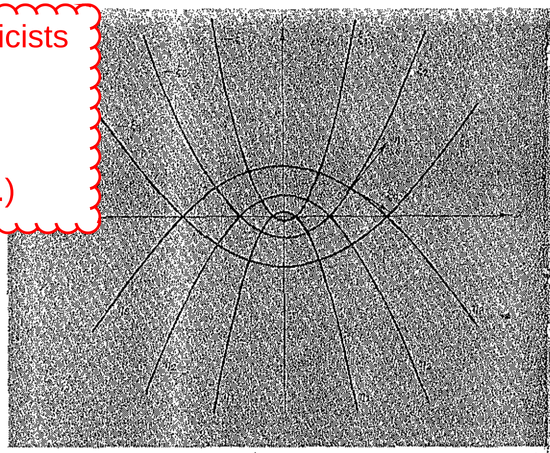


CHAPTER 2

COORDINATE SYSTEMS

In Chapter 1 we restricted ourselves almost completely to cartesian coordinate systems. A cartesian coordinate system offers the unique advantage that all three unit vectors, **i**, **j**, and **k**, are constant. We did introduce the radial distance **r** but even this was treated as a function of x, y, and z. Unfortunately, not all physical problems are well adapted to solution in cartesian coordinates. For instance, if we have a central force problem, $\mathbf{F} = \mathbf{r}_0 F(r)$, such as gravitational or electrostatic force, cartesian coordinates may be unusually inappropriate. Such a problem literally screams for the use of a coordinate system in which the radial distance is taken to be one of the coordinates, that is, spherical polar coordinates.

The point is that the coordinate system should be chosen to fit the problem, to exploit any constraint or symmetry present in it. Then, hopefully, it will be more readily soluble than if we had forced it into a cartesian framework. Quite often "more readily soluble" will mean that we have a partial differential equation that can be split into separate ordinary differential equations, often in "standard form" in the new coordinate system. This technique, the separation of variables, is discussed in Section 2.5.

We are primarily interested in coordinates in which the equation

$$\nabla^2\psi + k^2\psi = 0 \tag{2.1}$$

is separable. Equation 2.1 is much more general than it may appear. If

$k^2 = 0$	Eq. 2.1 $\rightarrow$	Laplace's equation,
$k^2 = (+)$ constant		Helmholtz' equation,
$k^2 = (-)$ constant		Diffusion equation (space part),
$k^2 =$ constant $\times$ kinetic energy		Schrödinger wave equation.

It has been shown (L. P. Eisenhart, *Phys. Rev.* **45**, 427 (1934)) that there are eleven coordinate systems in which Eq. 2.1 is separable, all of which can be considered particular cases of the confocal ellipsoidal system. In addition, we shall touch briefly on three other systems that are useful in solving Laplace's equation.

Naturally there is a price that must be paid for the use of a noncartesian coordinate system. We have not yet written expressions for gradient, divergence, or curl in any of the noncartesian coordinate systems. Such expressions are developed in very general form in Section 2.2. First we must develop a system of curvilinear coordinates, a general system that may be specialized to any of the fourteen particular systems of interest.

2.1 Curvilinear Coordinates

In cartesian coordinates we deal with three mutually perpendicular families of planes: $x =$ constant, $y =$ constant, and $z =$ constant. Imagine that we superimpose on this system three other families of surfaces. The surfaces of any one family need not be parallel to each other and they need not be planes. The three new families of surfaces need not be mutually perpendicular, but for simplicity we shall quickly impose this condition (Eq. 2.7). We may describe any point (x, y, z) as the intersection of three planes in cartesian coordinates or as the intersection of the three surfaces which form our new, curvilinear coordinates. Describing the curvilinear coordinate surfaces by $q_1 =$ constant, $q_2 =$ constant, $q_3 =$ constant, we may identify our point by (q_1, q_2, q_3) as well as by (x, y, z). This means that in principle we may write

$$\begin{aligned} x &= x(q_1, q_2, q_3), \\ y &= y(q_1, q_2, q_3), \\ z &= z(q_1, q_2, q_3), \end{aligned} \tag{2.2}$$

specifying x, y, z in terms of the q's and the inverse relations,

$$\begin{aligned} q_1 &= q_1(x, y, z), \\ q_2 &= q_2(x, y, z), \\ q_3 &= q_3(x, y, z). \end{aligned} \tag{2.3}$$

With each family of surface $q_i =$ constant, we can associate a unit vector $\mathbf{a}_i$ normal to the surface $q_i =$ constant and in the direction of increasing q_i.

The square of the distance between two neighboring points is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{ij} h_{ij}^2\, dq_i\, dq_j. \tag{2.4}$$

The coefficients h_{ij}^2, which we now proceed to investigate, may be viewed as specifying the nature of the coordinate system (q_1, q_2, q_3). Collectively, these coefficients are referred to as the *metric*.

The first step in the determination of h_{ij}^2 is the partial differentiation of Eq. 2.2 which yields

$$dx = \frac{\partial x}{\partial q_1}\, dq_1 + \frac{\partial x}{\partial q_2}\, dq_2 + \frac{\partial x}{\partial q_3}\, dq_3, \tag{2.5}$$

and similarly for dy and dz. Squaring and substituting into Eq. 2.4, we have

$$h_{ij}^2 = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}. \tag{2.6}$$

At this point we limit ourselves to orthogonal (mutually perpendicular surfaces) coordinate systems which means (cf. Exercise 2.1.1)

$$h_{ij} = 0, \qquad i \neq j. \tag{2.7}$$

Now, to simplify the notation, we write $h_{ii} = h_i$ so that

$$ds^2 = (h_1\, dq_1)^2 + (h_2\, dq_2)^2 + (h_3\, dq_3)^2. \tag{2.8}$$

Our specific coordinate systems are described in subsequent sections by specifying these scale factors h_1, h_2, and h_3. Conversely, the scale factors may be conveniently identified by the relation

$$ds_i = h_i\, dq_i \tag{2.9}$$

for any given dq_i, holding the other q's constant. Note that the three curvilinear coordinates q_1, q_2, q_3 need not be lengths. The scale factors h_i may depend on the q's and they may have dimensions. The *product* $h_i\, dq_i$ must have dimensions of length.

From Eq. 2.9 we may immediately develop the area and volume elements

$$d\sigma_{ij} = ds_i\, ds_j = h_i h_j\, dq_i\, dq_j \tag{2.10}$$

and

$$d\tau = ds_1\, ds_2\, ds_3 = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3. \tag{2.11}$$

The expressions in Eqs. 2.10 and 2.11 agree, of course, with the results of using the transformation equations, Eq. 2.2, and Jacobians.

EXERCISES

2.1.1 Show that limiting our attention to orthogonal coordinate systems implies that $h_{ij} = 0$ for $i \neq j$ (Eq. 2.7).

2.1.2 In the spherical polar coordinate system $q_1 = r, q_2 = \theta, q_3 = \varphi$. The transformation equations corresponding to Eq. 2.2 are

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta.$$

(a) Calculate the spherical polar coordinate scale factors: h_r, h_θ, and h_φ.
(b) Check your calculated scale factors by the relation $ds_i = h_i\, dq_i$.

2.1.3 The u-, v-, z-coordinate system frequently used in electrostatics and in hydrodynamics is defined by

$$xy = u,$$
$$x^2 - y^2 = v,$$
$$z = z.$$

This u-, v-, z-system is orthogonal.
(a) In words, describe briefly the nature of each of the three families of coordinate surfaces.
(b) Sketch the system in the xy-plane showing the intersections of surfaces of constant u and surfaces of constant v with the xy-plane.
(c) Indicate the directions of the unit vector u_0 and v_0 in all four quadrants.
(d) Finally, is this u-, v-, z-system right-handed or left-handed?

2.1.4 A *two* dimensional system is described by the coordinates q_1 and q_2. Show that the Jacobian

$$J\left(\frac{x, y}{q_1, q_2}\right) = h_1 h_2$$

in agreement with Eq. 2.10.

2.2 Differential Vector Operations

The starting point for developing the gradient, divergence, and curl operators in curvilinear coordinates is our interpretation of the gradient as the vector having the magnitude and direction of the maximum space rate of change (cf. Section 1.6). From this interpretation the component of $\nabla\psi(q_1, q_2, q_3)$ in the direction normal to the family of surfaces q_1 = constant is given by[1]

$$\nabla\psi\,|_1 = \frac{\partial\psi}{\partial s_1} = \frac{\partial\psi}{h_1\,\partial q_1}, \tag{2.12}$$

since this is the rate of change of ψ for varying q_1, holding q_2 and q_3 fixed. The quantity ds_1 is a differential length in the direction of increasing q_1 (cf. Eq. 2.9). In Section 2.1 we introduced a unit vector $\mathbf{a}_1$ to indicate this direction. By repeating Eq. 2.12 for q_2 and again for q_3 and adding vectorially the gradient becomes

$$\nabla\psi(q_1, q_2, q_3) = \mathbf{a}_1\frac{\partial\psi}{\partial s_1} + \mathbf{a}_2\frac{\partial\psi}{\partial s_2} + \mathbf{a}_3\frac{\partial\psi}{\partial s_3}$$
$$= \mathbf{a}_1\frac{\partial\psi}{h_1\,\partial q_1} + \mathbf{a}_2\frac{\partial\psi}{h_2\,\partial q_2} + \mathbf{a}_3\frac{\partial\psi}{h_3\,\partial q_3} \tag{2.13}$$

The divergence operator may be obtained from the second definition (Eq. 1.91) of Chapter 1 or equivalently from Gauss's theorem, Section 1.11. Let us use Eq. 1.91:

$$\nabla\cdot\mathbf{V}(q_1, q_2, q_3) = \lim_{\int d\tau\to 0}\frac{\int\mathbf{V}\cdot d\boldsymbol{\sigma}}{\int d\tau} \tag{2.14}$$

with a differential volume $h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$. Note that the positive directions have been chosen so that (q_1, q_2, q_3) or $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ form a right-handed set.

[1] Here the use of φ to label a function is avoided because it is conventional to use this symbol to denote an azimuthal coordinate.

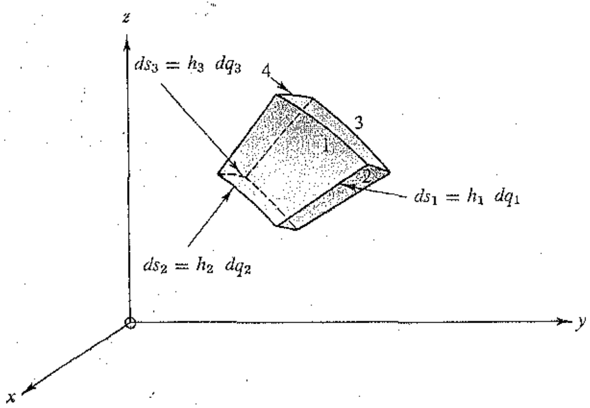


FIG. 2.1 Curvilinear volume element

The area integral for the two faces $q_1 =$ constant is given by

$$\left[V_1 h_2 h_3 + \frac{\partial}{\partial q_1}(V_1 h_2 h_3)\, dq_1\right] dq_2\, dq_3 - V_1 h_2 h_3\, dq_2\, dq_3$$

$$= \frac{\partial}{\partial q_1}(V_1 h_2 h_3)\, dq_1\, dq_2\, dq_3 \tag{2.15}$$

exactly as in Sections 1.7 and 1.10.[1] Adding in the similar results for the other two pairs of surfaces, we obtain

$$\int \mathbf{V}(q_1, q_2, q_3) \cdot d\boldsymbol{\sigma}$$

$$= \left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right] dq_1\, dq_2\, dq_3. \tag{2.16}$$

Division by our differential volume (Eq. 2.14) yields

$$\nabla \cdot \mathbf{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right]. \tag{2.17}$$

In Eq. 2.17 V_i is the component of $\mathbf{V}$ in the $\mathbf{a}_i$-direction, increasing q_i, that is, $V_i = \mathbf{a}_i \cdot \mathbf{V}$.

We may obtain the Laplacian by combining Eqs. 2.13 and 2.17, using $\mathbf{V} = \nabla\psi(q_1, q_2, q_3)$. This leads to

$$\nabla \cdot \nabla\psi(q_1, q_2, q_3)$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right]. \tag{2.18a}$$

Finally, to develop $\nabla \times \mathbf{V}$, let us apply Stokes's theorem (Section 1.12) and, as

[1] Since we take the limit $dq_1, dq_2, dq_3 \to 0$, the second and higher order derivatives will drop out.

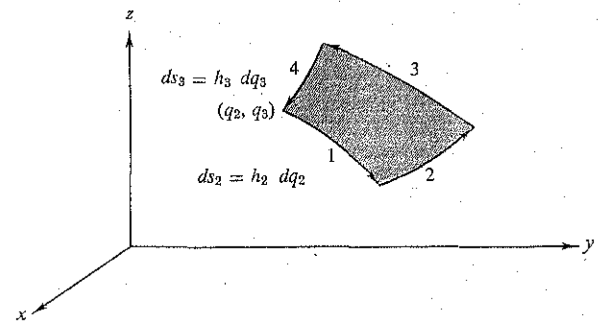


FIG. 2.2 Curvilinear surface element

with the divergence, take the limit as the surface area becomes vanishingly small. Working on one component at a time, we consider a differential surface element in the curvilinear surface $q_1 =$ constant. From

$$\int_s \nabla \times \mathbf{V} \cdot d\boldsymbol{\sigma} = \nabla \times \mathbf{V}\,|_1\, h_2 h_3\, dq_2\, dq_3 \tag{2.18b}$$

Stokes's theorem yields

$$\nabla \times \mathbf{V}\,|_1\, h_2 h_3\, dq_2\, dq_3 = \oint \mathbf{V} \cdot d\boldsymbol{\lambda}, \tag{2.19}$$

with the line integral lying in the surface $q_1 =$ constant. Following the loop (1, 2, 3, 4) of Fig. 2.2,

$$\oint \mathbf{V}(q_1, q_2, q_3) \cdot d\boldsymbol{\lambda} = V_2 h_2\, dq_2 + \left[V_3 h_3 + \frac{\partial}{\partial q_2}(V_3 h_3)\, dq_2\right] dq_3$$

$$- \left[V_2 h_2 + \frac{\partial}{\partial q_3}(V_2 h_2)\, dq_3\right] dq_2 - V_3 h_3\, dq_3$$

$$= \left[\frac{\partial}{\partial q_2}(h_3 V_3) - \frac{\partial}{\partial q_3}(h_2 V_2)\right] dq_2\, dq_3. \tag{2.20}$$

We pick up a positive sign when going in the positive direction on parts 1 and 2 and a negative sign on parts 3 and 4 because here we are going in the negative direction. From Eq. 2.19

$$\nabla \times \mathbf{V}\,|_1 = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(h_3 V_3) - \frac{\partial}{\partial q_3}(h_2 V_2)\right]. \tag{2.21}$$

The remaining two components of $\nabla \times \mathbf{V}$ may be picked up by cyclic permutation of the indices. As in Chapter 1, it is often convenient to write the curl in determinant form:

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \mathbf{a}_1 h_1 & \mathbf{a}_2 h_2 & \mathbf{a}_3 h_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}. \tag{2.22}$$

This completes the determination of ∇, $\nabla \cdot$, $\nabla \times$, and the Laplacian ∇^2 in curvilinear coordinates. Armed with these general expressions, we proceed to study the eleven systems in which Eq. 2.1 is separable (cf. Section 2.5) for $k^2 \neq 0$ and three special coordinate systems (bipolar, toroidal, and bispherical coordinates).

