Problem Set 3

1. Define $$\Phi(x) := \begin{cases} -|x| & , \quad n = 1 \\ -\frac{1}{2\pi}\log|x| & , \quad n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & , \quad n \geq 3 \end{cases}$$

Prove that $-\Delta \Phi(x) = \delta_0(x)$ in the rigorous distributional sense.

2. Define $$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & , \quad t > 0 \\ 0 & , \quad t \leq 0 \end{cases}$$

(a) Prove that $\Phi(x,t)$, as a distribution on $\mathbb{R}^n$ with parameter t, converges to $\delta_0(x)$ as $t \to 0^+$.

(b) Prove that $(\partial_t - \Delta)\,\Phi(x,t) = \delta_0(x,t)$, where $\delta_0(x,t)$ is the Dirac-delta distribution in $\mathbb{R}^{n+1}$.

3. Prove that $$(\Delta + k^2)\,\frac{e^{ikr}}{4\pi r} = -\delta_0(x) \quad \text{in } \mathbb{R}^3,$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.