u \in C^\infty(\bar\Omega) : u|_{\partial\Omega} = 0\}$. Be aware that this notation is not standard; but it is appropriate in this context. $C_0^\infty(\bar\Omega)$ could also refer to functions in $C^\infty(\bar\Omega)$ all of whose derivatives vanish on $\partial\Omega$, but we will not need to consider this space.

The "weak formulation" of (1,2a) or (1,2b) is based on the following theorem.

Theorem 1 If $\partial\Omega$ is of class C^1, $F \in C^1(\mathbb{R}^{2n+1})$, $\rho \in C(\mathbb{R})$, $u \in C^2(\bar\Omega)$, $f \in C(\bar\Omega)$, and $h, g \in C^1(\partial\Omega)$, then (2) is equivalent to

(3′) $$-\lambda\int_\Omega \rho(u)\,\phi - \int_\Omega F(x,u,\nabla u)\cdot\nabla\phi = \int_\Omega f\,\phi \qquad \forall\ \phi \in C_0^\infty(\bar\Omega)$$

and the pair (2, 2″) is equivalent to

(3″) $$-\lambda\int_\Omega \rho(u)\phi - \int_\Omega F(x,u,\nabla u)\cdot\nabla\phi - \int_{\partial\Omega} hu\,\phi = \int_\Omega f\,\phi - \int_{\partial\Omega} g\,\phi \quad \forall\ \phi \in C^\infty(\bar\Omega)$$

Proof Multiply (2) by $\phi \in C^\infty(\bar\Omega)$ and integrate over Ω to obtain

(4) $$-\lambda\int_\Omega \rho(u)\phi + \int_\Omega (\nabla\cdot F(u,\nabla u))\,\phi = \int_\Omega f\,\phi\,.$$

The divergence theorem yields

(5) $$\int_{\partial\Omega} F(u,\nabla u)\,\phi\cdot n = \int_\Omega \nabla\cdot(F(u,\nabla u)\,\phi) = \int_\Omega (\nabla\cdot F(u,\nabla u))\,\phi + \int_\Omega F(u,\nabla u)\cdot\nabla\phi\,.$$

Equation (2″) implies $\int_{\partial\Omega}(g-hu)\,\phi = \int_{\partial\Omega} F(u,\nabla u)\,\phi\cdot n$, and using this and (5) to replace the second term of (4), we obtain (3″). Assuming only (2) but not (2″), let ϕ be in $C_0^\infty(\bar\Omega)$ so that (5) yields

(6) $$\int_{\Omega} \left(\nabla \cdot F(u,\nabla u)\right)\phi = -\int_{\Omega} F(u,\nabla u)\cdot\nabla\phi \qquad (\phi \in C_0^\infty(\bar{\Omega})),$$

and substitution of this into (4) gives (3′).

Conversely, assume (3′) and let $x \in \Omega$ be given. For sufficiently small $\varepsilon > 0$, $\mu_\varepsilon(\cdot - x) \in C_c^\infty(\Omega)$ [μ_ε as defined before].
Equations (6) and (3′) imply (4) with $\phi = \mu_\varepsilon(\cdot - x)$. Taking $\varepsilon \to 0$ yields the limiting equality (3). Now assume (3″).
If $\phi \in C_c^\infty(\Omega)$, (3″) reduces to (3′) so that (3) holds, as we have already demonstrated. For $\phi \in C^\infty(\bar{\Omega})$, (3″) and (5) together yield

$$-\lambda\int_{\Omega} p(u)\phi + \int_{\Omega} \nabla F(u,\nabla u)\phi - \int_{\partial\Omega} F(u,\nabla u)\phi\cdot n - \int_{\partial\Omega} hu\,\phi = \int_{\Omega} f\phi - \int_{\partial\Omega} g\phi .$$

By using equation (3), this simplifies to

$$\int_{\partial\Omega} \left(F(u,\nabla u)\cdot n + hu - g\right)\phi = 0 .$$

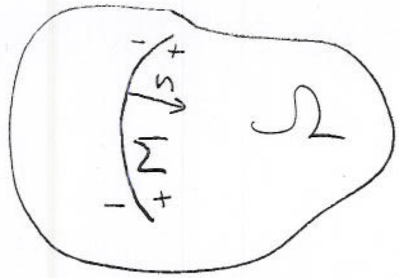
An argument similar to that used for $x \in \Omega$ shows that, for $x \in \partial\Omega$, one can use C^1 functions γ_ε^x on $\partial\Omega$ that converge to $\delta_x(\partial\Omega)$, in the sense that $\int_{\partial\Omega} \gamma_\varepsilon^x k \to k(x)$ as $\varepsilon \to 0$ for all continuous functions k on $\partial\Omega$, to obtain (2″).

