A simple <u>waveguide with an obstacle</u>.

We will develop the spectral theory for the wave equation in an infinite string with a localized defect that acts as a scatterer of waves.

The main arguments will be outlined first, omitting technical lemmas in order to elucidate the essential ideas.

1. <u>Undistorted waves</u>. The wave equation in a uniform string

$$\partial_{tt} u = c^2 \partial_{xx} u$$

admits harmonic solutions of the form $u = e^{i(kx - \omega t)}$, where $\omega = ct$. The (normalized) spatial factor

$$w_0(x;k) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

is a <u>generalized eigenfunction</u> of the positive "propagator" $-c^2\partial_{xx}$: $-c^2\partial_{xx} w_0 = (ck)^2 w_0$, or

ex $$\left(-\partial_{xx} - k^2\right) w_0(x;k) = 0$$

The usual Fourier transform is based on these undisturbed wave functions. For $\psi \in L^2(\mathbb{R})$,

$$\hat{\psi}_0(k) := (\mathcal{F}\psi)(k) = \int \psi(x)\, \bar{w}_0(x;k)\, dx .$$

2. <u>Distorted waves</u> due to an obstacle in the waveguide.
Let the density and rigidity of the string be modified in a finite region $[-L, L]$:

$$\rho\, \partial_{tt} u = c^2\, \partial_x(\tau\, \partial_x u),$$

$$\begin{cases} 0 < \sigma_- \le \sigma(x) = \rho^{-1}(x) \le \sigma_+ , \\ \sigma(x) = 1 \quad \text{for } |x| > L , \end{cases}$$

(the obstacle in $[-L, L]$)

$$\begin{cases} 0 < \tau_- \le \tau(x) \le \tau_+ , \\ \tau(x) = 1 \quad \text{for } |x| > L . \end{cases}$$

The new propagator is the positive operator $-\sigma(\partial_x \tau\, \partial_x)$.
We seek generalized eigenfunctions $w(x;k)$ that modify the functions $w_0(x;k)$ in a prescribed way:

(+)
$$-\sigma(\tau w')' - k^2 w = 0$$
$$w(x;k) = w_0(x,k) + u(x;k) \quad \text{(distorted waves)}$$
$$u(x;k) = \begin{cases} u_+ e^{i|k|x} & \text{for } x > L \\ u_- e^{-i|k|x} & \text{for } x < -L \end{cases} \quad \text{(radiating condition)}$$

These wave functions $w(x;k)$ are called <u>distorted wave functions</u> or <u>scattering fields</u> because an "incident field" $w_0(x;k)e^{-i\omega t}$ is scattered by the obstacle, producing a diffracted field that <u>radiates</u> away from the obstacle to $\pm\infty$. We take $\omega = c|k|$ in the time dynamics.

For $k>0$, $w(x,k)$ is interpreted as the field produced by the scattering by the obstacle of a harmonic wave produced by an oscillatory source at $x=-\infty$; and for $k<0$, $w(x,k)$ represents scattering of a field emanating from the right:

$$k>0: \quad \sqrt{2\pi}\, w(x;k) = \begin{cases} \left(e^{i|k|x} + r e^{-i|k|x}\right) e^{-i\omega t}, & x<-L \\ t e^{i|k|x} e^{-i\omega t}, & x>L \end{cases}$$

$$k<0: \quad \sqrt{2\pi}\, w(x;k) = \begin{cases} t e^{-i|k|x} e^{-i\omega t} & x<-L \\ \left(e^{-i|k|x} + r e^{i|k|x}\right) e^{-i\omega t}, & x>L. \end{cases}$$

The coefficients r and t refer to the <u>reflected</u> and <u>transmitted</u> part of the <u>incident</u> wave e^{ikx}.

