Preliminary notes on compact operators

Defn If $A : X \to X$ is a bounded linear operator on a Hilbert space X, the adjoint A^* is defined through the relation

$$(Au, v) = (u, A^* v) \quad \forall \; u, v \in X .$$

A^* is uniquely defined by the Riesz lemma.

Defn Let $C : X \to Y$ be a bounded linear operator between normed spaces X and Y. Then C is compact if, for each bounded sequence $\{x_i\}_{i=0}^{\infty}$ from X, the sequence $\{Cx_i\}_{i=0}^{\infty}$ admits a convergent subsequence in Y.

Theorem If $A : Y \to Z$ is bounded, $\hat{A} : W \to X$ is bounded, and $C : X \to Y$ is compact, then AC and $C\hat{A}$ are compact.

Theorem: The Rellich-Kondrachov Theorem (special case)
The inclusion $\iota : H^1(\Omega) \to L^2(\Omega)$ is compact.

Theorem If $C : X \to X$ is a compact operator in a Hilbert space X,

- the set of eigenvalues $\tilde{\sigma}(C)$ of C is finite OR is a sequence $\{\mu_i\}_{i=0}^{\infty}$ s. th. $\mu_i \to 0$ as $i \to \infty$ with eigenvalues repeated according to multiplicity.
- $\tilde{\sigma}(C^*) = \overline{\tilde{\sigma}(C)}$ and C^* is compact;
- $C - \mu$ has a bounded inverse for $\mu \notin \tilde{\sigma}(C) \cup \{0\}$;
- $\mathrm{Ran}(C - I) = [\mathrm{Null}(C^* - I)]^{\perp}$. In particular, $\mathrm{Ran}(C - \mu)$ is closed.

<u>Theorem</u> The Fredholm alternative (a form of it)

If $C : X \to X$ is compact and $\mu \neq 0$, then

(1) If $\mu \notin \tilde{\sigma}(C)^{\cup\{0\}}$, $(C-\mu)u = f$ has a unique solution

(2) If $\mu \in \sigma(C)$, $\mu \neq 0$, then $(C-\mu)u = f$ either

has no solution OR $f \perp \mathrm{Null}(C^* - \bar{\mu})$,

where $\mathrm{Null}(C^* - \mu)$ is finite-dimensional.

Let us turn now to solving the boundary-value problems posed in the L^2 sense by equations (9') and (9'').

Recall that $H_0^1(\Omega) = \mathrm{Null}(T_0) = \{ u \in H^1(\Omega) : T_0(u) = u|_{\partial\Omega} = 0 \}$,

so $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ since T_0 is bounded.

In the fllg, assume $\partial\Omega \subset C^1$, $0 < \Lambda^{-1} < \sigma(x)\xi\cdot\xi < \Lambda \quad \forall x \in \Omega$,

$$0 < \varepsilon_- < \varepsilon(x) < \varepsilon_+ \quad \forall x \in \Omega, \text{ and } -\lambda = \alpha > 0.$$

<u>Problem 1</u> Find $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega)$

and $$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \alpha \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} \quad \forall v \in H_0^1(\Omega)$$

<u>Theorem</u> Problem 1 has a unique solution u, and $\exists\, C > 0$ s.th.

$$\|u\|_{H^1(\Omega)} \leq C \left[\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} \right].$$

To prove this, let us split Problem 1 into to parts:

<u>Problem 1a</u> Find $u \in H_0^1(\Omega)$ s.th.

$$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \alpha \int_\Omega \varepsilon u \bar{v} = \int_\Omega f \bar{v} \quad \forall v \in H_0^1(\Omega)$$

<u>Problem 1b</u> Find $u \in H^1(\Omega)$ s. th. $T_0(u) = g \in H^{1/2}(\partial\Omega)$

and $$\int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \alpha \int_\Omega \varepsilon u \bar{v} = 0 \quad \forall v \in H_0^1(\Omega)$$

Set $a_D(u,v) = \int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \alpha \int \varepsilon u \bar{v}$, $u, v \in H^1(\Omega)$.

First, a_D satisfies the properties of an inner product. Second, by assumption, we have

$$0 < C_1 \|u\|^2_{H^1} < a_D(u,u) < C_2 \|u\|^2_{H^1},$$

so $(a_D(u,u))^{1/2}$ is equivalent to $\|u\|_{H^1}$ as a norm.