EXERCISES

2.2.1 With $\mathbf{a}_1$ a unit vector in the direction of increasing q_1, show that

(a) $\nabla \cdot \mathbf{a}_1 = \dfrac{1}{h_1 h_2 h_3} \dfrac{\partial(h_2 h_3)}{\partial q_1}$

(b) $\nabla \times \mathbf{a}_1 = \dfrac{1}{h_1}\left[\mathbf{a}_2 \dfrac{\partial h_1}{h_3\, \partial q_3} - \mathbf{a}_3 \dfrac{\partial h_1}{h_2\, \partial q_2}\right]$

2.2.2 Show that the orthogonal unit vectors $\mathbf{a}_i$ may be defined by

$$\mathbf{a}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial q_i} \qquad (a)$$

In particular show that $\mathbf{a}_i \cdot \mathbf{a}_i = 1$ leads to an expression for h_i in agreement with Eq. 2.6. Eq. (a) above may be taken as a starting point for deriving

$$\frac{\partial \mathbf{a}_i}{\partial q_j} = \mathbf{a}_j \frac{\partial h_j}{h_i\, \partial q_i}, \qquad i \neq j$$

and

$$\frac{\partial \mathbf{a}_i}{\partial q_i} = -\sum_{j \neq i} \mathbf{a}_j \frac{\partial h_i}{h_j\, \partial q_j}.$$

2.2.3 Develop arguments to show that ordinary dot and cross products (not involving ∇) in orthogonal curvilinear coordinates proceed as in cartesian coordinates *with no involvement of scale factors.*

2.2.4 Derive

$$\nabla\psi = \mathbf{a}_1 \frac{\partial\psi}{h_1 \partial q_1} + \mathbf{a}_2 \frac{\partial\psi}{h_2\, \partial q_2} + \mathbf{a}_3 \frac{\partial\psi}{h_3\, \partial q_3}$$

by direct application of Eq. 1.90,

$$\nabla\psi = \lim_{\int d\tau \to 0} \frac{\int \psi\, d\boldsymbol{\sigma}}{\int d\tau}.$$

Hint. Evaluation of the surface integral will lead to terms like $(h_1 h_2 h_3)^{-1}(\partial/\partial q_1)(\mathbf{a}_1 h_2 h_3)$. The results listed in Ex. 2.2.2 will be helpful.

2.3 Special Coordinate Systems—Rectangular Cartesian Coordinates

It has been emphasized that the choice of coordinate system may depend on constraints or symmetry conditions in the problem to be solved. It is perhaps convenient to list our fourteen systems, classifying them according to whether or not they have an axis of translation (perpendicular to a family of parallel plane surfaces) or an axis of rotational symmetry.

TABLE 2.1

Axis of Translation	Axis of Rotation	Neither
Cartesian (3 axes)		Confocal ellipsoidal
Circular cylindrical	Circular cylindrical	
	Spherical polar (3 axes)	
Elliptic cylindrical	Prolate spheroidal	
	Oblate spheroidal	
Parabolic cylindrical	Parabolic	
Bipolar	Toroidal	
	Bispherical	
		Conical
		Confocal paraboloidal

Table 2.1 contains fifteen entries—circular cylindrical coordinates with an axis of translation which is also an axis of rotational symmetry. The spacing in the table has been chosen to indicate relations between various coordinate systems. If we consider the two-dimensional version ($z = 0$) of a system with an axis of translation (left column) and rotate it about an axis of reflection symmetry, we generate the corresponding coordinate systems listed to the right in the center column. For instance, rotating the ($z = 0$)-plane of the elliptic cylindrical system about the major axis generates the prolate spheroidal system; rotating about the minor axis yields the oblate spheroidal system.

We do consider three systems with neither an axis of translation nor an axis of rotation. It might be noted that in this asymmetric group the confocal ellipsoidal system is sometimes taken as the most general system and almost all the others[1] are derived from it.

Rectangular cartesian coordinates. These are the cartesian coordinates on which Chapter 1 is based. In this simplest of all systems

$$h_1 = h_x = 1,$$
$$h_2 = h_y = 1, \qquad (2.23)$$
$$h_3 = h_z = 1.$$

The families of coordinate surfaces are three sets of parallel planes: $x =$ constant, $y =$ constant, and $z =$ constant. The cartesian coordinate system is unique in that all its h_i's are constant. This will be a significant advantage in treating tensors in

[1] Excluding the bipolar system and its two rotational forms, toroidal and bispherical.

Chapter 3. Note also that the unit vectors, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ or $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, have *fixed* directions.

From Eqs. 2.13, 2.17, 2.18, and 2.22 we reproduce the results of Chapter 1,

$$\nabla\psi = \mathbf{i}\frac{\partial\psi}{\partial x} + \mathbf{j}\frac{\partial\psi}{\partial y} + \mathbf{k}\frac{\partial\psi}{\partial z}, \tag{2.24}$$

$$\nabla\cdot\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}, \tag{2.25}$$

$$\nabla\cdot\nabla\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}, \tag{2.26}$$

$$\nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}. \tag{2.27}$$

2.4 Spherical Polar Coordinates (r, θ, φ)

Relabeling (q_1, q_2, q_3) as (r, θ, φ), the spherical polar coordinate system consists of the following:

1. Concentric spheres centered at the origin,

$$r = (x^2 + y^2 + z^2)^{1/2} = \text{constant}.$$

2. Right circular cones centered on the z-(polar) axis, vertices at the origin,

$$\theta = \arccos\frac{z}{(x^2+y^2+z^2)^{1/2}} = \text{constant}.$$

3. Half planes through the z-(polar) axis,

$$\varphi = \arctan\frac{y}{x} = \text{constant}.$$

By our arbitrary choice of definitions of θ, the polar angle, and φ, the azimuth angle, the z-axis is singled out for special treatment. The transformation equations corresponding to Eq. 2.2 are

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\theta, \end{aligned} \tag{2.28}$$

measuring θ from the positive z-axis and φ in the xy-plane from the positive x-axis. The ranges of values are $0 \le r < \infty$, $0 \le \theta \le \pi$, and $0 \le \varphi \le 2\pi$. From Eq. 2.6

$$\begin{aligned} h_1 &= h_r = 1, \\ h_2 &= h_\theta = r, \\ h_3 &= h_\varphi = r\sin\theta. \end{aligned} \tag{2.29}$$

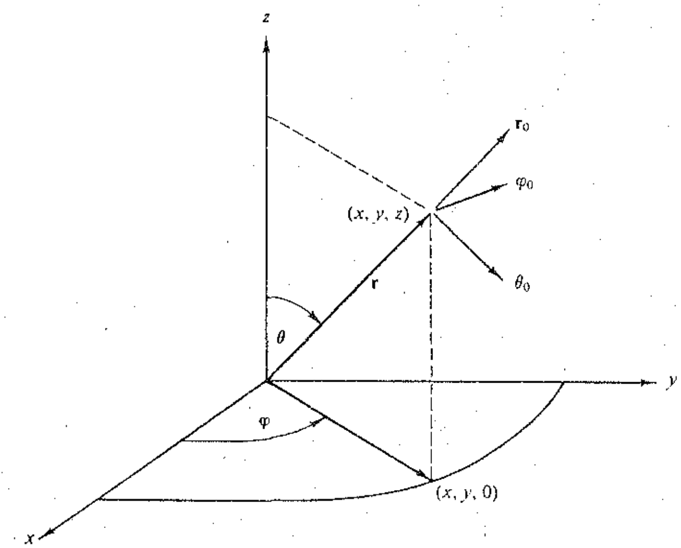


FIG. 2.3 Spherical polar coordinates

It must be emphasized that the unit vectors $\mathbf{r}_0$, $\boldsymbol{\theta}_0$, and $\boldsymbol{\varphi}_0$ vary in direction as the angles θ and φ vary. In terms of the fixed direction cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$,

$$\begin{aligned} \mathbf{r}_0 &= \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi + \mathbf{k}\cos\theta, \\ \boldsymbol{\theta}_0 &= \mathbf{i}\cos\theta\cos\varphi + \mathbf{j}\cos\theta\sin\varphi - \mathbf{k}\sin\theta, \\ \boldsymbol{\varphi}_0 &= -\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi. \end{aligned}$$

From Section 2.2, relabeling the curvilinear coordinate unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ as $\mathbf{r}_0$, $\boldsymbol{\theta}_0$, and $\boldsymbol{\varphi}_0$,

$$\nabla\psi = \mathbf{r}_0\frac{\partial\psi}{\partial r} + \boldsymbol{\theta}_0\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \boldsymbol{\varphi}_0\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}, \tag{2.30}$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2V_r\right) + r\frac{\partial}{\partial\theta}(\sin\theta\, V_\theta) + r\frac{\partial V_\varphi}{\partial\varphi}\right], \tag{2.31}$$

$$\nabla\cdot\nabla\psi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right], \tag{2.32}$$

$$\nabla\times\mathbf{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{r}_0 & r\boldsymbol{\theta}_0 & r\sin\theta\boldsymbol{\varphi}_0 \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ V_r & rV_\theta & r\sin\theta V_\varphi \end{vmatrix}. \tag{2.33}$$

Occasionally the vector Laplacian $\nabla^2\mathbf{V}$ is needed in spherical polar coordinates. It is best obtained by using the vector identity (Eq. 1.80) of Chapter 1. For future reference

$$\nabla^2\mathbf{V}|_r = \left(-\frac{2}{r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)V_r$$
$$+ \left(-\frac{2}{r^2}\frac{\partial}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}\right)V_\theta + \left(-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\varphi}\right)V_\varphi$$
$$= \nabla^2 V_r - \frac{2}{r^2}V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}V_\theta - \frac{2}{r^2\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}, \tag{2.34}$$

$$\nabla^2\mathbf{V}|_\theta = \nabla^2 V_\theta - \frac{1}{r^2\sin^2\theta}V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\varphi}{\partial\varphi}, \tag{2.35}$$

$$\nabla^2\mathbf{V}|_\varphi = \nabla^2 V_\varphi - \frac{1}{r^2\sin^2\theta}V_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial V_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\theta}{\partial\varphi}. \tag{2.36}$$

These expressions for the components of $\nabla^2\mathbf{V}$ are undeniably messy, but sometimes they are needed. There is no guarantee that nature will always be simple.

EXAMPLE 2.4.1

Using Eqs. 2.30–2.33, we can reproduce by inspection some of the results derived in Chapter 1 by laborious application of cartesian coordinates.

From Eq. 2.30

$$\nabla f(r) = \mathbf{r}_0\frac{df}{dr}, \tag{2.37}$$
$$\nabla r^n = \mathbf{r}_0 n r^{n-1}.$$

From Eq. 2.31

$$\nabla\cdot\mathbf{r}_0 f(r) = \frac{2}{r}f(r) + \frac{df}{dr},$$
$$\nabla\cdot\mathbf{r}_0 r^n = (n+2)r^{n-1}. \tag{2.38}$$

From Eq. 2.32

$$\nabla^2 f(r) = \frac{2}{r}\frac{df}{dr} + \frac{d^2 f}{dr^2}, \tag{2.39}$$
$$\nabla^2 r^n = n(n+1)r^{n-2}. \tag{2.40}$$

Finally, from Eq. 2.33

$$\nabla\times\mathbf{r}_0 f(r) = 0. \tag{2.41}$$

EXAMPLE 2.4.2

The computation of the magnetic vector potential of a single current loop in the xy plane involves the evaluation of

$$\mathbf{V} = \nabla\times[\nabla\times\boldsymbol{\varphi}_0 A_\varphi(r,\theta)]. \tag{2.41a}$$

In spherical polar coordinates this reduces as follows:

$$\mathbf{V} = \nabla\times\frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{r}_0 & r\boldsymbol{\theta}_0 & r\sin\theta\boldsymbol{\varphi}_0 \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ 0 & 0 & r\sin\theta A_\varphi(r,\theta) \end{vmatrix}$$
$$= \nabla\times\frac{1}{r^2\sin\theta}\left[\mathbf{r}_0\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) - r\boldsymbol{\theta}_0\frac{\partial}{\partial r}(r\sin\theta A_\varphi)\right]. \tag{2.41b}$$

Taking the curl a second time,

$$\mathbf{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{r}_0 & r\boldsymbol{\theta}_0 & r\sin\theta\boldsymbol{\varphi}_0 \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}(r\sin\theta A_\varphi) & -\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial r}(r\sin\theta A_\varphi) & 0 \end{vmatrix} \tag{2.41c}$$

By expanding the determinant

$$\mathbf{V} = -\boldsymbol{\varphi}_0\left\{\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_\varphi) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)\right]\right\}$$
$$= -\boldsymbol{\varphi}_0\left[\nabla^2 A_\varphi(r,\theta) - \frac{1}{r^2\sin^2\theta}A_\varphi(r,\theta)\right].$$

In Chapter 12 we shall see that $\mathbf{V}$ leads to the associated Legendre equation and that A_φ may be given by a series of associated Legendre polynomials.

EXERCISES

2.4.1 Resolve the spherical polar unit vectors into their cartesian components.

$$\mathbf{r}_0 = \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi + \mathbf{k}\cos\theta,$$
$$\boldsymbol{\theta}_0 = \mathbf{i}\cos\theta\cos\varphi + \mathbf{j}\cos\theta\sin\varphi - \mathbf{k}\sin\theta,$$
$$\boldsymbol{\varphi}_0 = -\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi.$$

2.4.2 (a) From the results of Exercise 2.4.1 calculate the partial derivatives of $\mathbf{r}_0$, $\boldsymbol{\theta}_0$, and $\boldsymbol{\varphi}_0$ with respect to r, θ, and φ.

(b) With ∇ given by

$$\mathbf{r}_0\frac{\partial}{\partial r} + \boldsymbol{\theta}_0\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{\varphi}_0\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

(greatest space rate of change), use the results of part (a) to calculate $\nabla\cdot\nabla\psi$. Here is an alternate derivation of the Laplacian.

2.4.3 Resolve the cartesian unit vectors into their spherical polar components.

$$\mathbf{i} = \mathbf{r}_0 \sin\theta \cos\varphi + \boldsymbol{\theta}_0 \cos\theta \cos\varphi - \boldsymbol{\varphi}_0 \sin\varphi,$$
$$\mathbf{j} = \mathbf{r}_0 \sin\theta \sin\varphi + \boldsymbol{\theta}_0 \cos\theta \sin\varphi + \boldsymbol{\varphi}_0 \cos\varphi,$$
$$\mathbf{k} = \mathbf{r}_0 \cos\theta - \boldsymbol{\theta}_0 \sin\theta.$$

2.4.4 The direction of one vector is given by the angles θ_1 and φ_1. For a second vector the corresponding angles are θ_2 and φ_2. Show that the cosine of the included angle γ is given by

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2).$$

C.f. Fig. 12.13

2.4.5 A vector $\mathbf{V}$ is tangential to the surface of a sphere. The curl of $\mathbf{V}$ is radial. What does this imply about the radial dependence of the spherical polar components of $\mathbf{V}$?

2.4.6 Modern physics lays great stress on the property of parity—whether a quantity remains invariant or changes sign under an inversion of the coordinate system.

(a) Show that the inversion (reflection through the origin) of a point (r, θ, φ) relative to *fixed* x-, y-, z-axes consists of the transformation

$$r \to r,$$
$$\theta \to \pi - \theta,$$
$$\varphi \to \pi + \varphi.$$

(b) Show that $\mathbf{r}_0$ and $\boldsymbol{\varphi}_0$ have odd parity (reversal of direction) and that $\boldsymbol{\theta}_0$ has even parity.

