Functions of bounded variation and the Stieltjes integral

Recall that the total variation of a function \( \sigma : [a,b] \to \mathbb{R} \) is defined by

\[
V^b_a(\sigma) = \sup \sum_{n=1}^{N} |\sigma(x_n) - \sigma(x_{n-1})|
\]

where the supremum is taken over all partitions \( a = x_0 < x_1 < \ldots < x_n = b \) of \( [a,b] \).

If \( V^b_a(\sigma) < \infty \), then \( \sigma \) is said to be of bounded variation on \( [a,b] \).

Recall that \( V^b_a(\sigma) < \infty \) if and only if \( \sigma \) is the difference of two increasing functions on \( [a,b] \):

\[
\sigma_+(x) = \sigma_-(x), \quad \sigma_+(x) = \sigma_-(x) + \sigma(x), \quad \sigma_+(x) = \sigma_-(x) - \sigma(x)
\]

[In this class, \( \sigma \) is increasing if \( x < y \Rightarrow \sigma(x) \leq \sigma(y) \).]

If \( V^b_a(\sigma) < \infty \), then \( \sigma \) has left and right limits at each point \( x \in [a,b] \):

\[
\sigma(x^-) = \lim_{y \to x^-} \sigma(y), \quad \sigma(x^+) = \lim_{y \to x^+} \sigma(y)
\]

and the points of discontinuity of \( \sigma \) \( (\sigma(x^-) \neq \sigma(x^+)) \) form a countable set.
The Riemann–Stieltjes integral

For a partition $P: a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$, define $\|P\| = \max_{i=1}^{N} |x_i - x_{i-1}|$. A sampled partition $\tilde{P} = (P, \tilde{x}_1, \ldots, \tilde{x}_N)$ is defined as $\|\tilde{P}\| = \|P\|$

The Riemann–Stieltjes integral of a function $f: [a, b] \rightarrow \mathbb{C}$ (or may be replaced with any complex vector space) with respect to a function $\sigma: [a, b] \rightarrow \mathbb{R}^n$ of bounded variation is defined to be

$$\int_{a}^{b} f(x) \, d\sigma(x) := \lim_{\|\tilde{P}\| \to 0} \sum_{i=1}^{N} f(\tilde{x}_i) \left(\sigma(x_i) - \sigma(x_{i-1})\right)$$

whenever this limit over all sampled partitions exists.

The Lebesgue–Stieltjes integral

Given a function $\sigma: [a, b] \rightarrow \mathbb{R}$ that is increasing, define a measure $\mu_\sigma$ to be the Lebesgue extension of the measure $m$ defined on subintervals of $[a, b]$ by

$$m((x, \beta]) = \sigma(\beta) - \sigma(\alpha)$$
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$$m([x, \beta]) = \sigma(\beta) - \sigma(\alpha)$$
Now let $\sigma$ be of bounded variation (not necessarily increasing), and let $\sigma_+$ be the "total variation function" of $\sigma$:

$$\sigma_+ (x) = \nu^x_+ (\sigma), \; x \in [a,b]$$

and put $\sigma_- (x) = \sigma_+ (x) - \sigma (x)$. This gives the canonical decomposition of $\sigma$ into the difference of two increasing functions,

$$\sigma (x) = \sigma_+ (x) - \sigma_- (x), \; x \in [a,b]$$

(one can verify that $\sigma_- \text{ is increasing}$).

The Lebesgue–Stieltjes integral of $f : [a,b] \to \mathbb{C}$ with respect to $\sigma$ is defined as

$$\int_a^b f(x) \, d\sigma (x) := \int_a^b f \, d\mu_+ - \int_a^b f \, d\mu_-$$

for all $\mu_+$ and $\mu_-$ integrable functions, which includes Borel-measurable functions.

Fact: If $f : [a,b] \to \mathbb{C}$ is continuous, then both forms of the Stieltjes integral exist and coincide.
The Hallé selection theorem. Let $\sigma_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of uniformly bounded increasing functions, say $|\sigma_n(x)| \leq M$ \( \forall x \in [a,b] \) \( \forall n \in \mathbb{N} \). Then there exists a subsequence \( \{ \sigma_{n_j} : j \in \mathbb{N} \} \) and an increasing function $\sigma : [a,b] \to \mathbb{R}$ such that $\sigma_{n_j}(x) \to \sigma(x)$ \( j \to \infty \) \( \forall x \in [a,b] \).

Proof. Let $S$ be a countable dense subset of $[a,b]$. Thus there is a subsequence \( \{ \sigma_{n_j} : j \in \mathbb{N} \} \) such that $\sigma_{n_j}(x)$ converges for each $x \in S$; denote the limit by $\sigma(x)$.