Thus, H^1 is complete in the norm $a_D(u,u)^{1/2}$, so it is a Hilbert space with inner product $a_D(u,v)$, which we may denote by $(H^1(\Omega), a_D)$ to emphasize the inner product that differs from the usual one. Also, $(H^1_0(\Omega), a_D)$ is a sub-Hilbert space of (H^1, a_D). Since $|\hat{f}(v)| = \left| \int_\Omega f \bar{v} \right| \leq \|f\|_{L^2} \|v\|_{H^1} \leq \text{const} \cdot a(v,v)^{1/2}$, the right-hand side in Problem 1a is a bounded conjugate-linear functional in $(H^1_0(\Omega), a_D)$. Thus, by the Riesz Lemma, there exists a unique function $u \in H^1_0(\Omega)$ such that $a(u,v) = \hat{f}(v)$ for all $v \in H^1_0(\Omega)$. This proves the unique solvability of Problem 1a.

To deal with Problem 1b, recall that, since (H_0^1, a_0) is a closed subspace of (H^1, a_0), there is a Hilbert-space direct sum

$$H^1(\Omega) = H_0^1(\Omega) \oplus_{a_0} H_0^1(\Omega)^{\perp} \quad ,$$

where $H_0^1(\Omega)^{\perp} = \left\{ u \in H^1(\Omega) : a_0(u,v) = 0 \quad \forall \; v \in H_0^1(\Omega) \right\}$.

Problem 1b can be parsed as the search for $u \in H_0^1(\Omega)^{\perp}$ such that $T_0 u = g$. Since T_0 is surjective, there exists $\tilde{u} \in H^1(\Omega)$ such that $T_0 \tilde{u} = g$. Let $\tilde{u} = u + u_0$, where $u_0 \in H_0^1(\Omega)$ and $u \in H_0^1(\Omega)^{\perp}$. By definition of u_0, we have $T_0 u = g$. Suppose that $T_0 w = g$ with $w \in H_0^1(\Omega)^{\perp}$. Then $T_0(u-w) = 0$ so that $u - w \in H_0^1(\Omega)$. Since $u-w \in H_0^1(\Omega)^{\perp}$ also, we have $u = w$, so that u is unique.

Now, putting the unique solutions of Problems 1a and 1b together as a sum, we obtain a unique solution of Problem 1.

Assume that h is such that $-\int_{\partial\Omega} (h\phi)\bar{\Phi} \geq 0 \quad \forall \quad \phi \in H^{1/2}(\partial\Omega)$.

We may say, for example, that $-h(x) = \beta > 0 \quad \forall \quad x \in \partial\Omega$.

Problem 2 Find $u \in H^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \sigma \nabla u \cdot \nabla \bar{v} + \alpha \int_\Omega \varepsilon u \bar{v} + \beta \int_{\partial\Omega} u \bar{v} = \int_\Omega f \bar{v} - \int_{\partial\Omega} g \bar{v} =: F(v)$$

Theorem Problem 2 has a unique solution u, and

$$\|u\|_{H^1(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right).$$

To prove this, observe again that

$$0 < C_1 \|u\|^2 \leq a(u,u) \leq C_2 \|u\|^2 \qquad \forall \quad u \in H^1(\Omega),$$

and $(H^1(\Omega), a)$ is a Hilbert space. The RHS in Problem 2 is a bounded linear functional on $(H^1(\Omega), a)$: As we have seen,

$$|F(v)| \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right) \|v\|_{H^1(\Omega)}.$$

Thus, by the Riesz lemma, $\exists!\ u \in H^1(\Omega)$ such that

$$a(u,v) = F(v) \quad \forall \ v \in H^1(\Omega),$$

with $\|u\|_{(H^1,a)} = \|F\|_{(H^1,a)^*}$, or

$$\|u\|_{H^1} \leq A_1 \|u\|_{(H^1,a)} \leq A_1 \|F\|_{(H^1,a)^*} = A_1 \sup_{v \neq 0} \frac{|F(v)|}{\|v\|_{(H^1,a)}}$$

$$\leq A_1 \sup \frac{|F(v)|}{C_1^{1/2} \|v\|_{H^1}} = A_2 \sup \frac{|F(v)|}{\|v\|_{H^1}} \leq A_3 \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right).$$

Suppose now that $\lambda \in \mathbb{R}$ is not necessarily negative.