2.4.7 Eq. 1.72 was a demonstration that

$$\boldsymbol{\omega} \cdot \nabla\mathbf{r} = \boldsymbol{\omega},$$

using cartesian coordinates. Verify this result using *spherical polar coordinates*. In the language of dyadics (Section 3.5), $\nabla\mathbf{r}$ is the indemfactor, a unit dyadic.

2.4.8 A particle is moving through space. Find the spherical coordinate components of it velocity and acceleration:

$$v_r = \dot r,$$
$$v_\theta = r\dot\theta,$$
$$v_\varphi = r \sin\theta\dot\varphi,$$
$$a_r = \ddot r - r\dot\theta^2 - r\sin^2\theta\dot\varphi^2,$$
$$a_\theta = r\ddot\theta + 2\dot r\dot\theta - r\sin\theta\cos\theta\dot\varphi^2,$$
$$a_\varphi = r\sin\theta\ddot\varphi + 2\dot r\sin\theta\dot\varphi + 2r\cos\theta\dot\theta\dot\varphi.$$

Hint.

$$\mathbf{r}(t) = \mathbf{r}_0(t)r(t)$$
$$= [\mathbf{i}\sin\theta(t)\cos\varphi(t) + \mathbf{j}\sin\theta(t)\sin\varphi(t) + \mathbf{k}\cos\theta(t)]r(t).$$

Note. Using the Lagrangian techniques of Section 17.3 these results may be obtained somewhat more elegantly. The dot in $\dot r$ means time derivative, $\dot r = dr/dt$. The notation was originated by Newton.

2.4.9 A particle m moves in response to a central force according to Newton's second law

$$m\ddot{\mathbf{r}} = \mathbf{r}_0 f(\mathbf{r}).$$

Show that $\mathbf{r}\times\dot{\mathbf{r}} = \mathbf{c}$, a constant and that the geometric interpretation of this leads to Kepler's second law.

2.4.10 Express $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ in spherical polar coordinates.

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$
$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}.$$

Hint. Equate ∇_{xyz} and $\nabla_{r\theta\varphi}$.

2.4.11 From Exercise 2.4.10 show that

$$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\varphi}.$$

This is the quantum mechanical operator corresponding to the z-component of angular momentum.

2.4.12 With the quantum mechanical angular momentum operator defined as $\mathbf{L} = -i(\mathbf{r}\times\nabla)$, show that

(a) $L_x + iL_y = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right),$

(b) $L_x - iL_y = -e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right).$

These are the raising and lowering operators of Sections 13.6 and 7.

2.4.13 Verify that $\mathbf{L}\times\mathbf{L} = i\mathbf{L}$ in spherical polar coordinates. $\mathbf{L} = -i(\mathbf{r}\times\nabla)$, the quantum mechanical angular momentum operator.

Hint: Use spherical polar coordinates for $\mathbf{L}$ but cartesian components for the cross product.

2.4.14 With $\mathbf{L} = -i\mathbf{r}\times\nabla$ verify the operator identities

(a) $\nabla = \mathbf{r}_0\frac{\partial}{\partial r} - i\frac{\mathbf{r}\times\mathbf{L}}{r^2},$

(b) $\mathbf{r}\nabla^2 - \nabla\left(1 + r\frac{\partial}{\partial r}\right) = i\nabla\times\mathbf{L}.$

This latter identity is useful in relating angular momentum and Legendre's differential equation, Ex. 8.2.3.

2.4.15 Show that the following three forms (spherical coordinates) of $\nabla^2\psi(r)$ are equivalent.

(a) $\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi(r)}{dr}\right],$

(b) $\frac{1}{r}\frac{d^2}{dr^2}\left[r\psi(r)\right],$

(c) $\frac{d^2\psi(r)}{dr^2} + \frac{2}{r}\frac{d\psi(r)}{dr}.$

The second form is particularly convenient in establishing a correspondence between spherical polar and cartesian descriptions of a problem.

2.4.16 One model of the solar corona assumes that the steady-state equation of heat flow

$$\nabla\cdot(k\nabla T) = 0$$

is satisfied. Here, k, the thermal conductivity, is proportional to $T^{5/2}$. Assuming that the temperature T is proportional to r^n, show that the heat flow equation is satisfied by $T = T_0(r_0/r)^{2/7}$.

2.4.17 A certain force field is given by

$$\mathbf{F} = \mathbf{r}_0 \frac{2P\cos\theta}{r^3} + \boldsymbol{\theta}_0 \frac{P}{r^3}\sin\theta, \qquad r \geq P/2,$$

(in spherical polar coordinates).
(a) Examine $\nabla \times \mathbf{F}$ to see if a potential exists.
(b) Calculate $\oint \mathbf{F} \cdot d\boldsymbol{\lambda}$ for a unit circle in the plane $\theta = \pi/2$. What does this indicate about the force being conservative or nonconservative?
(c) If you believe that $\mathbf{F}$ may be described by $\mathbf{F} = -\nabla\psi$, find ψ. Otherwise simply state that no acceptable potential exists.

2.4.18 (a) Show that $\mathbf{A} = -\boldsymbol{\varphi}_0(\cot\theta/r)$ is a solution of $\nabla \times \mathbf{A} = \mathbf{r}_0/r^2$.
(b) Show that this spherical polar coordinate solution agrees with the solution given for Exercise 1.13.5:

$$\mathbf{A} = \mathbf{i}\frac{yz}{r(x^2+y^2)} - \mathbf{j}\frac{xz}{r(x^2+y^2)}.$$

Note that the solution diverges for $\theta = 0, \pi$ corresponding to $x, y = 0$.
(c) Finally, show that $\mathbf{A} = -\boldsymbol{\theta}_0\varphi(\sin\theta/r)$ is a solution. Note that although this solution does not diverge ($r \neq 0$) it is no longer single-valued for all possible azimuth angles.

2.4.19 A magnetic vector potential is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}.$$

Show that this leads to the magnetic induction $\mathbf{B}$ of a point magnetic dipole, dipole moment $\mathbf{m}$.

Ans. For $\mathbf{m} = \mathbf{k}m$,

$$\nabla \times \mathbf{A} = \mathbf{r}_0 \frac{2m\cos\theta}{r^3} + \boldsymbol{\theta}_0 \frac{m\sin\theta}{r^3}.$$

Cf. Eqs. 12.146 and 12.147.

2.4.20 At large distances from its source, electric dipole radiation has fields

$$\mathbf{E} = a_E \sin\theta \frac{e^{i(kr-\omega t)}}{r}\boldsymbol{\theta}_0, \qquad \mathbf{B} = a_B \sin\theta \frac{e^{i(kr-\omega t)}}{r}\boldsymbol{\varphi}_0.$$

Show that Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{B} = \varepsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

are satisfied, if we take

$$a_E/a_B = \omega/k = c = (\varepsilon_0\mu_0)^{-1/2}.$$

Hint. Since r is large, terms of order r^{-2} may be dropped.

2.5 Separation of Variables

In cartesian coordinates the Helmholtz equation (Eq. 2.1) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0, \tag{2.42}$$

using Eq. 2.26 for the Laplacian. For the present let k^2 be a constant. Perhaps the simplest way of treating a partial differential equation such as 2.42 is to split it up into a set of ordinary differential equations. This may be done as follows: Let

$$\psi(x, y, z) = X(x)\,Y(y)\,Z(z) \tag{2.43}$$

and substitute back into Eq. 2.42. How do we know Eq. 2.43 is valid? The answer is very simple. We do not know it is valid! Rather we are proceeding in the spirit of let's try it and see if it works. If our attempt succeeds, then Eq. 2.43 will be justified. If it does not succeed, we shall find out soon enough and then we shall have to try another attack such as Green's functions, integral transforms, or brute force numerical analysis. With ψ assumed given by Eq. 2.43, Eq. 2.42 becomes

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} + k^2XYZ = 0. \tag{2.44}$$

Dividing by $\psi = XYZ$ and rearranging terms, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2}. \tag{2.45}$$

Equation 2.45 exhibits one separation of variables. The left-hand side is a function of x alone, whereas the right-hand side depends only on y and z. So Eq. 2.45 is a sort of paradox. A function of x is equated to a function of y and z, but x, y, and z are all independent coordinates. This independence means that the behavior of x as an independent variable is not determined by y and z. The paradox is resolved by setting each side equal to a constant, a constant of separation. We choose[1]

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2, \tag{2.46}$$

$$-k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} = -l^2. \tag{2.47}$$

Now, turning our attention to Eq. 2.47,

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 + l^2 - \frac{1}{Z}\frac{d^2Z}{dz^2}, \tag{2.48}$$

and a second separation has been achieved. Here we have a function of y equated to a function of z and the same paradox appears. We resolve it as before by equating each side to another constant of separation, $-m^2$,

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2; \tag{2.49}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2 + l^2 + m^2 = -n^2, \tag{2.50}$$

introducing a constant n^2 by $k^2 = l^2 + m^2 + n^2$ to produce a symmetric set of equations. Now we have three ordinary differential equations (2.46, 2.49, and

[1] The choice of sign, completely arbitrary here, will be fixed in specific problems by the need to satisfy specific boundary conditions.

2.50) to replace Eq. 2.42. Our assumption (Eq. 2.43) has succeeded and is thereby justified.

Our solution should be labeled according to the choice of our constants l, m, and n, that is,

$$\psi_{lmn}(x, y, z) = X_l(x)\, Y_m(y)\, Z_n(z). \tag{2.50a}$$

Subject to the conditions of the problem being solved and to the condition $k^2 = l^2 + m^2 + n^2$, we may choose l, m, and n as we like, and Eq. 2.50a will still be a solution of Eq. 2.1, provided $X_l(x)$ is a solution of Eq. 2.46, etc. We may develop the most general solution of Eq. 2.1 by taking a linear combination of solutions ψ_{lmn},

$$\Psi = \sum_{l,m,n} a_{lmn}\psi_{lmn}. \tag{2.50b}$$

The constant coefficients a_{lmn} are finally chosen to permit Ψ to satisfy the boundary conditions of the problem.

How is this possible? What is the justification for writing Eq. 2.50b? The justification is found in noting that $\nabla^2 + k^2$ is a linear (differential) operator. A linear operator $\mathscr{L}$ is defined as an operator with the following two properties:

$$\mathscr{L}(a\psi) = a\mathscr{L}\psi,$$

where a is a constant and

$$\mathscr{L}(\psi_1 + \psi_2) = \mathscr{L}\psi_1 + \mathscr{L}\psi_2.$$

As a consequence of these properties, any linear combination of solutions of a linear differential equation is also a solution. From its explicit form $\nabla^2 + k^2$ is seen to have these two properties (and is therefore a linear operator). Equation 2.50b then follows as a direct application of these two defining properties.[1]

A further generalization may be noted. The separation process just described would go through just as well for

$$k^2 = f(x) + g(y) + h(z) + k'^2, \tag{2.50c}$$

with k'^2 a new constant.

We would simply have

$$\frac{1}{X}\frac{d^2X}{dx^2} + f(x) = -l^2 \tag{2.50d}$$

replacing Eq. 2.46. The solutions X, Y, and Z would be different, but the technique of splitting the partial differential equation and of taking a linear combination of solutions would be the same.

In case the reader wonders what is going on here, this technique of separation of variables of a partial differential equation has been introduced to illustrate the usefulness of these coordinate systems. The solutions of the resultant ordinary differential equations are developed in Chapters 8 through 13.

Let us try to separate Eq. 2.1, again with k^2 constant, in spherical polar coordinates. Using Eq. 2.32, we obtain

[1] We are especially interested in linear operators because in quantum mechanics physical quantities are represented by linear operators operating in a complex, infinite dimensional Hilbert space.

$$\frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] = -k^2\psi. \tag{2.51}$$

Now, in analogy with Eq. 2.43 we try

$$\psi(r, \theta, \varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi). \tag{2.52}$$

By substituting back into Eq. 2.51 and dividing by $R\Theta\Phi$, we have

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -k^2. \tag{2.53}$$

Note that all derivatives are now ordinary derivatives rather than partials. By multiplying by $r^2\sin^2\theta$ we can isolate $(1/\Phi)(d^2\Phi/d\varphi^2)$ to obtain

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = r^2\sin^2\theta\left[-k^2 - \frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{1}{r^2\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]. \tag{2.54}$$

Equation 2.54 relates a function of φ alone to a function of r and θ alone. Since r, θ, and φ are independent variables, we equate each side of Eq. 2.54 to a constant. Here a little consideration can simplify the later analysis. In almost all physical problems φ will appear as an azimuth angle. This suggests a periodic solution rather than an exponential. With this in mind, let us use $-m^2$ as the separation constant. Any constant will do, but this one will make life a little easier. Then

$$\frac{1}{\Phi}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2 \tag{2.55}$$

and

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{r^2\sin^2\theta} = -k^2. \tag{2.56}$$

Multiplying Eq. 2.56 by r^2 and rearranging terms, we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + r^2k^2 = -\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta}. \tag{2.57}$$

Again the variables are separated. We equate each side to a constant Q and finally obtain

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + Q\Theta = 0, \tag{2.58}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{QR}{r^2} = 0. \tag{2.59}$$

Once more we have replaced a partial differential equation of three variables by three ordinary differential equations. The solutions of these ordinary differential equations are discussed in Chapters 11 and 12. In Chapter 12, for example, Eq. 2.58 is identified as the associated Legendre equation in which the constant Q becomes $l(l+1)$; l is an integer.

Again, our most general solution may be written

$$\psi_{Qm}(r, \theta, \varphi) = \sum_{Q,m} R_Q(r)\,\Theta_{Qm}(\theta)\,\Phi_m(\varphi). \tag{2.60a}$$

The restriction that k^2 be a constant is unnecessarily severe. The separation process will still be possible for k^2 as general as

$$k^2 = f(r) + \frac{1}{r^2} g(\theta) + \frac{1}{r^2 \sin^2\theta} h(\varphi) + k'^2 \tag{2.60b}$$

In the hydrogen atom problem, one of the most important examples of the Schrödinger wave equation with a closed form solution, we have $k^2 = f(r)$. Equation 2.59 for the hydrogen atom becomes the associated Laguerre equation. Separation of variables and an investigation of the resulting ordinary differential equations are taken up again in Section 8.2. Now we return to an investigation of the remaining special coordinate systems.

EXERCISES

2.5.1 By letting the operator $\nabla^2 + k^2$ act on the general form $a_1\psi_1(x, y, z) + a_2\psi_2(x, y, z)$, show that it is linear, that is, that $(\nabla^2 + k^2)(a_1\psi_1 + a_2\psi_2) = a_1(\nabla^2 + k^2)\psi_1 + a_2(\nabla^2 + k^2)\psi_2$.

2.5.2 Verify that

$$\nabla^2\psi(r, \theta, \varphi) + \left[k^2 + f(r) + \frac{1}{r^2} g(\theta) + \frac{1}{r^2 \sin^2\theta} h(\varphi)\right] \psi(r, \theta, \varphi) = 0$$

is separable (in spherical polar coordinates). The functions f, g, and h are functions only of the variables indicated; k^2 is a constant.

