Revisiting grad. and div. in L^2

Ω open bdd in $\mathbb{R}^n$ w/ C^1 boundary $\partial\Omega$.

Recall that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, so also $[C_c^\infty(\Omega)]^n$ is dense in $[L^2(\Omega)]^n$. The gradient operator $f \mapsto \nabla f$ is defined classically (pointwise) on $C_c^1(\Omega)$, and the divergence operator $F \mapsto \nabla \cdot F$ is defined classically on $[C_c^1(\Omega)]^n$.

If $f \in C^\infty(\Omega)$ and $\Phi \in [C_c^\infty(\Omega)]^n$ OR $F \in [C^\infty(\Omega)]^n$ and $\phi \in C_c^\infty(\Omega)$, then

(*1) $$\int_\Omega f \, \nabla \cdot \Phi + \int_\Omega \nabla f \cdot \Phi = 0$$

(*2) OR $$\int_\Omega (\nabla \cdot F)\phi + \int_\Omega F \cdot \nabla \phi = 0$$

This motivates the following definitions:

The gradient operator in $L^2(\Omega)$

Defn A function $f \in L^2(\Omega)$ is in $\mathcal{D}(\nabla)$ if

- the functional $[C_c^\infty(\Omega)]^n \to \mathbb{C} :: \Phi \mapsto \int_\Omega f \, \nabla \cdot \Phi$ is bounded in the L^2-norm,

or, equivalently if

- there exists a vector field $F \in [L^2(\Omega)]^n$ such that $\int_\Omega f \, \nabla \cdot \Phi + \int F \cdot \Phi = 0 \quad \forall \; \Phi \in [C_c^\infty(\Omega)]^n$.

For $f \in \mathcal{D}(\nabla)$, the function F, which is uniquely defined, ← by density of $[C_c^\infty(\Omega)]^n \subset (L^2)^n$ is the weak L^2-gradient of f: $F = \nabla f$,

and it coincides with the classical gradient for $f \in C^\infty(\Omega)$ and with the distributional gradient for $f \in \mathcal{D}(\nabla)$.

The divergence operator in $[L^2(\Omega)]^n$

A vector-valued function $F \in [L^2(\Omega)]^n$ is in $\mathcal{D}(\nabla\cdot)$ if

- the functional $C_c^\infty(\Omega) \to \mathbb{C} :: \phi \mapsto \int_\Omega F\cdot\nabla\phi$ is bounded in the L^2-norm,

or, equivalently, if

- there exists $f \in L^2(\Omega)$ s.th.

$$\int_\Omega f\phi + \int_\Omega F\cdot\nabla\phi = 0 \quad \forall\ \phi \in C_c^\infty(\Omega)$$

For $F \in \mathcal{D}(\nabla\cdot)$, the function f, which is uniquely defined by density of $C_c^\infty(\Omega) \subset L^2(\Omega)$, is the weak L^2-divergence of F and coincides with the distributional divergence: $f = \nabla\cdot F$; f also coincides with the classical divergence for $F \in (C^\infty(\Omega))^n$.

- The gradient operator $\nabla : \mathcal{D}(\nabla) \to [L^2(\Omega)]^n$ is the <u>adjoint</u> of the divergence operator restricted to $[C_c^\infty(\Omega)]^n$.
- The divergence operator $\nabla\cdot : \mathcal{D}(\nabla\cdot) \to L^2(\Omega)$ is the <u>adjoint</u> of the gradient operator restricted to $C_c^\infty(\Omega)$.

Notes on ∇ and $\nabla\cdot$

(1) For $F \in [L^2(\Omega)]^n$, the lin. fcn. al $\phi \mapsto \int_\Omega F\cdot\nabla\phi$ is bounded in the H^1-norm of ϕ, so F has a (distributional) divergence in $(H^1)^*$, but it is not in general represented by an L^2 function.

