A simple waveguide with an obstacle.

We will develop the spectral theory for the wave equation in an infinite string with a localized defect that acts as a scatterer of waves.

The main arguments will be outlined first, omitting technical lemmas in order to elucidate the essential ideas.

1. Undistorted waves. The wave equation in a uniform string

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

admits harmonic solutions of the form \( u = e^{i(kx - \omega t)} \), where \( \omega = ct \). The (normalized) spatial factor

\[ w_0(x; k) = \frac{1}{\sqrt{2\pi}} e^{ikx} \]

is a generalized eigenfunction of the positive "propagator"

\[ -c^2 \frac{\partial^2}{\partial x^2} - c^2 w_0 = (ck)^2 w_0 \quad \text{or} \]

\[ (-c^2 \frac{\partial^2}{\partial x^2} - k^2) w_0(x; k) = 0 \]

The usual Fourier transform is based on these undisturbed wave functions. For \( \psi \in L^2(\mathbb{R}) \),

\[ \hat{\psi}(k) := (\mathcal{F} \psi)(k) = \int \psi(x) \overline{w_0(x; k)} \, dx \].
2. Distorted waves due to an obstacle in the waveguide. Let the density and rigidity of the string be modified in a finite region $[-L, L]$:

$$\rho \ddot{u} + \sigma \dot{u} = c^2 \partial_x (\sigma \partial_x u),$$

$$\begin{cases} 0 < \sigma_- \leq \sigma(x) = \rho'(x) \leq \sigma_+ , \\ \sigma(x) = 1 \text{ for } |x| > L. \end{cases} \text{ (the obstacle in } [-L, L])$$

Positive

The new propagator is the operator $-\sigma(\partial_x \partial_x)$. We seek generalized eigenfunctions $W(x;jk)$ that modify the functions $w_0(x;jk)$ in a prescribed way:

$$-\sigma(\tau w')' - k^2 w = 0$$

$$W(x;jk) = W_0(x;jk) + w(x;jk) \quad \text{(distorted waves)}$$

$$u(x;jk) = \begin{cases} u_+ e^{ikx} & \text{for } x > L \\ u_- e^{-ikx} & \text{for } x < -L \end{cases} \quad \text{(radiating condition)}$$

These wave functions $W(x;jk)$ are called distorted wave functions or scattering fields because an "incident field" $w_0(x;jk)e^{-iwt}$ is scattered by the obstacle, producing a diffracted field that radiates away from the obstacle to $\pm \infty$. We take $w = c\ddot{u}$ in the time dynamics.
For $k > 0$, $W(x, k)$ is interpreted as the field produced by the scattering by the obstacle of a harmonic wave produced by an oscillatory source at $x = -\infty$; and for $k < 0$, $W(x, k)$ represents scattering of a field emanating from the right:

$$k > 0 : \sqrt{2\pi} W(x, k) = \begin{cases} (e^{ikL} + re^{-ikL})e^{-iw} & , x < -L \\ te^{-ikL}e^{-iw} & , x > L \end{cases}$$

$$k < 0 : \sqrt{2\pi} W(x, k) = \begin{cases} te^{-ikL}e^{-iw} & , x < -L \\ (e^{-ikL} + re^{ikL})e^{-iw} & , x > L \end{cases}$$

The coefficients $r$ and $t$ refer to the reflected and transmitted part of the incident wave $e^{ikx}$.

That the scattering problem (4) has a unique solution can be shown by means of ODE theory or within the context of Sobolev space theory and coercive forms. We leave this for later. For now, observe that the scattered field $W$ satisfies

\[ -\sigma(\tau w')' - k^2 w = \sigma(\tau w_0')' + ikw_0 \]

\[ = (\sigma(\tau w_0')' - w_0'') X_{[L,-L]} = F[w_0](x) \]

in which the left-hand side is viewed as a forcing function imposed by the incident field $w_0(x, k)$. This forcing is confined to the obstacle in $[-L, L]$.  

\(\ddots\)
Distorted Fourier transform relative to scattering states.

For \( \psi \in L^2(\mathbb{R}) \), define
\[
\hat{\psi}_\lambda(x) := \int \psi(x) \bar{w}(x; \lambda) \, dx.
\]

(1) This is a bounded operator from \( L^2 \) to \( L^2 \) and is unitarily equivalent to the usual Fourier transform.

3. Modified scattering states (modified distorted waves)

For \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda \geq 0 \), define the fields \( w(x; \lambda, \lambda) \) by

\[ w(x; \lambda, \lambda) = w_0(x; \lambda) + u(x; \lambda, \lambda) \]

\[ u(x; \lambda, \lambda) = \begin{cases} u_+ e^{i \sqrt{\lambda} x} & \text{for } x > L \\ u_- e^{-i \sqrt{\lambda} x} & \text{for } x < -L \end{cases} \]

in which \( \sqrt{\lambda} \) is in the first quadrant. If \( \text{Im} \lambda > 0 \), \( u(x; \lambda, \lambda) \to 0 \) as \( |x| \to \infty \). This problem has a unique solution.