Theorem 1 essentially says that, whenever (2) or (2, 2″) are satisfied in the classical sense, the equations (3′) or (3″) are equivalent — but (3′) and (3″) have meaning for a much wider class of functions, for which much less regularity is required. Thus, we call (3′) and (3″) the <u>weak formulation</u> of (2) and (2, 2″); these formulations allow a broader interpretation of the PDEs.

<u>Remark</u> We could have used $C_c^\infty(\Omega)$ instead of $C_0^\infty(\bar{\Omega})$; the latter will be more convenient for us. If we had instead considered an equation of higher order in which we needed to transfer more derivatives to the test function ϕ, $C_0^\infty(\bar{\Omega})$ would not suffice. Thus for the most general theory, $C_c^\infty(\Omega)$ should be used in this step.

Of the many implications of (3′), here is an important one.

Suppose the vector field $G(x)$ is of class C^1 in $\Omega\setminus\Sigma$ and is continuous up to Σ (a hypersurface in Ω) from the + and − sides (see picture), with limiting values G_+ and G_-.



Consider the condition

$$\int_\Omega G\cdot\nabla\phi + \int_\Omega f\,\phi = 0 \quad \forall\ \phi \in C_c^\infty(\Omega).$$

The divergence theorem yields, for $\phi \in C_c^\infty(\Omega)$,

$$(7) \qquad -\int_\Omega (\nabla\cdot\ )\phi - \int_\Sigma (G_+ - G_-)\phi + \int_\Omega f\,\phi = 0.$$

For all $\phi \in C_c^\infty(\Omega\setminus\Sigma)$, the middle term vanishes and we obtain $-\int_\Omega (\nabla\cdot G)\phi + \int_\Omega f\,\phi = 0$. This implies

$$\boxed{\nabla\cdot G = f \quad \text{in } \Omega\setminus\Sigma.}$$

Equation (7) now becomes $\int_\Sigma (G_+ - G_-)\phi\cdot n = 0 \quad \forall\ \phi \in C_c^\infty(\Omega)$, and thus $G_+\cdot n = G_-\cdot n$ on Σ, that is,

$$\boxed{\text{“the normal component of } \underline{G} \text{ is continuous across } \Sigma\text{”}}$$

This means that the flux of G across Σ is continuous, or there is no <u>sink or source</u> along Σ.

In particular, if $\int_\Omega \sigma \nabla u \cdot \nabla \phi + \int_\Omega f \phi = 0 \quad \forall \ \phi \in C_c^\infty(\Omega)$,

where σ has a jump discontinuity $\sigma_+ - \sigma_-$ across Σ, then

$$\nabla \cdot \sigma \nabla u = f \quad \text{in } \Omega \setminus \Sigma \qquad (\text{if } \sigma \nabla u \in C^1(\Omega \setminus \Sigma))$$

and $\sigma_+ \nabla u_+ = \sigma_- \nabla u_-$, so that $\nabla u_+ = \frac{\sigma_-}{\sigma_+} \nabla u_-$.

Thus, the flux $\sigma \nabla u$ is continuous across Σ, whereas ∇u is not.

If $\int_\Omega \sigma \nabla u \cdot \nabla \phi + \int_\Omega f \phi + \int_\Sigma g \phi = 0 \quad \forall \ \phi \in C_c^\infty(\Omega)$,

then $\begin{cases} \nabla \cdot \sigma \nabla u = f & \text{in } \Omega \setminus \Sigma, \\ \sigma_+ \nabla u_+ - \sigma_- \nabla u_- = g & \end{cases}$

The function g on Σ represents the extent to which the flux into Σ from the $-$ side differs from the flux out of Σ from the $+$ side. Thus Σ is a hypersurface sink or source.

Let us now restrict to the case that

$$\rho(u) = \varepsilon u \quad , \ \varepsilon \text{ a measurable function in } \Omega$$

$$F(u, \nabla u) = -\sigma \nabla u , \quad \sigma \text{ a measurable matrix function in } \Omega$$

Then (3') and (3") become

$$(8') \qquad \int_\Omega \sigma \nabla u \cdot \nabla \phi - \lambda \int \varepsilon u \phi = \int f \phi \qquad \forall \ \phi \in C_0^\infty(\bar{\Omega})$$

$$(8'') \qquad \int_\Omega (\sigma \nabla u \cdot \nabla \phi - \lambda \varepsilon u \phi) - \int_{\partial\Omega} h u \phi = \int_\Omega f \phi - \int_{\partial\Omega} g \phi \quad \forall \ \phi \in C^\infty(\bar{\Omega})$$

There is a natural linear structure associated with (8') and (8"), namely that of $H^1(\Omega)$. Let us assume that σ and ε are positive and bounded from below and above, i.e., $\exists\ \Lambda, E \in \mathbb{R}$, s.th

$$\Lambda^{-1} < \sigma(x) \xi \cdot \xi < \Lambda \qquad \forall \ x \in \Omega \ \forall \ \xi \in \mathbb{R}^n ,$$

$$E^{-1} < \varepsilon(x) < E \qquad \forall \ x \in \Omega .$$

In addition, let us assume that $f \in L^2(\Omega)$, $g \in H^{-1/2}(\partial\Omega)$, and $u \mapsto hu$ is a bounded linear operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$

Then, if we <u>restrict</u> u to live in $H^1(\Omega)$, the space of test functions in (8") can be <u>extended</u> to $H^1(\Omega)$, which is the closure of $C^\infty(\bar{\Omega})$ in the H^1 norm, as we have already seen.