(2) ∇ and $\nabla\cdot$ are unbounded. To wit:

Let $\psi \in C_c^\infty(\Omega)$ be given, and set $f_m(x) = \psi(x)e^{imK\cdot x}$ $(K\in\mathbb{Z}^n)$

Then $\nabla f_m(x) = (\nabla\psi(x))e^{imK\cdot x} + imK\,\psi(x)\,e^{imK\cdot x}$,

so $\|\nabla f_m\|_{L^2} \geq m|K|\,\|\psi\|_{L^2} - \|\nabla\psi\|_{L^2} \to \infty$ as $m\to\infty$,

whereas $\|f_m\|_{L^2} = \|\psi\|_{L^2}$ $\forall$ m. (Similar for $\nabla\cdot$)

(3) $\mathcal{D}(\nabla)$ is dense in $L^2(\Omega)$, and $\mathcal{D}(\nabla\cdot)$ is dense in $[L^2(\Omega)]^n$.

(4) $\mathcal{D}(\nabla) = H^1(\Omega)$, and the defn. of $\nabla : \mathcal{D}(\nabla) \to [L^2(\Omega)]^n$ coincides with ∇ in $H^1(\Omega)$.

(5) $[H^1(\Omega)]^n \subsetneq \mathcal{D}(\nabla\cdot)$, and $\nabla\cdot$ coincides with its definition in $[H^1(\Omega)]^n$ because $(*2)$ is satisfied for $F \in (H^1(\Omega))^n$.

<u>Example of inequality</u>: $\Omega = [-1,1]^2$, $F(x,y) = \langle x(x^2+y^2)^{-1/4}, -y(x^2+y^2)^{-1/4}\rangle \in \mathcal{D}(\nabla\cdot)$

and $\nabla\cdot F = 0$, but $F \notin (H^1(\Omega))^2$.

(6) The graph norm $\|\cdot\|_\nabla$ in $\mathcal{D}(\nabla) \subset L^2(\Omega)$ is defined through

$\|f\|_\nabla^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$; this is just the usual H^1-norm.

'Tis easy to prove that $\mathcal{D}(\nabla)$ is complete in this norm.

To wit: suppose that $\{f_\ell\}_{\ell=1}^\infty$ is a sequence from $\mathcal{D}(\nabla)$ that is Cauchy in the graph norm of ∇. Then $\{f_\ell\}$ and $\{\nabla f_\ell\}$ are Cauchy sequences in $L^2(\Omega)$ and $(L^2(\Omega))^n$, so $\exists$ $f \in L^2$ and $F \in (L^2(\Omega))^n$ s.th. $f_\ell \to f$ and $\nabla f_\ell \to F$ in the L^2-sense.

Using these limits in the relation

$$\int_\Omega f_\ell \, \nabla\cdot\Phi + \int_\Omega \nabla f_\ell \cdot \Phi = 0 \qquad \forall \; \Phi \in (C_c^\infty(\Omega))^n$$

yields the relation

$$\int_\Omega f \, \nabla\cdot\Phi + \int_\Omega F\cdot\Phi = 0 \qquad \forall \; \Phi \in (C_c^\infty(\Omega))^n ,$$

which, by defn. of ∇, means that $f \in \mathcal{D}(\nabla)$ and $F = \nabla f$.

Thus, $f_\ell \to f$ and $\nabla f_\ell \to \nabla f$ in L^2, so $\|f_\ell - f\|_\nabla \to 0$ as $\ell \to \infty$.

(7) The graph norm $\|\cdot\|_{\nabla\cdot}$ in $\mathcal{D}(\nabla\cdot) \subset (L^2(\Omega))^n$ is defined through

$$\|F\|_{\nabla\cdot}^2 = \|F\|_{L^2}^2 + \|\nabla\cdot F\|_{L^2}^2 .$$

Again, $\mathcal{D}(\nabla\cdot)$ is complete in its graph norm (and is a Hilbert space).

(8) $\mathcal{D}(T) = H^1(\Omega)$ admits a Hilbert-space decomposition

$$H^1(\Omega) = H^1_0(\Omega) + H^1_0(\Omega)^\perp$$

The interpretation of these component spaces is as follows:

$$u \in H^1_0(\Omega) \iff u\big|_{\partial\Omega} = 0 \qquad (\text{in the sense of } T_0)$$

$$u \in H^1_0(\Omega)^\perp \iff \Delta u + u = 0 \qquad (\text{in the weak sense})$$

The latter is seen through the relation

$$\int_\Omega \nabla u \cdot \nabla \bar{v} + \int_\Omega u \bar{v} = 0 \qquad \forall \quad v \in H^1_0(\Omega)$$

(since this holds for all $v \in C^\infty_c(\Omega)$, which is dense in $H^1_0(\Omega)$).