2.5.3 An atomic (quantum mechanical) particle is confined inside a rectangular box of sides a, b, and c. The particle is described by a wave function ψ which satisfies the Schrödinger wave equation

$$-\frac{\hbar^2}{2m} \nabla^2\psi = E\psi.$$

The wave function is required to vanish at each surface of the box (but not to be identically zero). This condition imposes constraints on the separation constants and therefore on the energy E. What is the smallest value of E for which such a solution can be obtained?

$$Ans.\ E = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

2.6 Circular Cylindrical Coordinates (ρ, φ, z)

From Fig. 2.4 we obtain the transformation relations

$$x = \rho \cos \varphi,$$
$$y = \rho \sin \varphi, \tag{2.61}$$
$$z = z,$$

using ρ for the perpendicular distance from the z-axis and saving r for the distance from the origin. According to these equations or from the length elements the scale factors are

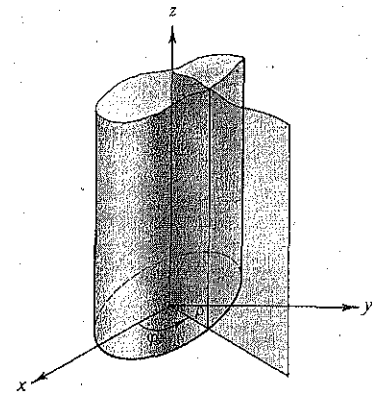


FIG. 2.4 Circular cylinder coordinates

$$h_1 = h_\rho = 1,$$
$$h_2 = h_\varphi = \rho, \tag{2.62}$$
$$h_3 = h_z = 1.$$

The families of coordinate surfaces shown in Fig. 2.4 are

1. Right circular cylinders having the z-axis as a common axis,

$$\rho = (x^2 + y^2)^{1/2} = \text{constant}.$$

2. Half planes through the z-axis,

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \text{constant}.$$

3. Planes parallel to the xy-plane, as in the cartesian system,

$$z = \text{constant}.$$

The limits on ρ, φ and z are

$$0 \leqslant \rho < \infty, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad \text{and} \quad -\infty < z < \infty.$$

From Eqs. 2.13, 2.17, 2.18, and 2.22,

$$\nabla\psi(\rho, \varphi, z) = \boldsymbol{\rho}_0 \frac{\partial\psi}{\partial\rho} + \boldsymbol{\varphi}_0 \frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} + \mathbf{k}\frac{\partial\psi}{\partial z}, \tag{2.63}$$

$$\nabla \cdot \mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z}, \tag{2.64}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}, \tag{2.65}$$

$$\nabla \times \mathbf{V} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{\rho}_0 & \rho\boldsymbol{\varphi}_0 & \mathbf{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\varphi & V_z \end{vmatrix}. \tag{2.66}$$

Finally, for problems such as circular wave guides or cylindrical cavity resonators the vector Laplacian $\nabla^2\mathbf{V}$ resolved in circular cylindrical coordinates is

$$\nabla^2\mathbf{V}|_\rho = \nabla^2 V_\rho - \frac{1}{\rho^2} V_\rho - \frac{2}{\rho^2}\frac{\partial V_\varphi}{\partial\varphi},$$

$$\nabla^2\mathbf{V}|_\varphi = \nabla^2 V_\varphi - \frac{1}{\rho^2} V_\varphi + \frac{2}{\rho^2}\frac{\partial V_\rho}{\partial\varphi}, \tag{2.67}$$

$$\nabla^2\mathbf{V}|_z = \nabla^2 V_z .$$

The basic reason for the form of the z-component is that the z-axis is a cartesian axis, that is,

$$\nabla^2(\boldsymbol{\rho}_0 V_\rho + \boldsymbol{\varphi}_0 V_\varphi + \mathbf{k} V_z) = \nabla^2(\boldsymbol{\rho}_0 V_\rho + \boldsymbol{\varphi}_0 V_\varphi) + \mathbf{k}\,\nabla^2 V_z$$
$$= \boldsymbol{\rho}_0 f(V_\rho, V_\varphi) + \boldsymbol{\varphi}_0 g(V_\rho, V_\varphi) + \mathbf{k}\,\nabla^2 V_z .$$

The operator ∇^2 operating on the $\boldsymbol{\rho}_0$, $\boldsymbol{\varphi}_0$ unit vectors stays in the $\boldsymbol{\rho}_0\,\boldsymbol{\varphi}_0$-plane. This behavior holds in all such cylindrical systems.

Example 2.6.1 Cylindrical Resonant Cavity

Consider a circular cylindrical cavity (radius a) with perfectly conducting walls. Electromagnetic waves will oscillate in such a cavity. If we assume our electric and magnetic fields have a time dependence $e^{-i\omega t}$, then Maxwell's equations lead to

$$\nabla \times \nabla \times \mathbf{E} = \omega^2 \varepsilon_0 \mu_0 \mathbf{E}. \qquad \text{(Cf. Example 1.9.2)} \tag{2.68}$$

With $\nabla \cdot \mathbf{E} = 0$ (vacuum, no charges),

$$\nabla^2\mathbf{E} + \alpha^2\mathbf{E} = 0,$$

where ∇^2 is the *vector* Laplacian and $\alpha^2 = \omega^2\varepsilon_0\mu_0$. In cylindrical coordinates E_z splits off, and we have the *scalar* Helmholtz equation

$$\nabla^2 E_z + \alpha^2 E_z = 0, \tag{2.69}$$

and the boundary condition $E_z\,(\rho = a) = 0$.

Using Eq. 2.65, Eq. 2.69 becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\varphi^2} + \frac{\partial^2 E_z}{\partial z^2} + \alpha^2 E_z = 0. \tag{2.70}$$

We try
$$E_z(\rho, \varphi, z) = P(\rho)\Phi(\varphi)Z(z)$$

to obtain
$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\varphi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + \alpha^2 = 0. \tag{2.71}$$

Splitting off the z-dependence with a separation constant $-k^2$,

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2.$$

For our cavity problem, $\sin kz$ and $\cos kz$ are the appropriate solutions (in that we can choose them to match the boundary conditions at the ends of the cavity). The exponentials $e^{\pm ikz}$ would be appropriate for a wave guide (traveling waves), cf. Section 11.3.

Using $\gamma^2 = \alpha^2 - k^2$, we isolate the φ dependence by multiplying by ρ^2. We set

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2,$$

with $\Phi(\varphi) = e^{\pm im\varphi}$, $\sin m\varphi$, $\cos m\varphi$. Then the remaining ρ dependence is

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + (\gamma^2\rho^2 - m^2)\,P = 0. \tag{2.72}$$

This is Bessel's equation. The solutions are developed in Chapter 11. This particular example, with Bessel functions, is continued as Example 11.2.2.

EXERCISES

2.6.1 Resolve the circular cylindrical unit vectors into their cartesian components.

$$\boldsymbol{\rho}_0 = \mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi,$$
$$\boldsymbol{\varphi}_0 = -\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi,$$
$$\mathbf{k}_0 = \mathbf{k}.$$

2.6.2 Resolve the cartesian unit vectors into their circular cylindrical components.

$$\mathbf{i} = \boldsymbol{\rho}_0\cos\varphi - \boldsymbol{\varphi}_0\sin\varphi,$$
$$\mathbf{j} = \boldsymbol{\rho}_0\sin\varphi + \boldsymbol{\varphi}_0\cos\varphi,$$
$$\mathbf{k} = \mathbf{k}_0.$$

2.6.3 A particle is moving through space. Find the circular cylindrical components of its velocity and acceleration.

$$v_\rho = \dot\rho, \qquad a_\rho = \ddot\rho - \rho\dot\varphi^2,$$
$$v_\varphi = \rho\dot\varphi, \qquad a_\varphi = \rho\ddot\varphi + 2\dot\rho\dot\varphi,$$
$$v_z = \dot z, \qquad a_z = \ddot z.$$

Hint.

$$\mathbf{r}(t) = \boldsymbol{\rho}_0(t)\rho(t) + \mathbf{k}z(t)$$
$$= [\mathbf{i}\cos\varphi(t) = \mathbf{j}\sin\varphi(t)]\rho(t) + \mathbf{k}z(t).$$

Note. $\dot{\rho} = d\rho/dt$, $\ddot{\rho} = d^2\rho/dt^2$, etc.

2.6.4 Show that the Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0$$

is still separable in circular cylindrical coordinates if k^2 is generalized to $k^2 + f(\rho) + (1/\rho^2)g(\varphi) + h(z)$.

2.6.5 Solve Laplace's equation $\nabla^2\psi = 0$, in cylindrical coordinates for $\psi = \psi(\rho)$.

Ans. $\psi = k \ln \dfrac{\rho}{\rho_0}$

2.6.6 In right circular cylindrical coordinates a particular vector function is given by

$$\mathbf{V}(\rho, \varphi) = \boldsymbol{\rho}_0 V_\rho(\rho, \varphi) + \boldsymbol{\varphi}_0 V_\varphi(\rho, \varphi).$$

Show that $\nabla \times \mathbf{V}$ has only a z-component. Note that this result will hold for any vector confined to a surface $q_3 =$ constant as long as the products h_1V_1 and h_2V_2 are each independent of q_3.

2.6.7 A conducting wire along the z-axis carries a current I. The resulting magnetic vector potential is given by

$$\mathbf{A} = \mathbf{k}\frac{\mu I}{2\pi}\ln\left(\frac{1}{\rho}\right).$$

Show that the magnetic induction $\mathbf{B}$ is given by

$$\mathbf{B} = \boldsymbol{\varphi}_0 \frac{\mu I}{2\pi\rho}.$$

2.6.8 A force is described by

$$\mathbf{F} = -\mathbf{i}\frac{y}{x^2+y^2} + \mathbf{j}\frac{x}{x^2+y^2}.$$

(a) Express $\mathbf{F}$ in circular cylindrical coordinates.
Operating entirely in circular cylindrical coordinates for (b) and (c),
(b) calculate the curl of $\mathbf{F}$ and
(c) calculate the work done by $\mathbf{F}$ in encircling the unit circle once counterclockwise.
(d) How do you reconcile the results of (b) and (c)?

2.6.9 A transverse electromagnetic wave (TEM) in a coaxial wave guide has an electric field $\mathbf{E} = \mathbf{E}(\rho, \varphi)e^{i(kz-\omega t)}$ and a magnetic induction field of $\mathbf{B} = \mathbf{B}(\rho, \varphi)e^{i(kz-\omega t)}$. Since the wave is transverse neither $\mathbf{E}$ nor $\mathbf{B}$ has a z component. The two fields satisfy the *vector* Laplacian equation

$$\nabla^2\mathbf{E}(\rho, \varphi) = 0$$
$$\nabla^2\mathbf{B}(\rho, \varphi) = 0.$$

(a) Show that $\mathbf{E} = \boldsymbol{\rho}_0 E_0(a/\rho)e^{i(kz-\omega t)}$ and $\mathbf{B} = \boldsymbol{\varphi}_0 B_0(a/\rho)e^{i(kz-\omega t)}$ are solutions. Here a is the radius of the inner conductor.
(b) Assuming a vacuum inside the wave guide, verify that Maxwell's equations are satisfied with

$$B_0/E_0 = k/\omega = \mu_0\varepsilon_0(\omega/k) = 1/c.$$

2.6.10 A calculation of the magnetohydrodynamic pinch effect involves the evaluation of $(\mathbf{B}\cdot\nabla)\mathbf{B}$. If the magnetic induction $\mathbf{B}$ is taken to be $\mathbf{B} = \boldsymbol{\varphi}_0 B_\varphi(\rho)$, show that

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = -\boldsymbol{\rho}_0 B_\varphi{}^2/\rho.$$

2.6.11 (a) Explain why ∇^2 in plane polar coordinates follows from ∇^2 in circular cylindrical coordinates with $z =$ constant.
(b) Explain why taking ∇^2 in spherical polar coordinates and restricting θ to $\pi/2$ does NOT lead to the plane polar form of ∇^2.

Note. $\nabla^2(\rho, \varphi) = \dfrac{\partial^2}{\partial\rho^2} + \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial\varphi^2}.$

2.7 Elliptic Cylindrical Coordinates (u, v, z)

One reasonable way of classifying the separable coordinate systems is to start with the confocal ellipsoidal system (Section 2.15) and derive the other systems as degenerate cases. Details of this procedure will be found in Morse and Feshbach's *Methods of Mathematical Physics*, Chapter 5. Here, to emphasize the application

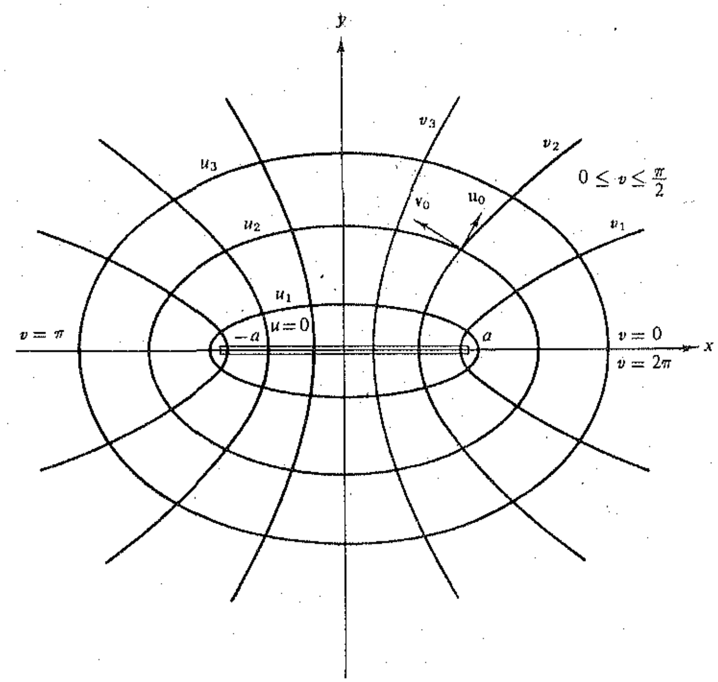


FIG. 2.5 Elliptic cylindrical coordinates

rather than a derivation, we take the coordinate systems in order of symmetry properties, proceeding with those that have an axis of translation. All of those with an axis of translation are essentially two-dimensional systems with a third dimension (the z-axis) tacked on.

For the elliptic cylindrical system we have

$$\begin{aligned} x &= a \cosh u \cos v, \\ y &= a \sinh u \sin v, \\ z &= z. \end{aligned} \tag{2.73}$$

The families of coordinate surfaces are the following:

1. Elliptic cylinders, $u =$ constant, $0 \leqslant u < \infty$.
2. Hyperbolic cylinders, $v =$ constant, $0 \leqslant v \leqslant 2\pi$.
3. Planes parallel to the xy-plane, $z =$ constant, $-\infty < z < \infty$.

This may be seen by inverting Eq. 2.73. Squaring each side,

$$x^2 = a^2 \cosh^2 u \cos^2 v, \tag{2.74}$$

$$y^2 = a^2 \sinh^2 u \sin^2 v, \tag{2.75}$$

from which

$$\frac{x^2}{a^2 \cosh^2 u} + \frac{y^2}{a^2 \sinh^2 u} = 1, \tag{2.76}$$

$$\frac{x^2}{a^2 \cos^2 v} - \frac{y^2}{a^2 \sin^2 v} = 1. \tag{2.77}$$

Holding u constant, Eq. 2.76 yields a family of ellipses with the x-axis the major one. For $v =$ constant, Eq. 2.77 gives hyperbolas with focal points on the x-axis.

The scale factors are

$$\begin{aligned} h_1 &= h_u = a(\sinh^2 u + \sin^2 v)^{1/2}, \\ h_2 &= h_v = a(\sinh^2 u + \sin^2 v)^{1/2}, \\ h_3 &= h_z = 1. \end{aligned} \tag{2.78}$$

We shall meet this system again as a two-dimensional system in Chapter 6 when we take up conformal mapping.