For equation (8'), the test functions can be extended to the closure of $C_0^\infty(\bar{\Omega})$ in the H^1 norm, which is the subspace of $H^1(\Omega)$ that has vanishing trace on $\partial\Omega$. It is denoted by $H_0^1(\Omega) := \text{Null } T_0$.

Remark It can be shown that $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$; because of this, we could have used $C_c^\infty(\Omega)$ instead of $C_0^\infty(\bar{\Omega})$ in (3') and (8').

PDE in the L^2 setting

If we replace ϕ with $\bar{v}$ for $v \in H^1(\Omega)$ [or $H_0^1(\Omega)$] in (8'') [or (8')], we obtain the weak formulation of (3) in the L^2 setting.

Weak PDE formulation of (3)

(9') Find $u \in H^1(\Omega)$ with $T_0(u) = g \in H^{1/2}(\partial\Omega)$ such that

$$a_D(u,v) := \int_\Omega \sigma \nabla u \cdot \nabla \bar{v} - \lambda \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} =: \tilde{f}_1(v) \quad \forall v \in H_0^1(\Omega).$$

(9'') Find $u \in H^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \sigma \nabla u \cdot \nabla \bar{v} - \lambda \int_\Omega \varepsilon u \bar{v} - \int_{\partial\Omega} h u \bar{v} = \int_\Omega f \bar{v} - \int_{\partial\Omega} g \bar{v} =: f_2(v) \quad \forall v \in H^1(\Omega).$$

Problem (9') is the weak form of (2,2') in the L^2 sense (as opposed to L^p, $p \neq 2$), and problem (9'') is the weak form of (2,2'') in the L^2 sense.

Let us verify that this makes sense. Because σ and ε are bounded, we have $|a_0(u,v)| =$

$$\left|\int_\Omega \sigma \nabla u \cdot \nabla \bar{v}\right| \le \|\sigma\|_\infty \|\nabla u\|_{L^2} \|\nabla \bar{v}\|_{L^2} \le \Lambda \|u\|_{H^1} \|v\|_{H^1_0},$$

and, for $f \in L^2(\Omega)$, the RHS also acts conjugate linearly and continuously on $H^1_0(\Omega)$:

$$|\hat{f}_1(v)| \le \left|\int_\Omega f \bar{v}\right| \le \|f\|_{L^2} \|v\|_{L^2} \le \|f\|_{L^2} \|v\|_{H^1}$$

Assuming that the map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) :: w \mapsto hw$ is bounded, and recalling that $H^{-1/2}(\partial\Omega)$ acts as the dual of $H^{1/2}(\partial\Omega)$, we interpret $\int_{\partial\Omega} hu\bar{v}$ as representing the pairing of $hu \in H^{-1/2}(\partial\Omega)$ with $v \in H^{1/2}(\partial\Omega)$, so that

$$\left|\int_{\partial\Omega} hu\bar{v}\right| \le \|hu\|_{H^{-1/2}(\partial\Omega)} \|v\|_{H^{1/2}(\partial\Omega)}$$

$$\le \|h\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \|u\|_{H^{1/2}(\partial\Omega)} \|v\|_{H^{1/2}(\partial\Omega)} \le C \|h\| \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

In addition, if $g \in H^{-1/2}(\partial\Omega)$,

$$\left|\int_{\partial\Omega} g\bar{v}\right| \leq \|g\|_{H^{-1/2}} \|v\|_{H^{1/2}} \leq C\|g\|_{H^{-1/2}} \|v\|_{H^1(\Omega)} .$$

In summary, we obtain

$$|a_D(u,v)| \leq C_1 \|u\|_{H^1} \|v\|_{H^1} ,$$

$$|a(u,v)| \leq C_2 \|u\|_{H^1} \|v\|_{H^1} ,$$

$$|\hat{f}_1(v)| \leq C_3 \|v\|_{H^1} ,$$

$$|\hat{f}_2(v)| \leq C_4 \|v\|_{H^1} .$$

The pairings a_D and a are <u>sesquilinear</u> because they are linear in the first argument and conjugate-linear in the second. Both are bounded in both arguments.

Also, both $\hat{f}_1$ and $\hat{f}_2$ are bounded conjugate-linear functionals.