Since $H^1_0(\Omega) = \text{Null}(T_0)$ and T_0 is bounded and surjective, T_0 restricted to $H^1_0(\Omega)^\perp$ is bounded and bijective. It also has a bounded inverse, which we may take to be the right inverse R of T_0 that we defined earlier.

Thus R associates with $g \in H^{1/2}(\partial\Omega)$, the unique function $u \in H^1(\Omega)$ that satisfies the weak form of the boundary-value problem

$$\begin{cases} \Delta u + u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} .$$

(4) Functions in $\mathcal{D}(\nabla\cdot)$ do not in general possess boundary values in $(H^{1/2}(\partial\Omega))^n$, but they do have boundary <u>normal components</u> in $H^{-1/2}(\partial\Omega)$.

The defn. of $F\cdot n|_{\partial\Omega}$ is motivated by the relation

$$(F,u) := \int_\Omega (\nabla\cdot F)u + \int_\Omega F\cdot\nabla u = \int_{\partial\Omega} u\, F\cdot n$$

for u, F of class C^∞. The LHS is a bounded bilinear function in $\mathcal{D}(\nabla\cdot)\times\mathcal{D}(\nabla)$, which we denote by (F,u):

$$|(F,u)| \le \|\nabla\cdot F\|_{L^2}\|u\|_{L^2} + \|F\|_{L^2}\|\nabla u\|_{L^2} \le 2\|F\|_{\nabla\cdot}\|u\|_\nabla .$$

Thus, since $(C^\infty(\bar\Omega))^n$ is dense in $\mathcal{D}(\nabla\cdot)$ and $C_0^\infty(\bar\Omega)$ is dense in $H_0^1(\Omega)\subset H^1(\Omega)$, we find that

$$(F,u) = 0 \quad \forall\ u\in H_0^1(\Omega),\ F\in\mathcal{D}(\nabla\cdot).$$

Because of this, for fixed $F\in\mathcal{D}(\nabla\cdot)$, (F,u) depends only on the boundary values of u in $H^{1/2}(\partial\Omega)$. Recall that, for $g\in H^{1/2}(\partial\Omega)$, $Rg\in H_0^1(\Omega)^\perp$ and R is bounded. Thus

$$|(F,Rg)| \le \|F\|_{\nabla\cdot}\|Rg\|_\nabla \le \text{const.}\|F\|_{\nabla\cdot}\|g\|_{H^{1/2}(\partial\Omega)} .$$

Therefore the map

$$T_1 : F \longmapsto (F, R(\cdot))$$

is a bounded linear operator from $\mathcal{D}(\nabla\cdot)$ to $H^{-1/2}(\partial\Omega)$.

Moreover, for $F \in [C^\infty(\bar{\Omega})]^n$,

$$(T_1 F)(g) = (F, Rg) = \int_{\partial\Omega} g F\cdot n ,$$

and because $[C^\infty(\bar{\Omega})]^n$ is dense in $\mathcal{D}(\nabla\cdot)$, it makes sense to extend this notation to all of $\mathcal{D}(\nabla\cdot)$ and write

$$T_1 F =: F\cdot n \quad , \quad F \in \mathcal{D}(\nabla\cdot)$$

Now, for $u \in H^1(\Omega)$, write $u = u_0 + \tilde{u}$, with $u_0 \in H_0^1(\Omega)$ and $\tilde{u} \in H_0^1(\Omega)^\perp$. Then

$$\begin{aligned}(F,u) &= (F,u_0) + (F,\tilde{u}) = 0 + (F,\tilde{u}) \\ &= (F, RT_0\tilde{u}) = (F, RT_0 u) = (T_1F)(T_0 u) \\ &= \int_{\partial\Omega} (T_0 u) F\cdot n\end{aligned}$$

In other words,

$$\int_\Omega (\nabla\cdot F) u + \int_\Omega F\cdot \nabla u = \int_{\partial\Omega} u F\cdot n ,$$

in which the trace $T_0 u$ of u is simply denoted by u on $\partial\Omega$.