The importance of \( w(x; \lambda, \lambda) \) lies in the equation

\[
(-\sigma(\partial_x \partial_x) - \lambda) w(x; \lambda, \lambda) = (k^2 - \lambda) w_0(x; \lambda, \lambda).
\]
Modified distorted Fourier transform:

For \( \psi \in L^2(\mathbb{R}) \),

\[
\hat{\psi}_1(k, \lambda) = \int \psi(x) \overline{w(x, ik, \lambda)} \, dx
\]

4. Formula for the resolvent of \(-\sigma \partial^2\partial_\tau\)

Because of (*) we have

\[
(-\sigma \partial^2 \partial_\tau - \lambda) \frac{w(xsk, \lambda)}{k^2 - \lambda} = w_0(x, ik)
\]

Thus,

\[
(-\sigma \partial^2 \partial_\tau - \lambda) \int \frac{\hat{\psi}_0(k)}{k^2 - \lambda} \, w(xsk, \lambda) \, dk
\]

\[
\frac{1}{\lambda} \int \hat{\psi}_0(k) \, (-\sigma \partial^2 \partial_\tau - \lambda) \, w(xsk, \lambda) \, dk = \psi(x)
\]

This means that

\[
(R_\lambda \psi)(x) = \int \frac{\hat{\psi}_0(k)}{k^2 - \lambda} \, w(xsk, \lambda) \, dk
\]

Next, we find the Fourier transform of \( R_\lambda \psi \).
Compute \((R_\lambda \varphi)_0(k) = \frac{\varphi(\lambda k)}{\lambda^2 - \lambda} \) for \( \lambda \mathbf{n} \lambda < 0 \)

Let \( \varphi \in L^2(\mathbb{R}) \) be given:

\[
(R_\lambda \varphi, \varphi) = (\varphi, R_\lambda \varphi) = \int \varphi(x) (R_\lambda \varphi)(x) \, dx
\]

\[
= \int \varphi(x) \left( \frac{\varphi_0(k)}{\lambda^2 - \lambda} \right) \overline{\varphi}(x) \, dx \, dk
\]

\[
= \int \frac{\varphi_0(k)}{\lambda^2 - \lambda} \int \varphi(x) \overline{\varphi}(x) \, dx \, dk
\]

\[
= \int \frac{\varphi_0(k)}{\lambda^2 - \lambda} \varphi(k, \lambda) \, dk = \left( \frac{\varphi(\lambda k)}{\lambda^2 - \lambda} \right) \varphi_0(k)
\]

By the unitarity of the Fourier transform,

\[
\sqrt{(R_\lambda \varphi)_0^*(k)} = \frac{\varphi(k, \lambda)}{\lambda^2 - \lambda}
\]

5. Obtain the spectral measures for \(-\infty < \lambda < \infty\).

First, we derive a formula for the function \( \sigma(t, \varphi) \)

Let \( R_\lambda \) be the resolvent of a self-adjoint operator, and let \( a < b \) be given:

\[
\lim_{\epsilon \to 0} \int_a^b \| R_{\lambda - \epsilon i \tau} \varphi \|^2 \, d\tau = \lim_{\epsilon \to 0} \int_a^b \int_{-\infty}^{\infty} \frac{1}{\epsilon - (z \pm i \epsilon)} d(E_{\lambda \pm \epsilon} R_{\lambda - \epsilon i \tau}) \, dz \, d\tau
\]
\[ \lim_{\varepsilon \to 0} \int_{a}^{b} \varepsilon \left( \frac{1}{t-(\tau \pm \varepsilon)} \right) d\left( R_{\tau \pm \varepsilon} E_{\psi}, \chi \right) dt \]

\[ = \lim_{\varepsilon \to 0} \int_{a}^{b} \varepsilon \left( \frac{1}{t-(\tau \pm \varepsilon)} \right) d\int_{t-(\tau \pm \varepsilon)}^{\infty} \frac{1}{s-(\tau \pm \varepsilon)} d\left( E_{\chi} E_{\psi}, \chi \right) ds \]

\[ = \lim_{\varepsilon \to 0} \int_{a}^{b} \varepsilon \left( \frac{1}{t-(\tau \pm \varepsilon)} \right) d\int_{t-(\tau \pm \varepsilon)}^{\infty} \frac{1}{s-(\tau \pm \varepsilon)} d\left( E_{\chi} E_{\psi}', \chi \right) ds \\
\]