(Proof of this is an exercise.) Define, \( \forall x \in [a,b] \),

$$\underline{\sigma}(x) := \lim_{j \to \infty} \sigma_{n_j}(x), \quad \overline{\sigma}(x) := \lim_{j \to \infty} \sup \sigma_{n_j}(x).$$

Since $\forall x \in S$, $\underline{\sigma}(x) = \sigma(x) = \overline{\sigma}(x)$ and since $S$ is dense in $[a,b]$, we find that $\underline{\sigma}(x) = \overline{\sigma}(x)$ at all points at which both $\underline{\sigma}$ and $\overline{\sigma}$ are continuous. At all such points, set $\sigma(x) = \underline{\sigma}(x) = \overline{\sigma}(x) = \lim_{j \to \infty} \sigma_{n_j}(x)$.

Set $S' = \{ x \in [a,b] : \sigma$ or $\overline{\sigma}$ is discontinuous at $x \}$. Since both $\underline{\sigma}$ and $\overline{\sigma}$ are increasing, $S'$ is countable.

Let \( \{ \sigma_{n_j} : j \in \mathbb{N} \} \) be a subsequence of \( \{ \sigma_{n_j} : j \in \mathbb{N} \} \) such that $\sigma_{n_j}(x)$ converges for each $x \in S'$, and again denote the limit by $\sigma(x)$, so that

$$\sigma(x) = \lim_{j \to \infty} \sigma_{n_j}(x) \quad \forall x \in [a,b],$$

$\sigma$ is increasing because each function $\sigma_{n_j}$ is increasing. \( \blacksquare \)
Recall the following fact (related to the Riesz representation theorem for $C[a,b]$, the space of continuous functions on $[a,b]$):

**Fact.** If $f: [a,b] \to \mathbb{C}$ is continuous and $\sigma: [a,b] \to \mathbb{R}$ is of bounded variation, then

$$\left| \int_a^b f(x) \, \sigma(x) \right| \leq V_0(\sigma) \| f \|_{L^1}. $$

The Helly convergence theorem.

**Let $\tau_n: [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions that converges pointwise to a function $\tau$, suppose that $V_0(\tau_n) \leq M \forall n \in \mathbb{N}$, and let $f: [a,b] \to \mathbb{C}$ be continuous. Then $V_0(\tau) \leq M$ and

$$\int_a^b f(x) \, \tau_n(x) \to \int_a^b f(x) \, \tau(x) \quad \text{as } n \to \infty.$$**

**Proof.** To show $V_0(\tau) \leq M$, let $P$ be any partition of $[a,b]$. Then

$$\sum_{i=1}^N |\sigma(x_i) - \tau(x_i)| = \lim_{n \to \infty} \sum_{i=1}^N |\sigma_n(x_i) - \tau_n(x_i)| \leq M,$$

and thus $V_0(\tau) \leq M$.

Let $\varepsilon > 0$ be given, and let $g: [a,b] \to \mathbb{C}$ be a step function characterized by a partition $a = x_0 < x_1 < \ldots < x_m = b$, and the numbers $g_k \in \mathbb{C}$, $k = 1, \ldots, m$, with $g(x) = g_k$ for $x_{k-1} \leq x < x_k$, $k = 1, \ldots, m$. Let the $\delta x_k$ and $\delta g_k$ be chosen such that two properties hold:
(1) $\sigma$, $\sigma_n \forall n \in \mathbb{N}$ are continuous at each $x_k$, $k \to 0, m$

(2) $\|f-g\|_{\sup} < \frac{\varepsilon}{3M}$

Property (1) is possible since each $\sigma_n$ as well as $\sigma$ has only countably many points of discontinuity. Property (2) is possible because $f$ is continuous.

Then we obtain

$$\left| \int_a^b f \, d\sigma - \int_a^b g \, d\sigma \right| \leq \int_a^b (\sigma) \|f-g\|_{\sup} < \frac{\varepsilon}{3}$$

$$\left| \int_a^b f \, d\sigma_n - \int_a^b g \, d\sigma_n \right| < \frac{\varepsilon}{3}$$

Because $\sigma_n$ and $\sigma$ are continuous at $x_k$, $k \to 0, m$

$$\int_a^b g \, d\sigma = \sum_{k=1}^{m} g(x_k) M \cdot |E(x_k, x_{k+1})| + g_m(\sigma(b) - \sigma(b-1))$$

$$= \sum_{k=1}^{m} g(x_k) (\sigma(x_k) - \sigma(x_{k-1}))$$

and likewise $\int_a^b g \, d\sigma_n = \sum_{k=1}^{m} g(x_k) (\sigma_n(x_k) - \sigma_n(x_{k-1})) \forall n \in \mathbb{N}$.

By assumption, $\sigma_n \to \sigma$ pointwise, so $\exists n_0 \text{ s.t. if } n \geq n_0 \text{ then}$

$$\left| \int_a^b g \, d\sigma - \int_a^b g \, d\sigma_n \right| < \frac{\varepsilon}{3}, \text{ and we obtain}$$

$$\left| \int_a^b f \, d\sigma - \int_a^b f \, d\sigma_n \right| < \varepsilon \forall n \geq n_0.$$