Problem Set 1

1. Advection equations with nonconstant advection rates

in the absence of external sources are naturally written as

conservation laws:

\[
\begin{align*}
    u_t + \nabla_x \cdot (\vec{c}(x) u) &= 0 & \forall x \in \mathbb{R}^n \\
    u(x,0) &= f(x) \\
\end{align*}
\]

in which \( f \) is the initial concentration of stuff.
This equation can be rewritten as

\[
    u_t + \vec{c}(x) \cdot \nabla u = - (\nabla \vec{c}(x)) u
\]

Find the explicit solution of the 1D problem

\[
\begin{align*}
    u_t + (x^2 u)_x &= 0 \\
    u(x,0) &= f(x) \\
\end{align*}
\]

Use the method of characteristic curves by finding the curves
\( x(t) \) with \( x(0) = y \) such that the left-hand-side of the PDE in the form (1) is equal to \( \frac{du}{dt} \) along the curve. Then solve for \( u \) as a function of \( y \) and \( t \), and ultimately for \( u \) as a function of \( x \) and \( t \). [It is useful to check your result!]

Discuss the domain of validity of the solution.
2. Find the dispersion relation \( \omega = \omega(k) \) that relates frequency to wavenumber of solutions \( e^{ikx - \omega t} \) to the linearized KdV equation

\[
U_t + cU_x + \varepsilon U_{xxx} = 0
\]

and determine the phase velocity and group velocity of wave packets as a function of the wavenumber \( k \).

3. Prove that the solution of the 1D heat equation

\[
U_t = \Delta u
\]

\[
U(x,0) = f(x)
\]

is even in \( x \) for all \( t \) whenever \( f \) is even and that the solution is odd in \( x \) for all \( t \) whenever \( f \) is odd.

Find the fundamental solution of the heat equation on the half-line \([0, \infty)\) with the stipulation \( u(0,t) = 0 \) \( \forall \ t > 0 \), that is, find the function

\[
\Phi_0(x,y,t), \quad x, y > 0, \ t > 0
\]

such that the solution of the initial-boundary-value problem

\[
U_t = \Delta u \quad x > 0, \ t > 0
\]

\[
U(x,0) = f(x) \quad x > 0
\]

\[
U(0,t) = 0 \quad t > 0
\]

is equal to

\[
U(x,t) = \int_0^\infty \Phi(x,y,t) f(y) \, dy
\]