\[ = \lim_{\varepsilon \to 0} \int_{a}^{b} \varepsilon \left( \frac{1}{t-(\tau \pm \varepsilon)^2} \right) d\left( E_{\chi} E_{\psi}, \chi \right) dt \\
\]

\[ = \lim_{\varepsilon \to 0} \int_{a}^{b} \varepsilon \left( \frac{1}{(t-\tau)^2 + \varepsilon^2} \right) d\sigma(t', \psi) dt \\
\]

\[ = \int_{a}^{b} \lim_{\varepsilon \to 0} \varepsilon \left( \frac{1}{(t-\tau)^2 + \varepsilon^2} \right) d\sigma(t', \psi) dt \\
\]

\[ = \int_{a}^{b} \left( \pi \chi_{a,b} + \frac{\pi}{2} \left( \chi_{a,b} + \chi_{b,a} \right) \right) d\sigma(t', \psi) \\
\]

\[ = \frac{\pi}{2} \left[ \sigma(b, 0; \psi) + \sigma(b, 0; \psi) - \sigma(a, 0; \psi) \right] \\
\]

\[ = \left\{ \begin{array}{ll} \\
\end{array} \right. \\
\]

\[ = \Pi \left( \sigma(b, \psi) - \sigma(a, \psi) \right) \]

\[ \text{if } \sigma \text{ is continuous} \]

\[ \text{av. of } \Pi \text{ for limits} \]

\[ (x) \quad \sum_{x \in \mathbb{R}} \left\{ \int_{a}^{b} \varepsilon \left( \frac{1}{(t-\tau)^2 + \varepsilon^2} \right) d\sigma(t', \psi) \right\} dt \]

\[ = \Pi \left( \sigma(b, \psi) - \sigma(a, \psi) \right) \]
\[ y(x) = 5z \]

So with \( l^2 = 2 \), we have \( \psi(t) = \frac{1}{4} \int_{0}^{\pi} (\tilde{e}(t))^{2} dt \).

Since \( l^2 \geq 0 \), we may write \( a = 0 \) and \( b = 0 \).

Now use the fact (needs to be proved) that \( \lim_{t \to 0} \frac{\tilde{e}(t)}{l^2} \) is continuous in \( t \) and is \( \lesssim c \) for \( s \geq 0 \), and that

\[ \lim_{t \to 0} \int \frac{\tilde{e}(t)}{l^2} dt = \lim_{t \to 0} \int \frac{\tilde{e}(t)}{l^2} \] by the formula \((R \tilde{e})^{2}(t) = \frac{\tilde{e}(t)}{l^2} - \frac{\tilde{e}(t)}{l^2} \).

Now, using the formula \((R \tilde{e})^{2}(t) = \frac{\tilde{e}(t)}{l^2} - \frac{\tilde{e}(t)}{l^2} \), we compute

\[ \lim_{t \to 0} \int \frac{\tilde{e}(t)}{l^2} dt = \lim_{t \to 0} \int \frac{\tilde{e}(t)}{l^2} \]
Le. Letting $a \rightarrow 0$ and $b \rightarrow \infty$, we obtain

$$\int_{-\infty}^{\infty} |\hat{e}_i(k)|^2 \, dk = \int_{\infty}^{\infty} d\sigma(\lambda, \varepsilon, \varepsilon) = (e_i, e_i)$$

so that $\varepsilon \mapsto \vec{\varepsilon}$ preserves the norm. In fact, this map is invertible, as we now show.

From the result

$$\int_{a \leq k^2 \leq b} |\hat{v}_i(k)|^2 \, dk = \int_a^b d\sigma(\lambda, \varepsilon, \varepsilon) = ((E_b - E_a)\varepsilon, \varepsilon)$$

and the polarization procedure, we obtain

$$((E_b - E_a)f, \psi) = \int_{a \leq k^2 \leq b} \hat{f}_i(k) \hat{\psi}(k) \, dk \quad \text{for} \quad \psi \in C^\infty_c(\mathbb{R})$$

$$= \int_{a \leq k^2 \leq b} \hat{f}_x(k) \int_{-\infty}^{\infty} \overline{\psi(x)} w(x; k) \, dx \, dk$$

$$= \int_{-\infty}^{\infty} \overline{\psi(x)} \int_{a \leq k^2 \leq b} \hat{f}_i(k) w(x; k) \, dk \, dx$$

$$\Rightarrow (E_b - E_a)f(x) = \int_{a \leq k^2 \leq b} \hat{f}_i(k) w(x; k) \, dk$$

Taking $a \rightarrow 0$ and $b \rightarrow \infty$, we obtain

$$f(x) = \int_{-\infty}^{\infty} \hat{f}_i(k) w(x; k) \, dk$$