For essentials on the Fourier transform, see Folland Ch. 0. A pdf copy is available on the course website.

Goal in next 4 pages: Find an explicit spectral representation for the derivative operator on $\mathbb{R}$.

Multiplication operator in $L^2(\mathbb{R})$

Define an operator T in L^2 by

$$\begin{cases} \mathcal{D}(T) = \{ f \in L^2 : kf(k) \in L^2 \} = \{ f \in L^2 : \int k^2 |f(k)|^2 dk < \infty \} \\ (Tf)(k) = kf(k) \quad \forall \; f \in \mathcal{D}(T). \end{cases}$$

Fact T is self-adjoint.

Proof To see that T is symmetric, let $f, g \in \mathcal{D}(T)$ be given. Then

$$(Tf, g) = \int_{-\infty}^{\infty} kf(k)\overline{g(k)}\, dk = \int_{-\infty}^{\infty} f(k)\overline{kg(k)}\, dk = (f, Tg).$$

Now we show that $\mathcal{D}(T^*) = \mathcal{D}(T)$. Let $g \in \mathcal{D}(T^*)$ be given, meaning that

$$(Tf, g) < C\|f\| \quad \forall \; f \in \mathcal{D}(T).$$

Since $\mathcal{D}(T)$ is dense in L^2, $\exists \; h \in L^2$ (by the Riesz Lemma) such that

$$\int_{-\infty}^{\infty} kf(k)\overline{g(k)}\, dk = \int_{-\infty}^{\infty} f(k)\overline{h(k)}\, dk,$$

or $$\int_{-\infty}^{\infty} f(k)\left(\overline{kg(k)} - \overline{h(k)}\right) dk = 0 \quad \forall \; f \in \mathcal{D}(T).$$

Since $\mathcal{D}(T) \supset C_c^\infty(\mathbb{R})$, a theorem of Lebesgue tells us that

$$kg(k) = h(k) \quad \text{a.e. } \mathbb{R}.$$

This means that $kg(k) \in L^2$, so that $g \in \mathcal{D}(T)$. ∎

<u>Schwartz space $\mathcal{S}(\mathbb{R})$</u>

$$\mathcal{S} = \{ f : \mathbb{R} \to \mathbb{C} : (1+|x|)^n \partial^m f(x) < C \quad \forall\, m,n \in \mathbb{N} \}$$

<u>Fact</u> Define $T_0 = T|_{\mathcal{S}}$, that is $\mathcal{D}(T_0) = \mathcal{S}$ and $(Tf)(k) = f(k)$ $\forall f \in \mathcal{S}$. Then $\overline{T_0} = T$.

<u>Proof</u> We will prove that $T_0^* = T$ (so that T_0^* is self-adjoint). Then it will follow that

$$\overline{T_0} = T_0^{**} = T_0^* = T.$$

Since $\mathcal{D}(\overline{T_0}) = \mathcal{S} \subset \mathcal{D}(T)$, $\mathcal{D}(T) = \mathcal{D}(T^*) \subset \mathcal{D}(T_0^*)$.

Now suppose that $g \in \mathcal{D}(T_0^*)$. Then $(T_0 f, g) = (f, T_0^* g)$ $\forall f \in \mathcal{S}$. Thus

$$0 = \int k f(k) \overline{g(k)}\, dk - \int f(k) \overline{(T_0^* g)(k)}\, dk$$

$$= \int f(k) \left(\overline{kg(k)} - \overline{(T_0^* g)(k)} \right) dk \quad \forall f \in \mathcal{S}.$$

This implies that $(T_0^* g)(k) = kg(k)$ in L^2, so $g \in \mathcal{D}(T)$ and $T_0^* g = Tg$. ∎

Since the closure of $T|_{\mathcal{S}}$ is equal to T, we say that $\mathcal{S}$ is a core of T, and since T is self-adjoint, $T|_{\mathcal{S}}$ is called essentially self-adjoint.

on L^2 as well as on $\mathcal{S}$ in the L^2-norm

Since the Fourier transform is a unitary operator from L^2 to L^2, we infer that the closure of the differential operator $-i\partial_0$ on $\mathcal{S}$ is self-adjoint.

$$\overline{-i\partial_0} =: -i\partial, \text{ self-adjoint on its domain.}$$

The domain of $-i\partial$ is called $H^1(\mathbb{R})$, the Sobolev space of L^2 functions with weak derivatives in L^2. Since $-i\partial = (-i\partial_0)^*$,

$$g \in H^1(\mathbb{R}) \iff \exists\ g^* \in L^2 \text{ s.th.}$$

$$\int -i\partial f\,\overline{g} = \int f\,\overline{g^*} \quad \forall f \in \mathcal{S},$$

and g^* is called the weak derivative (times $-i$) of g.

Note The weak (distributional) gradient and divergence and other differential operators in the Hilbert space setting can be defined as adjoints, and their norms are the graph norm of the operator. For example,

$$\|f\|_{H^1(\mathbb{R})} = \|\langle f, -i\partial f\rangle\| = \left(\|f\|_{L^2}^2 + \|\partial f\|_{L^2}^2\right)^{1/2}$$

Let us see how the Fourier transform provides a spectral representation for $-i\partial$. For $f \in L^2$,

$\mathcal{F}f = \hat{f}$

$$\begin{cases} \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \\ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk \end{cases}$$

Using the rules $(-i\partial f)^\wedge(k) = k\hat{f}(k)$
and $(f, g) = (\hat{f}, \hat{g})$, we find

$$(-i\partial f, g) = \int_{-\infty}^{\infty} -i\partial f \, \bar{g} = \int_{-\infty}^{\infty} k \hat{f}(k) \overline{\hat{g}(k)} \, dk = \int_{-\infty}^{\infty} k \, d\sigma(k; f, g),$$

where $\sigma(k; f, g) = \int_{-\infty}^{k} \hat{f}(s) \overline{\hat{g}(s)} \, ds$

$$= \int_{-\infty}^{\infty} \left(\hat{f}(s) 1_{-\infty}^{k}(s)\right) \overline{\hat{g}(s)} \, ds = (E_k f, g).$$

Thus the spectral projections for $-i\partial$ are

$$\boxed{E_k f = \left(\hat{f} 1_{-\infty}^{k}\right)^\vee = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k} \hat{f}(s) e^{isx} \, ds}$$