EXERCISES

2.7.1 Let $\cosh u = q_1$, $\cos v = q_2$, $z = q_3$. Find the new scale factors h_{q_1} and h_{q_2}.

$$h_{q_1} = a\left(\frac{q_1^2 - q_2^2}{q_1^2 - 1}\right)^{1/2},$$

$$h_{q_2} = a\left(\frac{q_1^2 - q_2^2}{1 - q_2^2}\right)^{1/2}.$$

2.7.2 Show that the Helmholtz equation in elliptic cylindrical coordinates separates into
(a) the linear oscillator equation for the z dependence,
(b) Mathieu's equation

$$\frac{d^2g}{dv^2} + (b - 2q\cos 2v)\, g = 0,$$

and
(c) Mathieu's modified equation

$$\frac{d^2f}{du^2} - (b - 2q\cosh 2u) f = 0.$$

2.8 Parabolic Cylindrical Coordinates (ξ, η, z)

The transformation equations,

$$\begin{aligned} x &= \xi\eta, \\ y &= \tfrac{1}{2}(\eta^2 - \xi^2), \\ z &= z, \end{aligned} \tag{2.79}$$

generate two sets of orthogonal parabolic cylinders (Fig. 2.6). By solving Eq. 2.79 for ξ and η we obtain the following:

1. Parabolic cylinders, $\xi =$ constant,[1] $-\infty < \xi < \infty$.
2. Parabolic cylinders, $\eta =$ constant, $0 \leqslant \eta < \infty$.
3. Planes parallel to the xy-plane, $z =$ constant, $-\infty < z < \infty$.

From Eq. 2.6 the scale factors are

$$\begin{aligned} h_1 &= h_\xi = (\xi^2 + \eta^2)^{1/2}, \\ h_2 &= h_\eta = (\xi^2 + \eta^2)^{1/2}, \\ h_3 &= h_z = 1. \end{aligned} \tag{2.80}$$

2.9 Bipolar Coordinates (ξ, η, z)

This is an oddball coordinate system. It is not a degenerate case of the confocal ellipsoidal coordinates. Equation 2.1 is not completely separable in this system even for $k^2 = 0$ (cf. Exercise 2.9.2). It is included here as an example of how an unusual coordinate system may be chosen to fit a problem.

[1] The parabolic cylinder $\xi =$ constant is invariant to the sign of ξ. We must let ξ (or η) go negative to cover negative values of x.

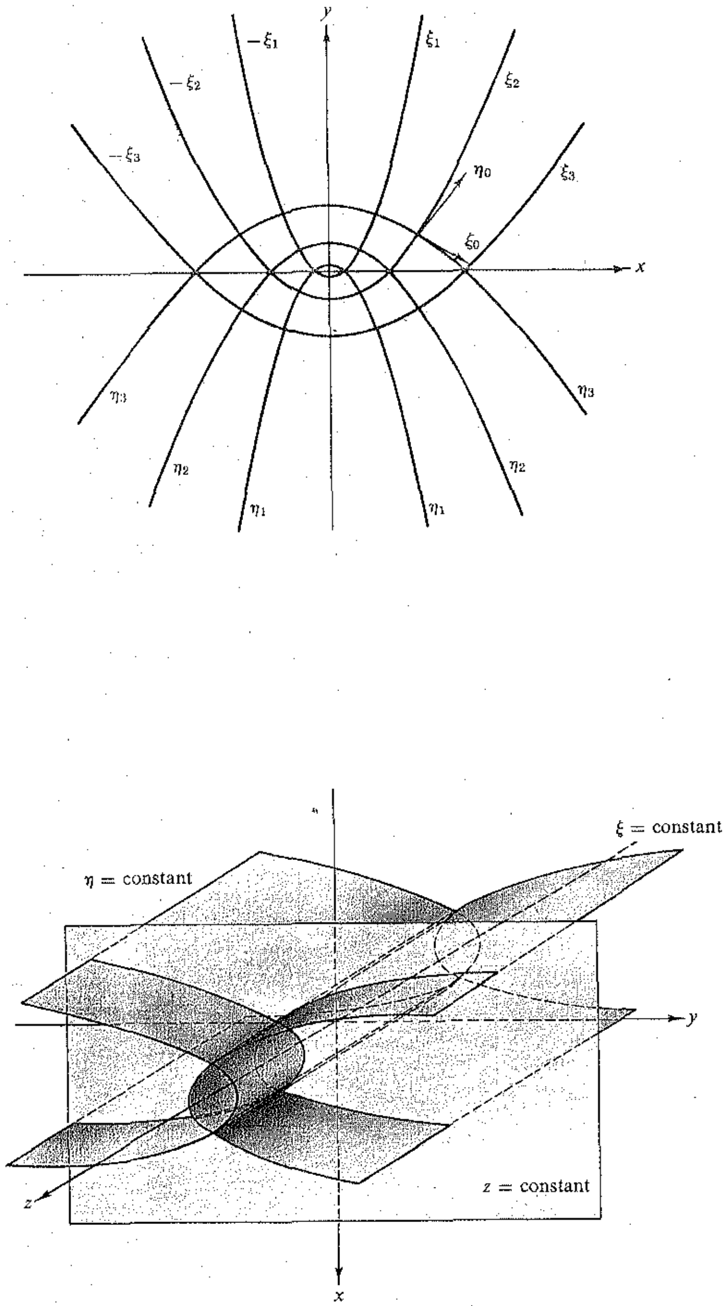


FIG. 2.6 Parabolic cylindrical coordinates. (Top) Cross section

The transformation equations are

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}, \tag{2.81a}$$

$$y = \frac{a \sin \xi}{\cosh \eta - \cos \xi}, \tag{2.81b}$$

$$z = z. \tag{2.81c}$$

Dividing Eq. 2.81a by 2.81b, we obtain

$$\frac{x}{y} = \frac{\sinh \eta}{\sin \xi}. \tag{2.82}$$

Using Eq. 2.82 to eliminate ξ from Eq. 2.81a, we have

$$(x - a \coth \eta)^2 + y^2 = a^2 \operatorname{csch}^2 \eta. \tag{2.83}$$

Using Eq. 2.82 to eliminate η from Eq. 2.81b, we have

$$x^2 + (y - a \cot \xi)^2 = a^2 \csc^2 \xi. \tag{2.84}$$

From Eqs. 2.83 and 2.84 we may identify the coordinate surfaces as follows:

1. Circular cylinders, center at $y = a \cot \xi$,

$$\xi = \text{constant}, \qquad 0 \leqslant \xi \leqslant 2\pi.$$

2. Circular cylinders, center at $x = a \coth \eta$,

$$\eta = \text{constant}, \qquad -\infty < \eta < \infty.$$

3. Planes parallel to xy-plane,

$$z = \text{constant}, \qquad -\infty < z < \infty.$$

When $\eta \to \infty$, $\coth \eta \to 1$ and $\operatorname{csch} \eta \to 0$. Equation 2.83 has a solution $x = a$, $y = 0$. Similarly, when $\eta \to -\infty$, a solution is $x = -a$, $y = 0$, the circle degenerating to a point, the cylinder to a line. The family of circles (in the xy-plane) described by Eq. 2.84 passes through both of these points. This follows from noting that $x = \pm a$, $y = 0$ are solutions of Eq. 2.84 for any value of ξ.

The scale factors for the bipolar system are

$$h_1 = h_\xi = \frac{a}{\cosh \eta - \cos \xi},$$

$$h_2 = h_\eta = \frac{a}{\cosh \eta - \cos \xi}, \tag{2.85}$$

$$h_3 = h_z = 1.$$

To see how the bipolar system may be useful let us start with the three points $(a, 0)$, $(-a, 0)$, and (x, y) and the two distance vectors ρ_1 and ρ_2 at angles of θ_1

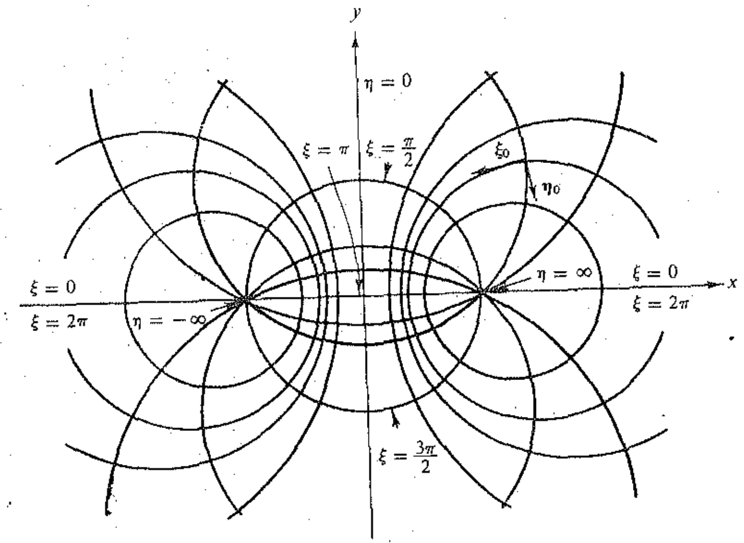


FIG. 2.7 Bipolar coordinates

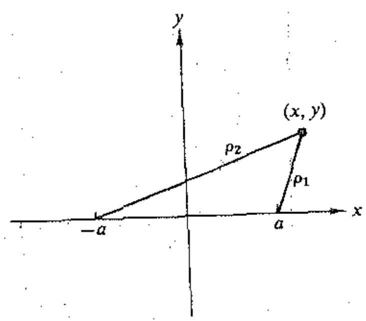


FIG. 2.8

and θ_2 from the positive x-axis. From Fig. 2.8

$$\begin{aligned}\rho_1^2 &= (x-a)^2 + y^2,\\ \rho_2^2 &= (x+a)^2 + y^2,\end{aligned} \tag{2.86}$$

and

$$\begin{aligned}\tan\theta_1 &= \frac{y}{x-a},\\ \tan\theta_2 &= \frac{y}{x+a}.\end{aligned} \tag{2.87}$$

We define[1]

$$\eta_{12} = \ln\frac{\rho_2}{\rho_1}, \tag{2.88a}$$

$$\xi_{12} = \theta_1 - \theta_2. \tag{2.88b}$$

By taking $\tan\xi_{12}$ and Eq. 2.87

$$\begin{aligned}\tan\xi_{12} &= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} \\ &= \frac{y/(x-a) - y/(x+a)}{1 + y^2/(x-a)(x+a)}.\end{aligned} \tag{2.89}$$

[1] The notation ln is used to indicate $\log_e$.

From Eq. 2.89, Eq. 2.84 follows directly. This identifies ξ as $\xi_{12} = \theta_1 - \theta_2$. Solving Eq. 2.88$a$ for ρ_2/ρ_1 and combining this with Eq. 2.86, we get

$$e^{2\eta_{12}} = \frac{\rho_2^2}{\rho_1^2} = \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \tag{2.90}$$

Multiplication by $e^{-\eta_{12}}$ and use of the definitions of hyperbolic sine and cosine produces Eq. 2.83, which identifies η as $\eta_{12} = \ln(\rho_2/\rho_1)$. The following example exploits this identification.

EXAMPLE 2.9.1

An infinitely long straight wire carries a current I in the negative z-direction. A second wire, parallel to the first, carries a current I in the positive z-direction. Using

$$d\mathbf{A} = \frac{\mu_0}{4\pi} I \frac{d\boldsymbol{\lambda}}{r}, \tag{2.91}$$

find $\mathbf{A}$, the magnetic vector potential, and $\mathbf{B}$, the magnetic inductance.

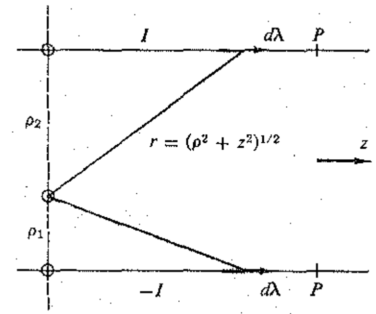


FIG. 2.9 Antiparallel electric currents

From Eq. 2.91 $\mathbf{A}$ has only a z-component. Integrating over each wire from 0 to P and taking the limit as $P \to \infty$, we obtain

$$A_z = \frac{\mu_0 I}{4\pi}\lim_{P\to\infty}\left(2\int_0^P \frac{dz}{\sqrt{\rho_2^2 + z^2}} - 2\int_0^P \frac{dz}{\sqrt{\rho_1^2 + z^2}}\right), \tag{2.92}$$

$$\begin{aligned}A_z &= \frac{\mu_0 I}{4\pi}\lim_{P\to\infty} 2\left[\ln\left(z + \sqrt{\rho_2^2 + z^2}\right)\Big|_0^P - \ln\left(z + \sqrt{\rho_1^2 + z^2}\right)\Big|_0^P\right],\\ &= \frac{\mu_0 I}{4\pi}\left(\lim_{P\to\infty} 2\ln\frac{P + \sqrt{\rho_2^2 + P^2}}{P + \sqrt{\rho_1^2 + P^2}} - 2\ln\frac{\rho_2}{\rho_1}\right).\end{aligned} \tag{2.93}$$

This reduces to

$$A_z = -\frac{\mu_0 I}{2\pi}\ln\frac{\rho_2}{\rho_1} = -\frac{\mu_0 I}{2\pi}\eta. \tag{2.94}$$

So far there has been no need for bipolar coordinates. Now, however, let us calculate the magnetic inductance $\mathbf{B}$ from $\mathbf{B} = \nabla \times \mathbf{A}$. From Eqs. 2.22 and 2.85

$$\begin{aligned}\mathbf{B} &= \frac{(\cosh\eta - \cos\xi)^2}{a^2}\begin{vmatrix} h_\xi \boldsymbol{\xi}_0 & h_\eta \boldsymbol{\eta}_0 & \mathbf{k} \\ \dfrac{\partial}{\partial\xi} & \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial z} \\ 0 & 0 & \dfrac{-\mu_0 I}{2\pi}\eta \end{vmatrix},\\ &= -\boldsymbol{\xi}_0\,\frac{(\cosh\eta - \cos\xi)}{a}\cdot\frac{\mu_0 I}{2\pi}.\end{aligned} \tag{2.95}$$

The magnetic field has only a ξ_0-component. The reader is urged to try to compute **B** in some other coordinate system.

We shall return to bipolar coordinates in Sections 2.13 and 2.14 to derive the toroidal and bispherical coordinate systems.

EXERCISES

2.9.1 Show that specifying the radius of each of two parallel cylinders and the center to center distance fixes a particular bipolar coordinate system in the sense that η_1 (first circle), η_2 (second circle) and a are uniquely determined.

2.9.2 (a) Show that Laplace's equation, $\nabla^2 \psi(\xi, \eta, z,) = 0$ is *not* completely separable in bipolar coordinates.
(b) Show that a complete separation is possible if we require that $\psi = \psi(\xi, \eta)$, that is, if we restrict ourselves to a two-dimensional system.

2.9.3 Find the capacitance per unit length of two conducting cylinders of radii b and c and of infinite length, with axes parallel and a distance d apart.

$$C = \frac{2\pi\varepsilon_0}{\eta_1 - \eta_2}.$$

2.9.4 As a limiting case of Exercise 2.9.3, find the capacitance per unit length between a conducting cylinder and a conducting infinite plane parallel to the axis of the cylinder.

$$C = \frac{2\pi\varepsilon_0}{\eta_1}.$$

2.9.5 A parallel wire wave guide (transmission line) consists of two infinitely long conducting cylinders defined by $\eta = \pm\eta_1$.
(a) Show that

$$\eta_1 = \cosh^{-1}\left\{\frac{\text{center–center distance}}{\text{cylinder diameter}}\right\}.$$

(b) From Example 2.9.1 and Exercise 2.9.3 we expect a TEM mode with electric and magnetic fields of the form

$$\mathbf{E} = \eta_0 \frac{1}{h_1} E_0 e^{i(kz-\omega t)}$$

$$\mathbf{H} = -\xi_0 \frac{1}{h_1} H_0 e^{i(kz-\omega t)}.$$

Show that $E_0 = V_0/\eta_1$ where $2V_0$ is the maximum voltage difference between the cylinders.
(c) With $H_0 = (\varepsilon_0/\mu_0)^{1/2}E_0$, show that Maxwell's equations are satisfied.
(d) By integrating the time averaged Poynting vector

$$\mathbf{P} = \tfrac{1}{2}(\mathbf{E} \times \mathbf{H}^*)$$

calculate the rate at which energy is propagated along this transmission line.