Set
$$\begin{cases} a(u,v) = \int \sigma \nabla u \cdot \nabla \bar{v} + \int \varepsilon u \bar{v} \\ b(u,v) = \int \varepsilon u \bar{v} \end{cases} \qquad \forall\ u,v \in H^1(\Omega)$$

Then Problem 2 becomes

<u>Problem 3</u> Find $u \in H^1(\Omega)$ such that

$$a(u,v) - (1+\lambda)\, b(u,v) = F(v) \qquad \forall\ v \in H^1(\Omega).$$

Again, $a(u,v)$ is an inner product on $H^1(\Omega)$ and (H^1, a) is complete in the norm $a(u,u)^{1/2}$. If $(H^1)^*$ is the conjugate dual of H^1, then, by the Riesz lemma, there exist bounded linear operators

$$\varphi : H^1 \to (H^1)^* :: u \mapsto a(u,\cdot),$$
$$\psi : H^1 \to (H^1)^* :: u \mapsto (u,\cdot),$$

each of which has a <u>bounded inverse</u> [The Riesz Lemma declares a Hilbert-space isomorphism between a Hilbert space and its dual.]

- Define $A = \psi^{-1}\varphi : H^1(\Omega) \to H^1(\Omega)$. This means that

$$a(u,v) = (Au, v) \qquad \forall\ u,v \in H^1(\Omega).$$

A is bounded with a bounded inverse.

- Let $\hat{F} \in H^1(\Omega)$ be such that

$$(\hat{F}, v) = F(v) \quad \forall \; v \in H^1(\Omega).$$

In $L^2(\Omega)$, the inner products $\int_\Omega \varepsilon u \bar{v}$ and $\int_\Omega u\bar{v}$ give rise to equivalent norms because $0 < \varepsilon_- < \varepsilon(x) < \varepsilon_+ \quad \forall \; x \in \Omega$. By the R-K theorem, the inclusion $\iota : H^1(\Omega) \to L^2(\Omega; \varepsilon)$ is compact.

Since $|b(u,v)| = \left| \int \varepsilon u \bar{v} \right| \leq \varepsilon_+ \|u\|_{L^2} \|v\|_{H^1}$, there is a bounded linear operator $\jmath : L^2 \to H^1$ such that

$$b(u,v) = (\jmath u, v) \quad \forall \; u, v \in L^2 \supset H^1$$

Set $B = \jmath \circ \iota : H^1(\Omega) \longrightarrow H^1(\Omega)$, which is compact since ι is compact and $\jmath$ is bounded.

Problem 3 now becomes,

Find $u \in H^1(\Omega)$ such that

$$(Au, v) - \lambda (Bu, v) = (\hat{F}, v) \quad \forall \; v \in H^1(\Omega),$$

or, equivalently,

Find $u \in H^1(\Omega)$ such that

$$Au - \lambda Bu = \hat{F},$$

OR $\boxed{u - \lambda A^{-1} B u = A^{-1}(\hat{F})}$

If $\lambda = 0$, then there is evidently a unique solution $u = A^{-1}(\hat{F})$. Otherwise, we consider

$$(*) \qquad \left(\tfrac{1}{\lambda} - A^{-1}B\right)u = \tfrac{1}{\lambda}A^{-1}(\hat{F}) .$$

Now, $A^{-1}B$ is compact because B is compact and A^{-1} is bounded. Thus, whenever $\frac{1}{\lambda}$ is <u>not</u> in the set of eigenvalues $\{\mu_1, \mu_2, \dots\}$ ($\mu_j \to 0$ as $j \to \infty$) of $A^{-1}B$, there is a <u>unique solution</u>

$$u = \left(\tfrac{1}{\lambda} - A^{-1}B\right)^{-1}\tfrac{1}{\lambda}A^{-1}(\hat{F}) = (1 - \lambda A^{-1}B)^{-1}A^{-1}(\hat{F}) .$$

So, for a countable set of values

$$\lambda = \mu_j^{-1} \to \infty \quad \text{as } j \to \infty,$$

(*) and therefore also Problem 3 either has <u>no solution</u> or an infinitude of solutions

$$u = \mathrm{Null}(1 - \lambda A^{-1}B) + u_{partic} ,$$

where u is a particular solution.