Goal in next 4 pages: Find an explicit spectral representation for the derivative operator on $L^2$.

**Multiplication operator in $L^2(\mathbb{R})$:**

Define an operator $T$ in $L^2$ by

$$\mathcal{S}(T) = \{ f \in L^2 : k f(k) \in L^2 \} = \{ f \in L^2 : \int k^2 |f(k)|^2 \, dk < \infty \}$$

$$Tf(k) = k f(k) \quad \forall f \in \mathcal{S}(T).$$

**Fact** $T$ is self-adjoint.

**Proof** To see that $T$ is self-adjoint, let $f, g \in \mathcal{S}(T)$ be given. Then

$$\langle Tf, g \rangle = \int_{-\infty}^{\infty} k f(k) \overline{g(k)} \, dk = \int_{-\infty}^{\infty} f(k) k \overline{g(k)} \, dk = \langle f, Tg \rangle.$$ 

Now we show that $\mathcal{S}(T^*) = \mathcal{S}(T)$. Let $g \in \mathcal{S}(T^*)$ be given, meaning that

$$\langle Tf, g \rangle = \|f\| \quad \forall f \in \mathcal{S}(T).$$

Since $\mathcal{S}(T)$ is dense in $L^2$, $\exists h \in L^2$ (by the Riesz Lemma) such that

$$\int_{-\infty}^{\infty} k f(k) \overline{g(k)} \, dk = \int_{-\infty}^{\infty} f(k) \overline{h(k)} \, dk,$$

or

$$\int_{-\infty}^{\infty} f(k) (k \overline{g(k)} - \overline{h(k)}) \, dk = 0 \quad \forall f \in \mathcal{S}(T).$$
Since \( S(T) \supset C_c(\mathbb{R}) \), a theorem of Lebesgue tells us that
\[
kg(k) = h(k) \quad \text{are \, i.e. \, } \mathbb{R}.
\]
This means that \( kg(k) \in L^2 \), so that \( g \in S(T) \).

Schwartz space \( S(\mathbb{R}) \)
\[
S = \{ f : \mathbb{R} \to \mathbb{C} : (1+|x|^m)2^nf(x) < C \quad \forall \, m, n \in \mathbb{N} \}.
\]

Fact: Define \( T_0 = T|_S \), that is \( S(T_0) = S \) and \( (Tf)(k) = f(k) \quad \forall \, f \in S \). Then \( T_0^* = T \).

Proof: We will prove that \( T_0^{**} = T \) (so that \( T_0^{**} \) is self-adjoint). Then it will follow that
\[
T_0 = T_0^{**} = T_0^* = T.
\]
Since \( S(T_0) = S \subseteq S(T) \), \( S(T) = S(T^*) \subseteq S(T_0^*) \).

Now suppose that \( g \in S(T_0^*) \). Then \( (T_0^*g)(k) = (f, T_0^*g) \quad \forall \, f \in S \).
Thus
\[
\sigma = \int k f(k) \overline{g(k)} \, dk - \int f(k) \overline{(T_0^*g)(k)} \, dk
= \int f(k) \overline{(kg(k) - (T_0^*g)(k))} \, dk \quad \forall \, f \in S.
\]
This implies that \( (T_0^*g)(k) = kg(k) \) in \( L^2 \), so \( g \in S(T) \) and \( T_0^*g = Tg \).
Since the closure of $Tg$ is equal to $T$, we say that $S$ is a core of $T$, and since $T$ is self-adjoint, $Tg$ is called essentially self-adjoint.

Since the Fourier transform is a unitary operator from $L^2$ to $L^2$, we infer that the closure of the differential operator $-i\partial_x$ on $S$ is self-adjoint.

$$-i\partial_x = -i\partial_x^*$$, self-adjoint on its domain.

The domain of $-i\partial_x$ is called $H^1(\mathbb{R})$, the Sobolev space of $L^2$ functions with weak derivatives in $L^2$.

Since $-i\partial_x = (-i\partial_x)^*$,

$$g \in H^1(\mathbb{R}) \iff \exists \ g^* \in L^2 \ \text{s.t.} \ \int -i\partial_x g = \int f g^* \quad \forall f \in S,$$

and $g^*$ is called the weak derivative (times $-i$) of $g$.

Note: The weak (distributional) gradient and divergence and other differential operators in the Hilbert space setting can be defined as adjoints, and their norms are the graph norm of the operator. For example,

$$||f||_{H^1(\mathbb{R})} = ||\langle f, -i\partial_x f \rangle|| = (||f||_{L^2}^2 + ||\partial_x f||_{L^2}^2)^{1/2}$$
Let us see how the Fourier transform provides a spectral representation for $-i\mathcal{D}$. For $f \in L^2$,

$$\hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

Using the rules $(-i\mathcal{D}f)^*(k) = k \hat{f}(k)$ and $(f, g) = (\hat{f}, \hat{g})$, we find

$$(-i\mathcal{D}f, g) = \int_{-\infty}^{\infty} -i\mathcal{D}f \overline{g} = \int_{-\infty}^{\infty} k \hat{f}(k) \hat{g}(k) dk = \int_{-\infty}^{\infty} k dk (\mathcal{D}, \mathcal{D}),$$

where $\mathcal{D} = \int_{-\infty}^{\infty} \hat{f}(s) \overline{g}(s) ds$

$$= \int_{-\infty}^{\infty} (\hat{f}(s) \overline{\hat{g}(s)}) ds = (E_{\mathcal{D}f, \mathcal{D}g}).$$

Thus the spectral projections for $-i\mathcal{D}$ are

$$E_k f = (\hat{f} \mathcal{P}_k)^* = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s) e^{iks} ds$$