Ans. Power $= 2\pi(\varepsilon_0/\mu_0)^{1/2}(V_0^2/\eta_1)$.

2.10 Prolate Spheroidal Coordinates (u, v, φ)

Let us start with the elliptic coordinates of Section 2.7 as a two-dimensional system. We can generate a three-dimensional system by rotating about the major or minor elliptic axes and introducing φ as an azimuth angle (Fig. 2.10). Rotating first about the major axis gives us prolate spheroidal coordinates with the following coordinate surfaces:

1. Prolate spheroids,

$$u = \text{constant}, \qquad 0 \leqslant u < \infty.$$

2. Hyperboloids of two sheets,

$$v = \text{constant}, \qquad 0 \leqslant v \leqslant \pi.$$

3. Half planes through the z-axis,

$$\varphi = \text{constant}, \qquad 0 \leqslant \varphi \leqslant 2\pi.$$

The transformation equations are

$$\begin{aligned} x &= a \sinh u \sin v \cos \varphi, \\ y &= a \sinh u \sin v \sin \varphi, \\ z &= a \cosh u \cos v. \end{aligned} \tag{2.96}$$

Note that we have permuted our cartesian axes to make the axis of rotational symmetry the z-axis. The scale factors for this system are

$$\begin{aligned} h_1 = h_u &= a(\sinh^2 u + \sin^2 v)^{1/2}, \\ &= a(\cosh^2 u - \cos^2 v)^{1/2}, \\ h_2 = h_v &= a(\sinh^2 u + \sin^2 v)^{1/2}, \\ h_3 = h_\varphi &= a \sinh u \sin v. \end{aligned} \tag{2.97}$$

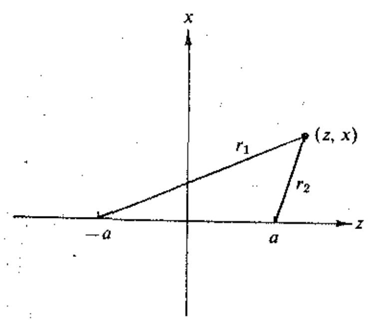


FIG. 2.11

The prolate spheroidal coordinates are rather important in physics, primarily because of their usefulness in treating "two-center" problems. The two centers will correspond to the two focal points, $(0, 0, a)$ and $(0, 0, -a)$, of the ellipsoids and hyperboloids of revolution. As shown in Fig. 2.11, we label the distance from the left focal point to the point (z, x), r_1, and the corresponding distance from the right focal point r_2.

$$r_1 + r_2 = \text{constant}, \qquad \text{for fixed } u.$$

The point (z, x) is described in terms of u and v by Eqs. 2.96. The azimuth is irrelevant here. From the properties of the

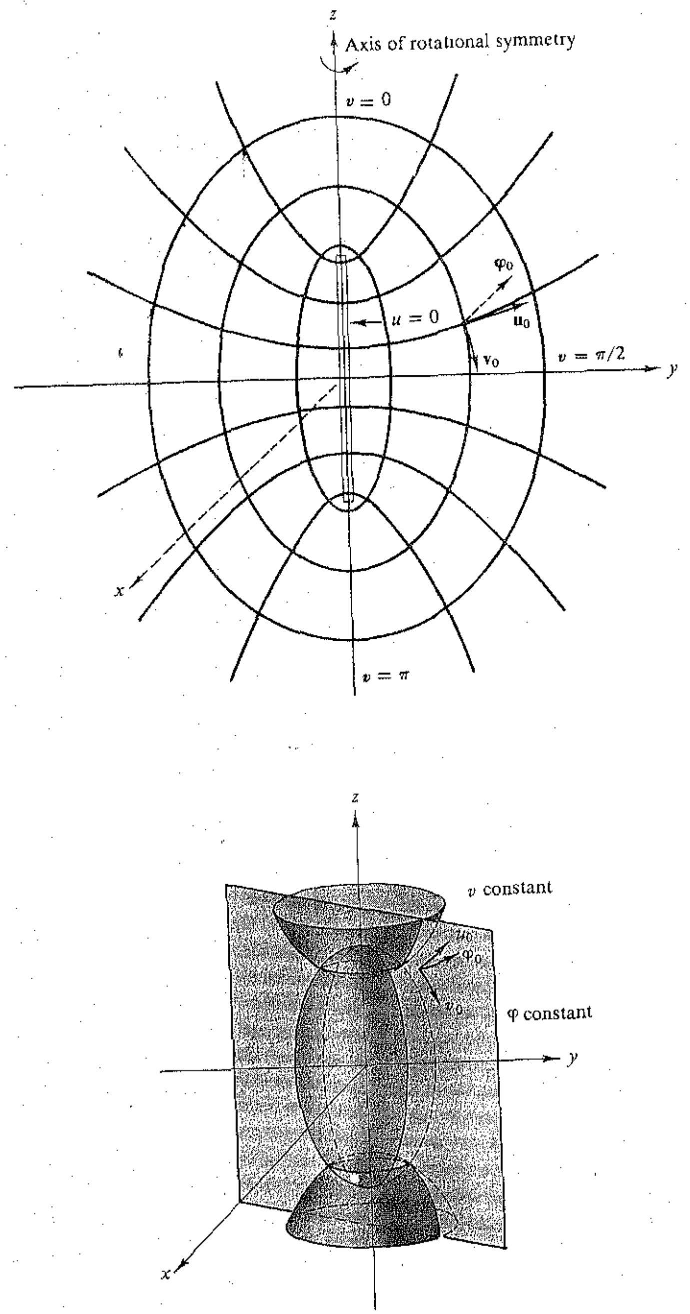


FIG. 2.10 Prolate spheroidal coordinates. (*Top*) Cross section

ellipse and hyperbola we know

$$
\begin{aligned}
r_1 + r_2 &= \text{constant}, \qquad \text{for fixed } u, \\
r_1 - r_2 &= \text{constant}, \qquad \text{for fixed } v.
\end{aligned}
\tag{2.98}
$$

Using

$$
\begin{aligned}
r_1 &= [(a + z)^2 + x^2]^{1/2}, \\
r_2 &= [(a - z)^2 + x^2]^{1/2},
\end{aligned}
\tag{2.99}
$$

and Eq. 2.96, we find

$$
\begin{aligned}
r_1 &= a(\cosh u + \cos v), \\
r_2 &= a(\cosh u - \cos v),
\end{aligned}
\tag{2.100}
$$

or

$$
\begin{aligned}
\frac{r_1 + r_2}{2a} &= \cosh u \\
\frac{r_1 - r_2}{2a} &= \cos v
\end{aligned}
\tag{2.101}
$$

This means u is a function of the sum of the distances from the two centers, whereas v is a function of the difference of the distances from the two centers.

To facilitate this application of the coordinate system we change the variables by introducing

$$
\begin{aligned}
\xi_1 &= \cosh u, \qquad 1 \leqslant \xi_1 < \infty, \\
\xi_2 &= \cos v, \qquad -1 \leqslant \xi_2 \leqslant 1, \\
\xi_3 &= \varphi, \qquad 0 \leqslant \xi_3 \leqslant 2\pi.
\end{aligned}
\tag{2.102}
$$

Note carefully that

$$
h_{\xi_1} = h_{\cosh u} \neq h_u. \tag{2.103}
$$

New variables involve new scale factors.

EXAMPLE 2.10.1

The hydrogen molecule ion is a system composed of two protons which we take to be fixed at the focal points and one electron. The Schrödinger wave equation for this system is

$$
\frac{\hbar^2}{2M}\nabla^2\psi - \frac{e^2}{r_1}\psi - \frac{e^2}{r_2}\psi + \frac{e^2}{r_{12}}\psi = E\psi. \tag{2.104}
$$

The variables r_1 and r_2 are defined in Fig. 2.11, and r_{12}, the proton-proton distance, is just $2a$. The problem is to separate the variables in Eq. 2.104.

In choosing the prolate spheroidal coordinates, ξ_1, ξ_2, ξ_3, our first step is to calculate the scale factors. From Eqs. 2.96 and 2.102

$$
h_{\xi_1} = a\left(\frac{\xi_1^2 - \xi_2^2}{\xi_1^2 - 1}\right)^{1/2}, \qquad h_{\xi_2} = a\left(\frac{\xi_1^2 - \xi_2^2}{1 - \xi_2^2}\right)^{1/2}, \tag{2.105}
$$

$$
h_{\xi_3} = a(\xi_1^2 - 1)^{1/2}(1 - \xi_2^2)^{1/2}.
$$

Using these scale factors and Eq. 2.18a, we find

$$\nabla^2\psi = \frac{1}{a^2}\left\{\frac{1}{(\xi_1^2-\xi_2^2)}\frac{\partial}{\partial\xi_1}\left[(\xi_1^2-1)\frac{\partial\psi}{\partial\xi_1}\right] + \frac{1}{(\xi_1^2-\xi_2^2)}\frac{\partial}{\partial\xi_2}\left[(1-\xi_2^2)\frac{\partial\psi}{\partial\xi_2}\right] + \frac{1}{(\xi_1^2-1)(1-\xi_2^2)}\frac{\partial^2\psi}{\partial\xi_3^2}\right\}. \tag{2.106}$$

From Eq. 2.100

$$\frac{e^2}{r_1} + \frac{e^2}{r_2} = \frac{e^2 2a\xi_1}{a^2(\xi_1^2-\xi_2^2)}. \tag{2.107}$$

By substituting Eqs. 2.106 and 2.107 into Eq. 2.104 and using the now standard procedure,

$$\psi(\xi_1, \xi_2, \xi_3) = f_1(\xi_1)\, f_2(\xi_2)\, f_3(\xi_3), \tag{2.108}$$

we can quickly isolate the azimuthal (ξ_3) dependence to obtain

$$-\frac{\hbar^2}{2Ma^2}\left\{\frac{1}{(\xi_1^2-\xi_2^2)}\frac{1}{f_1}\frac{d}{d\xi_1}\left[(\xi_1^2-1)\frac{df_1}{d\xi_1}\right] + \frac{1}{(\xi_1^2-\xi_2^2)}\frac{1}{f_2}\frac{d}{d\xi_2}\left[(1-\xi_2^2)\frac{df_2}{d\xi_2}\right]\right\} - \frac{2e^2}{a}\frac{\xi_1}{(\xi_1^2-\xi_2^2)} - E'$$
$$= \frac{\hbar^2}{2Ma^2}\frac{1}{(\xi_1^2-1)(1-\xi_2^2)}\frac{1}{f_3}\frac{d^2 f_3}{d\xi_2^2}. \tag{2.109}$$

Here we have used $E' = E - e^2/r_{12}$, a constant. As in Sections 2.5 and 2.6, we set

$$\frac{1}{f_3}\frac{d^2 f_3}{d\xi_3^2} = -m^2. \tag{2.110}$$

Equation 2.109 may be simplified to yield

$$\frac{1}{f_1}\frac{d}{d\xi_1}\left[(\xi_1^2-1)\frac{df_1}{d\xi_1}\right] + \frac{1}{f_2}\frac{d}{d\xi_2}\left[(1-\xi_2^2)\frac{df_2}{d\xi_2}\right] + \frac{4Mae^2\xi_1}{\hbar^2} + \frac{2Ma^2E'}{\hbar^2}(\xi_1^2-\xi_2^2)$$
$$= m^2\left[\frac{1}{\xi_1^2-1} + \frac{1}{1-\xi_2^2}\right]. \tag{2.111}$$

The variables ξ_1 and ξ_2 separate by inspection, and we have one second-order differential equation for $f_1(\xi_1)$ and another for $f_2(\xi_2)$.

An example of the use of prolate spheroidal coordinates in electrostatics appears in Section 12.11.

EXERCISES

2.10.1 Using $\xi = \cosh u$, $\eta = \cos v$, show that the volume element in prolate spheroidal coordinates obtained by direct transformation of

$$d\tau = a^3(\sinh^2 u + \sin^2 v)\sinh u \sin v\, du\, dv\, d\varphi$$

is

$$d\tau = -a^3(\xi^2 - \eta^2)\, d\xi\, d\eta\, d\varphi.$$

(The minus sign will be taken out by reversing the limits of integration over η.)

2.10.2 Using prolate spheroidal coordinates, set up the volume integral representing the volume of a given prolate ellipsoid using (a) u, v, φ and (b) ξ, η, φ. Evaluate the integrals and show that your results are equivalent to the usual result given in terms of the semi axes,

$$V = \frac{4}{3}\pi a_0^2 b_0,$$

where a_0 is the semiminor axis and b_0, the semimajor axis.

2.10.3 In the quantum mechanical analysis of the hydrogen molecule by the Heitler-London method we encounter the integral

$$I_{HL} = \frac{1}{\pi a_0^3}\int e^{-(r_1+r_2)/a_0}\, d\tau,$$

in which the volume integral is over all space. Introduce prolate spheroidal coordinates and evaluate the integral. *Ans.* $I_{HL} = \left(1 + \frac{2a}{a_0} + \frac{4a^2}{3a_0^2}\right)e^{-2a/a_0}$.

2.11 Oblate Spheroidal Coordinates (u, v, φ)

When the elliptic coordinates of Section 2.7 (taken as a two-dimensional set) are rotated about the minor elliptic axis, we generate another three-dimensional spheroidal system, the oblate spheroidal coordinate system. Again φ is the azimuthal angle. The coordinate surfaces are the following:

1. Oblate spheroids,
$$u = \text{constant}, \qquad 0 \leqslant u < \infty.$$
2. Hyperboloids of one sheet,
$$v = \text{constant},^1 \qquad -\frac{\pi}{2} \leqslant v \leqslant \frac{\pi}{2}.$$
3. Half planes through the z-axis,
$$\varphi = \text{constant}, \qquad 0 \leqslant \varphi \leqslant 2\pi.$$

The transformation equations relating to cartesian coordinates may be written

$$x = a\cosh u \cos v \cos\varphi$$
$$y = a\cosh u \cos v \sin\varphi$$
$$z = a\sinh u \sin v.$$

[1] Note that v has a range of only π in contrast to the range of 2π for elliptic cylindrical coordinates (Section 2.7). The negative values of v generate negative values of z.

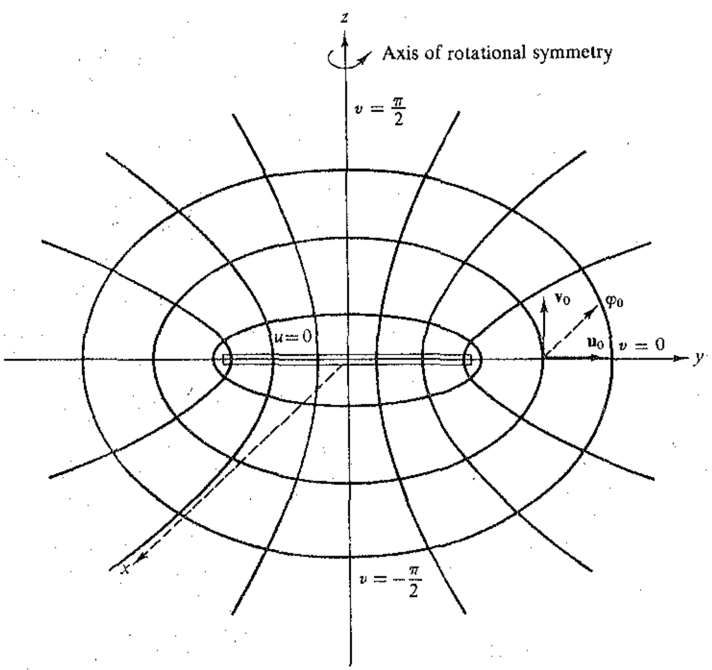


FIG. 2.12 Oblate spheroidal coordinates. Cross section

The scale factors become

$$\begin{aligned} h_1 = h_u &= a\,(\sinh^2 u + \sin^2 v)^{1/2} \\ &= a(\cosh^2 u - \cos^2 v)^{1/2}, \\ h_2 = h_v &= a(\sinh^2 u + \sin^2 v)^{1/2}, \\ h_3 = h_\varphi &= a\cosh u \cos v. \end{aligned} \tag{2.112}$$

Since holding u constant results in an oblate spheroid which is a good approximation to a planetary surface, this coordinate system has been useful in describing the earth's gravitational field. (J. P. Vinti, *Phys. Rev. Letters* **3**, 8 (1959)). Both prolate and oblate spheroidal coordinates are used in Section 12.11 to illustrate Legendre functions of the second kind.

Note carefully that if we require φ to advance from x to y as usual and if we insist on the order (u, v, φ), this system is left-handed! This will introduce an over-all (-1) in the expression for the curl. To get back to a right-handed system it is necessary to use only (v, u, φ),

$$\mathbf{v}_0 \times \mathbf{u}_0 = +\boldsymbol{\varphi}_0$$

or let $v \to (\pi/2) - v$ in the transformation equations.

EXERCISES

2.11.1 Separate Laplace's equation in oblate spheroidal coordinates. Solve the φ-dependent differential equation.

2.11.2 A thin conducting metal disk of radius a carries a total electric charge Q. Find the capacitance of the disk and the distribution of charge over the surface of the disk.

$$C = 8a\varepsilon_0,$$

$$\sigma = \frac{Q}{4\pi a\sqrt{a^2 - r^2}} \qquad \text{(on each side).}$$

2.12 Parabolic Coordinates (ξ, η, φ)

In Section 2.8 two sets of orthogonal confocal parabolas were described. Imagine that we have taken the system shown in the xy-plane (Fig. 2.6) and have rotated about the y-axis the axis of symmetry for each set of parabolas. This generates two sets of orthogonal confocal paraboloids. By permuting the coordinates (cyclically) so that the axis of rotation is the z-axis we have the following:

1. Paraboloids about the positive z-axis,

$$\xi = \text{constant}, \qquad 0 \leqslant \xi < \infty.$$

2. Paraboloids about the negative z-axis,

$$\eta = \text{constant}, \qquad 0 \leqslant \eta < \infty.$$

3. Half planes through the z-axis,

$$\varphi = \text{constant}, \qquad 0 \leqslant \varphi \leqslant 2\pi.$$

Measuring the azimuth from the x-axis in the xy-plane, as usual, we obtain

$$\begin{aligned} x &= \xi\eta\cos\varphi, \\ y &= \xi\eta\sin\varphi, \\ z &= \tfrac{1}{2}(\eta^2 - \xi^2). \end{aligned} \tag{2.113}$$

From Eq. 2.113 we find the scale factors

$$\begin{aligned} h_1 = h_\xi &= (\xi^2 + \eta^2)^{1/2}, \\ h_2 = h_\eta &= (\xi^2 + \eta^2)^{1/2}, \\ h_3 = h_\varphi &= \xi\eta. \end{aligned} \tag{2.114}$$

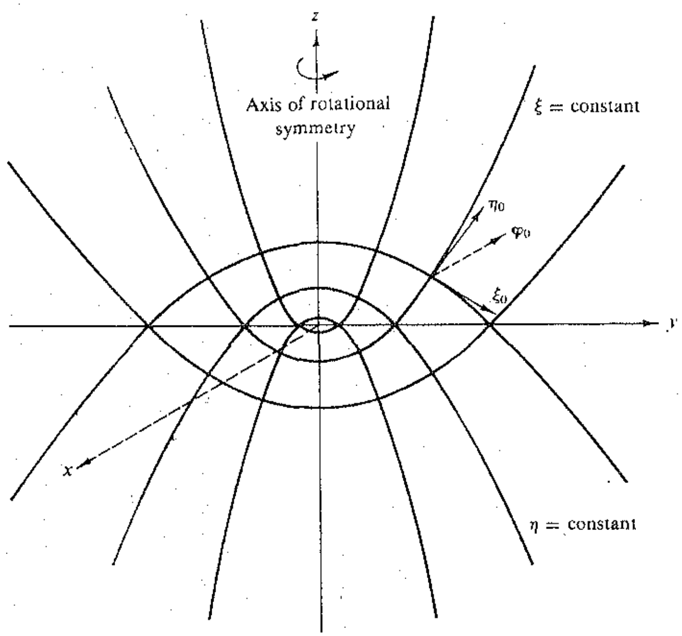


FIG. 2.13 Parabolic coordinates

From Fig. 2.13 it is seen that

$$\xi_0 \times \eta_0 = -\varphi_0,$$

that is, the parabolic system (ξ, η, φ) given here is a left-handed system. Equations 2.113 imply that ξ and η each have dimensions of (length)$^{1/2}$. For this reason some writers prefer to use $\xi^{1/2}$ in place of our ξ and $\eta^{1/2}$ in place of our η. Others have interchanged ξ and η.

The parabolic coordinates have found an application in the analysis of the Stark effect,[1] the shift of energy levels which results when an atom is placed in an *electric* field.

EXAMPLE 2.12.1 THE STARK EFFECT

The presence of the external electric field E_0, along the positive z-axis, adds a potential energy term $-eE_0 z$ to the Schrödinger wave equation. For hydrogen we have

$$-\frac{\hbar^2}{2M}\nabla^2\psi - \frac{e^2}{r}\psi - eE_0 z\psi = E\psi. \tag{2.115}$$

[1] H. A. Bethe and E. S. Salpeter. *Quantum Mechanics of One- and Two-Electron Atoms*. New York: Academic Press (1957).

Once more the problem is to separate the variables.

Using Eqs. 2.18*a* and 2.114, we obtain

$$\nabla^2\psi = \frac{1}{\xi\eta(\xi^2+\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[\xi\eta\frac{\partial\psi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\xi\eta\frac{\partial\psi}{\partial\eta}\right]\right\} + \frac{1}{\xi^2\eta^2}\frac{\partial^2\psi}{\partial\varphi^2}. \tag{2.116}$$

We also find

$$r = \frac{\xi^2+\eta^2}{2}. \tag{2.117}$$

Using Eqs. 2.116, 2.117, and $\psi = f(\xi)\,g(\eta)\,\Phi(\varphi)$, Eq. 2.115 becomes

$$\frac{\hbar^2}{2M}\frac{1}{(\xi^2+\eta^2)}\left[\frac{1}{\xi f}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) + \frac{1}{\eta g}\frac{d}{d\eta}\left(\eta\frac{dg}{d\eta}\right)\right]$$
$$+\frac{\hbar^2}{2M}\frac{1}{\xi^2\eta^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + \frac{2e^2}{\xi^2+\eta^2} + \frac{eE_0}{2}(\eta^2-\xi^2) + E = 0. \tag{2.118}$$

Setting

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2, \tag{2.119}$$

Eq. 2.118 may readily be split into the two equations:

$$\frac{\hbar^2}{2M}\left[\frac{1}{\xi f}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) - \frac{m^2}{\xi^2}\right] + E\xi^2 - \frac{eE_0\xi^4}{2} + A = 0 \tag{2.120}$$

and

$$\frac{\hbar^2}{2M}\left[\frac{1}{\eta g}\frac{d}{d\eta}\left(\eta\frac{dg}{d\eta}\right) - \frac{m^2}{\eta^2}\right] + E\eta^2 + \frac{eE_0\eta^4}{2} + B = 0. \tag{2.121}$$

The constants A and B are arbitrary except for the constraint $A + B = 2e^2$.

Other applications of parabolic coordinates are included in the problems.

EXERCISES

2.12.1 Find h_ξ^2, h_η^2, and h_φ^2 if the parabolic coordinates (ξ, η, φ) are related to the usual cartesian coordinates by

$$x = \sqrt{\xi\eta}\cos\varphi, \qquad y = \sqrt{\xi\eta}\sin\varphi,$$
$$z = \tfrac{1}{2}(\xi - \eta).$$

2.12.2 Using the ξ, η, φ defined in Ex. 2.12.1, derive the Stark effect equation corresponding to Eq. 2.120. The resulting equation appears in Ex. 8.4.11 (series solution) and 13.2.12 (Laguerre polynomials).

2.12.3 A concept of particular importance in atomic and nuclear physics is that of parity, the property of a wave function being either even or odd under inversion of the coordinates. In cartesian coordinates this inversion or parity operator P acting on (x, y, z) gives

$$P(x, y, z) = (-x, -y, -z).$$

Write out the corresponding operator equations in the following coordinate systems:
(a) Spherical polar (r, θ, φ)
(b) Circular cylindrical (ρ, φ, z)
(c) Prolate spheroidal (u, v, φ)
(d) Prolate spheroidal (ξ, η, φ)
(e) Oblate spheroidal (u, v, φ)
(f) Parabolic (ξ, η, φ)

2.12.4 (a) The wave equation for the hydrogenlike atom is

$$-\frac{\hbar^2}{2m}\nabla^2 u + Vu = Eu,$$

where V, the potential energy of the electron is

$$V = -\frac{Ze^2}{r}$$

and E is the total energy, a number. Show that variables can be separated by using parabolic coordinates.
(b) Show that the variables also separate in prolate spheroidal coordinates with the nucleus at one of the foci.

2.13 Toroidal Coordinates (ξ, η, φ)

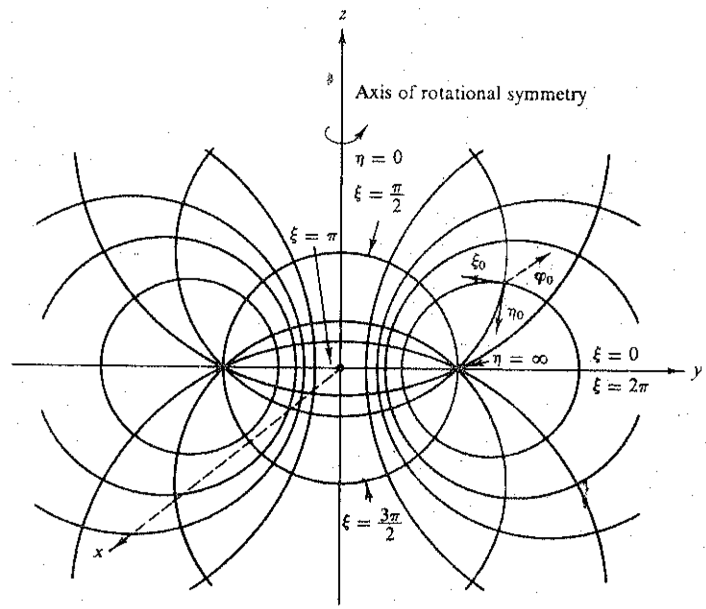
This system is formed by rotating the xy-plane of the bipolar system (Section 2.9) about the y-axis of Fig. 2.7. The circles centered on the y-axis (ξ = constant) yield spheres, whereas the circles centered on the x-axis (η = constant) form toroids. By relabeling the coordinates so that the axis of rotation is again the z-axis, the transformation equations are

$$x = \frac{a \sinh \eta \cos \varphi}{\cosh \eta - \cos \xi},$$

$$y = \frac{a \sinh \eta \sin \varphi}{\cosh \eta - \cos \xi}, \qquad (2.122)$$

$$z = \frac{a \sin \xi}{\cosh \eta - \cos \xi}.$$

From these equations the scale factors are



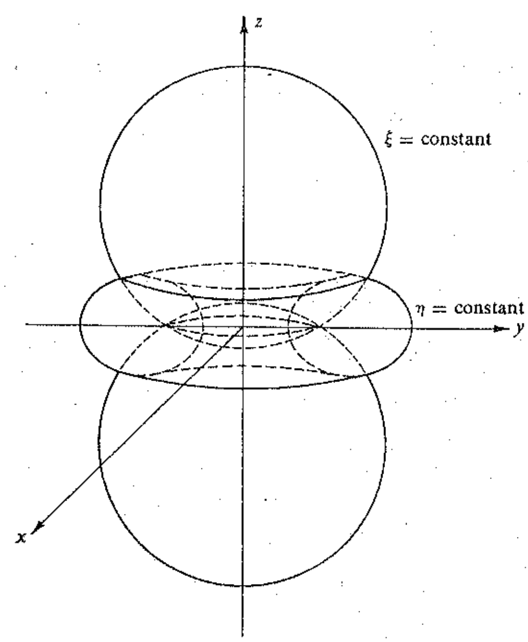


FIG. 2.14 Toroidal coordinates. (*Top*) **Cross section**

$$h_1 = h_\xi = \frac{a}{\cosh\eta - \cos\xi},$$

$$h_2 = h_\eta = \frac{a}{\cosh\eta - \cos\xi}, \tag{2.123}$$

$$h_3 = h_\varphi = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}.$$

The coordinate surfaces formed by the rotation are the following:

1. Spheres centered at $(0, 0, a\cot\xi)$ with radii, $a|\csc\xi|$,

$$\xi = \text{constant}, \qquad 0 \leqslant \xi \leqslant 2\pi.$$

$$2az\cot\xi = x^2 + y^2 + z^2 - a^2. \tag{2.124}$$

2. Toroids,

$$\eta = \text{constant}, \qquad 0 < \eta < \infty.$$

The cross sections are circles displaced a distance $a\coth\eta$ from the z-axis and of radii $a\operatorname{csch}\eta$,

$$4a^2(x^2 + y^2)\coth^2\eta = (x^2 + y^2 + z^2 + a^2)^2. \tag{2.125}$$

3. Half planes through the z-axis,

$$\varphi = \text{constant}, \qquad 0 \leqslant \varphi \leqslant 2\pi.$$

Laplace's equation is not completely separable in toroidal coordinates. This coordinate system has some physical applications (such as describing vortex rings) but they are rare and the system is seldom used.

Again, as in the two preceding sections, note that (ξ, η, φ) yields a left-handed set. To transform to a right-handed system perhaps the simplest way is to take the coordinates in the order (η, ξ, φ).

EXERCISES

2.13.1 Show that the surface area of a toroid defined by Fig. 2.15 is $(2\pi a) \times (2\pi b) = 4\pi^2 ab$.

2.13.2 As a step in solving Laplace's equation in toroidal coordinates, assume the potential $\psi(\xi, \eta, \varphi)$ to have the form

$$\psi(\xi, \eta, \varphi) = \sqrt{\cosh\eta - \cos\xi}\; X(\xi)N(\eta)\Phi(\varphi).$$

Assume further that (a) $X(\xi) = \sin n\xi, \cos n\xi$, (b) $\Phi(\varphi) = \sin m\varphi, \cos m\varphi$, with n and m integers. What is the basis for these forms for $X(\xi)$ and $\Phi(\varphi)$? Show that Laplace's

equation reduces to

$$\frac{1}{\sinh\eta}\frac{d}{d\eta}\left[\sinh\eta\,\frac{dN}{d\eta}\right] - \frac{m^2}{\sinh^2\eta}N - (n^2 - \tfrac{1}{4})N = 0.$$

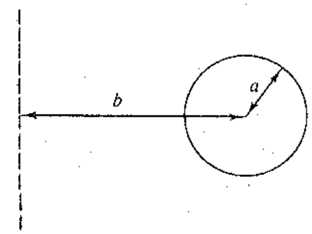


FIG. 2.15

2.14 Bispherical Coordinates (ξ, η, φ)

Returning to the bipolar coordinates of Section 2.9, a rotation of the xy-plane shown in Fig. 2.7, about the x-axis generates two families of orthogonal intersecting spheres. Adding planes of constant azimuth, this is our bispherical system with transformation equations:

$$x = \frac{a\sin\xi\cos\varphi}{\cosh\eta - \cos\xi},$$

$$y = \frac{a\sin\xi\sin\varphi}{\cosh\eta - \cos\xi}, \tag{2.126}$$

$$z = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}.$$

Once more the axis of rotation has been relabeled the z-axis. The scale factors become

$$h_1 = h_\xi = \frac{a}{\cosh\eta - \cos\xi},$$

$$h_2 = h_\eta = \frac{a}{\cosh\eta - \cos\xi}, \tag{2.127}$$

$$h_3 = h_\varphi = \frac{a\sin\xi}{\cosh\eta - \cos\xi}.$$

The coordinate surfaces are the following:

1. A fourth-order surface of revolution about the z-axis,

$$\xi = \text{constant}, \qquad 0 < \xi < \frac{\pi}{2}, \qquad \text{“dimples” on } z\text{-axis},$$

$$\xi = \frac{\pi}{2}, \qquad \text{sphere},$$

$$\frac{\pi}{2} < \xi < \pi, \qquad \text{cusps on } z\text{-axis.}$$

2. Spheres of radius $a|\operatorname{csch} \eta|$ centered at $(0, 0, a \coth \eta)$,

$$\eta = \text{constant}, \qquad -\infty < \eta < \infty.$$

3. Half planes through the z-axis,

$$\varphi = \text{constant}, \qquad 0 \leqslant \varphi \leqslant 2\pi.$$

Laplace's equation is partly separable in this system, though the general equation (2.1), $k^2 \neq 0$, is not separable. The bispherical coordinate system has been found to be useful in specialized electrostatic problems such as the capacitance between a conducting sphere and a nearby conducting plane (cf. Exercise 2.14.1).

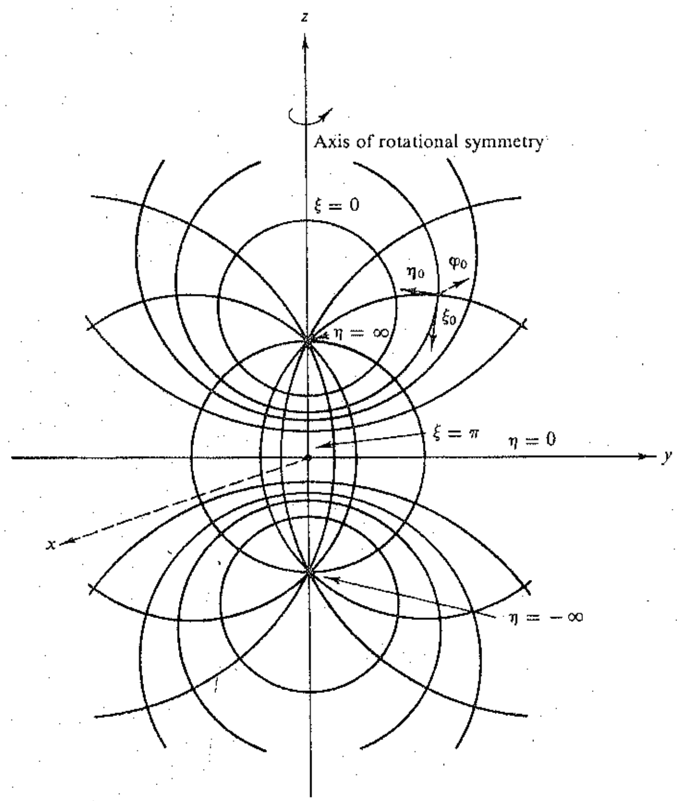


FIG. 2.16 Bispherical coordinates

EXERCISES

2.14.1 Show that Laplace's equation is separable in bispherical coordinates (to within a factor $\sqrt{\cosh \eta - \cos \xi}$).

Hint. Let $\psi(\xi, \eta, \varphi) = \sqrt{\cosh \eta - \cos \xi}\, X(\xi) N(\eta) \Phi(\varphi)$.

2.14.2 Using bispherical coordinates find the capacitance between a conducting sphere and (nonintersecting) conducting plane.

2.15 Confocal Ellipsoidal Coordinates (ξ_1, ξ_2, ξ_3)

This very general coordinate system has the following three families of coordinate surfaces:

1. Ellipsoids (no two axes are equal), $\xi_1 = \text{constant}$,

$$\frac{x^2}{a^2 - \xi_1} + \frac{y^2}{b^2 - \xi_1} + \frac{z^2}{c^2 - \xi_1} = 1. \tag{2.128}$$

2. Hyperboloids of one sheet, $\xi_2 = \text{constant}$,

$$\frac{x^2}{a^2 - \xi_2} + \frac{y^2}{b^2 - \xi_2} - \frac{z^2}{\xi_2 - c^2} = 1. \tag{2.129}$$

3. Hyperboloids of two sheets, $\xi_3 = \text{constant}$,

$$\frac{x^2}{a^2 - \xi_3} - \frac{y^2}{\xi_3 - b^2} - \frac{z^2}{\xi_3 - c^2} = 1. \tag{2.130}$$

The constants a, b, c are parameters which describe the ellipsoids and hyperboloids subject to the constraints

$$a^2 > \xi_3 > b^2 > \xi_2 > c^2 > \xi_1. \tag{2.131}$$

In Eqs. 2.128, 2.129, and 2.130, the minus signs resulting from these constraints were shown explicitly.

The transformation equations are

$$\begin{aligned} x^2 &= \frac{(a^2 - \xi_1)(a^2 - \xi_2)(a^2 - \xi_3)}{(a^2 - b^2)(a^2 - c^2)}, \\ y^2 &= \frac{(b^2 - \xi_1)(b^2 - \xi_2)(\xi_3 - b^2)}{(a^2 - b^2)(b^2 - c^2)}, \\ z^2 &= \frac{(c^2 - \xi_1)(\xi_2 - c^2)(\xi_3 - c^2)}{(a^2 - c^2)(b^2 - c^2)}. \end{aligned} \tag{2.132}$$

After an undue amount of algebra, the scale factors are found to be

$$h_1 = h_{\xi_1} = \frac{1}{2}\left[\frac{(\xi_2-\xi_1)(\xi_3-\xi_1)}{(a^2-\xi_1)(b^2-\xi_1)(c^2-\xi_1)}\right]^{1/2}$$

$$h_2 = h_{\xi_2} = \frac{1}{2}\left[\frac{(\xi_3-\xi_2)(\xi_2-\xi_1)}{(a^2-\xi_2)(b^2-\xi_2)(c^2-\xi_2)}\right]^{1/2} \tag{2.133}$$

$$h_3 = h_{\xi_3} = \frac{1}{2}\left[\frac{(\xi_3-\xi_1)(\xi_3-\xi_2)}{(a^2-\xi_3)(\xi_3-b^2)(\xi_3-c^2)}\right]^{1/2}$$

As with the equations of the coordinate surfaces, the symmetry of this set has been sacrificed by requiring each factor to be positive.

From the transformation equations (2.132) it will be seen that a given point $P(\xi_1, \xi_2, \xi_3)$ corresponds to eight possible points $(\pm x, \pm y, \pm z)$, the cartesian coordinates appearing as squares. This eightfold multiplicity may be resolved by introducing an appropriate sign convention for ξ_1, ξ_2 and ξ_3 or by bringing in elliptic functions or related functions.

Although this coordinate system has been useful in problems of mathematical physics, its very generality makes it cumbersome and awkward to use. Since this text proports to be an introduction, we shall restrict ourselves to ellipsoids with axes of rotational symmetry.

2.16 Conical Coordinates (ξ_1, ξ_2, ξ_3)

This is one of the more unusual (and less useful) degenerate forms of the confocal ellipsoidal coordinate system of the preceding section. The coordinate surfaces are the following:

1. Spheres centered at the origin, radii ξ_1, ξ_1 = constant,

$$x^2 + y^2 + z^2 = \xi_1^2. \tag{2.134}$$

2. Cones of elliptic cross section with apexes at the origin and axes along the z-axis, ξ_2 = constant,

$$\frac{x^2}{\xi_2^2} + \frac{y^2}{\xi_2^2 - b^2} = \frac{z^2}{c^2 - \xi_2^2}. \tag{2.135}$$

3. Elliptic cones, apexes at the origin, axes along the x-axis, ξ_3 = constant,

$$\frac{x^2}{\xi_3^2} = \frac{y^2}{b^2 - \xi_3^2} + \frac{z^2}{c^2 - \xi_3^2}. \tag{2.136}$$

As in Section 2.15, the parameters b and c satisfy constraints

$$c^2 > \xi_2^2 > b^2 > \xi_3^2. \tag{2.137}$$

Inverting the set of equations (2.134), (2.135), and (2.136), the transformation equations

$$x^2 = \left(\frac{\xi_1\xi_2\xi_3}{bc}\right)^2,$$

$$y^2 = \frac{\xi_1^2(\xi_2^2 - b^2)(b^2 - \xi_3^2)}{b^2(c^2 - b^2)}, \tag{2.138}$$

$$z^2 = \frac{\xi_1^2(c^2 - \xi_2^2)(c^2 - \xi_3^2)}{c^2(c^2 - b^2)}$$

are obtained. Then by Eq. 2.6 the scale factors are

$$h_1 = h_{\xi_1} = 1,$$

$$h_2 = h_{\xi_2} = \left[\frac{\xi_1^2(\xi_2^2 - \xi_3^2)}{(\xi_2^2 - b^2)(c^2 - \xi_2^2)}\right]^{1/2} \tag{2.139}$$

$$h_3 = h_{\xi_3} = \left[\frac{\xi_1^2(\xi_2^2 - \xi_3^2)}{(b^2 - \xi_3^2)(c^2 - \xi_3^2)}\right]^{1/2}$$

This really oddball coordinate system has been almost completely ignored. Recently, however, it was found useful to describe the angular momentum eigenfunctions of an asymmetric rotor.[1]

2.17 Confocal Parabolic Coordinates (ξ_r, ξ_2, ξ_3)

Except for the bipolar, toroidal, and bispherical coordinate systems, all the coordinate systems in this chapter are derivable from the confocal ellipsoidal coordinates (Section 2.15). The last of these degenerate or special systems is the confocal paraboloidal system. Here the coordinate surfaces are the following:

1. Confocal paraboloids of elliptic cross section extending along the negative z-axis, ξ_1 = constant,

$$\frac{x^2}{a^2 - \xi_1} + \frac{y^2}{b^2 - \xi_1} + 2z + \xi_1 = 0. \tag{2.140}$$

2. Hyperbolic paraboloids, ξ_2 = constant,

$$\frac{x^2}{a^2 - \xi_2} - \frac{y^2}{\xi_2 - b^2} + 2z + \xi_2 = 0. \tag{2.141}$$

3. Confocal paraboloids of elliptic cross section extending along the positive z-axis, ξ_3 = constant,

$$\frac{x^2}{\xi_3 - a^2} + \frac{y^2}{\xi_3 - b^2} - 2z - \xi_3 = 0. \tag{2.142}$$

As in Sections 2.15 and 2.16, there are constraints on the parameters and variables

$$\xi_3 > a^2 > \xi_2 > b^2 > \xi_1. \tag{2.143}$$

[1] R. D. Spence, *Am. J. Phys.* **27**, 329 (1959).

The transformation equations are

$$x^2 = \frac{(a^2 - \xi_1)(a^2 - \xi_2)(\xi_3 - a^2)}{a^2 - b^2},$$
$$y^2 = \frac{(b^2 - \xi_1)(\xi_2 - b^2)(\xi_3 - b^2)}{a^2 - b^2}, \qquad (2.144)$$
$$z = \tfrac{1}{2}(a^2 + b^2 - \xi_1 - \xi_2 - \xi_3)$$

with resulting scale factors

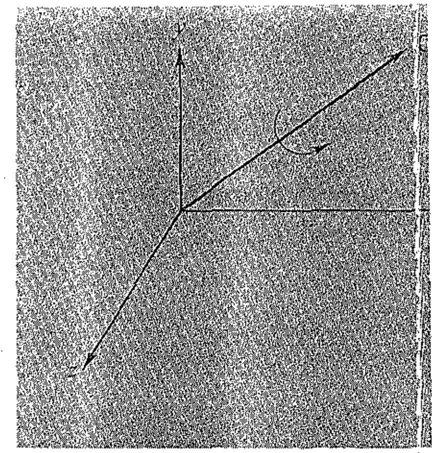
$$h_1 = h_{\xi_1} = \frac{1}{2}\left[\frac{(\xi_2 - \xi_1)(\xi_3 - \xi_1)}{(a^2 - \xi_1)(b^2 - \xi_1)}\right]^{1/2}$$
$$h_2 = h_{\xi_2} = \frac{1}{2}\left[\frac{(\xi_3 - \xi_2)(\xi_2 - \xi_1)}{(a^2 - \xi_2)(\xi_2 - b^2)}\right]^{1/2} \qquad (2.145)$$
$$h_3 = h_{\xi_3} = \frac{1}{2}\left[\frac{(\xi_3 - \xi_1)(\xi_3 - \xi_2)}{(\xi_3 - a^2)(\xi_3 - b^2)}\right]^{1/2}$$

Applications of this system have been developed in electromagnetic theory[1] but within the scope of this book the system is of little interest.

REFERENCES

MORSE, P. M., and H. FESHBACH. *Methods of Theoretical Physics*. New York: McGraw-Hill (1953). Chapter 5 includes a description of most of the coordinate systems presented here. Note carefully that Morse and Feshbach are not above using left-handed coordinate systems even for cartesian coordinates. Elsewhere in this excellent (and difficult) book are many examples of the use of the various coordinate systems in solving physical problems.

[1] J. C. Maxwell. *A Treatise on Electricity and Magnetism*. Vol. I, 3rd Ed. Oxford: Oxford University Press (1904), Chapter X.



CHAPTER 3

TENSOR ANALYSIS

3.1 Introduction, Definitions

Tensors are important in many areas of physics, including general relativity and electromagnetic theory. One of the more prolific sources of tensor quantities is the anisotropic solid. Here the elastic, optical, electrical, and magnetic properties may well involve tensors. The elastic properties of the anisotropic solid are considered in some detail in Section 3.6. As an introductory illustration, let us consider the flow of electric current. We can write Ohm's law in the usual form

$$\mathbf{J} = \sigma \mathbf{E}, \qquad (3.1)$$

with current density $\mathbf{J}$ and electric field $\mathbf{E}$, both vector quantities.[1] If we have an isotropic medium, σ, the conductivity, is a scalar, and for the x-component, for example,

$$J_1 = \sigma E_1. \qquad (3.2)$$

However, if our medium is anisotropic, as in many crystals, or a plasma in the presence of a magnetic field, the current density in the x-direction may depend on the electric fields in the y- and z-directions as well as on the field in the x-direction. Assuming a linear relationship, we must replace Eq. 3.2 with

$$J_1 = \sigma_{11} E_1 + \sigma_{12} E_2 + \sigma_{13} E_3, \qquad (3.3)$$

and, in general,

$$J_i = \sum_k \sigma_{ik} E_k. \qquad (3.4)$$

For ordinary three-dimensional space the scalar conductivity σ has given way to a set of nine elements, σ_{ik}.

[1] Another example of this type of physical equation appears in Section 4.6.