(!) That the scattering problem (†) has a <u>unique solution</u> can be shown by means of ODE theory or within the context of Sobolev space theory and coercive forms. We leave this for later. For now, observe that the <u>scattered field</u> u satisfies

induced forcing $F[w_0](x)$

$$-\sigma(\tau u')' - k^2 u = \sigma(\tau w_0')' + k^2 w_0$$
$$= \left(\sigma(\tau w_0')' - w_0''\right) \chi_{[-L,L]} =: F[w_0](x)$$

in which the left-hand side is viewed as a forcing function imposed by the incident field $w_0(x,k)$. This <u>forcing is confined to the obstacle</u> in $[-L,L]$.

<u>Distorted Fourier transform</u> relative to scattering states.

For $\psi \in L^2(\mathbb{R})$, define

$$\hat{\psi}_1(k) := \int \psi(x)\,\bar{w}(x;k)\,dx$$

(!) This is a bounded operator from L^2 to L^2 and is unitarily equivalent to the usual Fourier transform

3. <u>Modified scattering states</u> (<u>modified distorted waves</u>)

For $\lambda \in \mathbb{C}$ with $\Im m\lambda \geq 0$, define the fields $w(x;k,\lambda)$ by

λ in place of k^2

$$-\sigma(\tau u')' - \lambda u = F[w_0] \quad \left(= \sigma(\tau w_0')' - w_0''\right)$$

$$w(x;k,\lambda) = w_0(x;k) + u(x;k,\lambda)$$

$$u(x;k,\lambda) = \begin{cases} u_+ e^{i\sqrt{\lambda}x} & \text{for } x > L \\ u_- e^{-i\sqrt{\lambda}x} & \text{for } x < -L \end{cases},$$

in which $\sqrt{\lambda}$ is in the closed first quadrant. If $\Im m\lambda > 0$, $u(x;k,\lambda) \to 0$ as $|x| \to \infty$. This

(!) problem has a <u>unique solution</u>.

The importance of $w(x;k,\lambda)$ lies in the equation

$$(*) \longrightarrow \quad \left(-\sigma(\partial_x \tau \partial_x) - \lambda\right) w(x;k,\lambda) = (k^2 - \lambda)\, w_0(x;k).$$

Modified distorted Fourier transform:

For $\psi \in L^2(\mathbb{R})$,

$$\hat{\psi}_1(k,\lambda) := \int \psi(x)\, \overline{w}(x;k,\lambda)\, dx$$

4. Formula for the resolvent of $-\sigma\partial\tau\partial$

Because of $(*)$, we have

$$(-\sigma\partial\tau\partial - \lambda)\frac{w(x;k,\lambda)}{k^2-\lambda} = w_0(x;k)$$

Thus,

$$(-\sigma\partial\tau\partial - \lambda)\int \frac{\hat{\psi}_0(k)}{k^2-\lambda} w(x;k,\lambda)\, dk$$

(!) needs to be justified

$$\overset{!}{=} \int \hat{\psi}_0(k)\, \frac{(-\sigma\partial\tau\partial-\lambda)\, w(x;k,\lambda)}{k^2-\lambda}\, dk$$

$$= \int \hat{\psi}_0(k)\, w_0(x;k)\, dk = \psi(x).$$

This means that

$$\checkmark \quad (R_\lambda\psi)(x) = \int \frac{\hat{\psi}_0(k)}{k^2-\lambda} w(x;k,\lambda)\, dk.$$

Next, we find the Fourier transform of $R_{\bar{\lambda}}\psi$.

52

Compute $(R_{\bar{\lambda}}\varphi)^{\wedge}_{0}(k) = \dfrac{\hat{\varphi}_1(k,\lambda)}{k^2-\bar{\lambda}}$ for $\operatorname{Im}\lambda < 0$

Let $\psi \in L^2(\mathbb{R})$ be given:

$$(R_{\bar{\lambda}}\varphi, \psi) = (\varphi, R_\lambda \psi) = \int \varphi(x)\overline{(R_\lambda\psi)(x)}\,dx$$

(!)

$$= \int \varphi(x) \int \overline{\frac{\hat{\psi}_0(k)}{k^2-\lambda}}\, \bar{w}(x;k,\lambda)\,dk\,dx$$

$$= \int \overline{\frac{\hat{\psi}_0(k)}{k^2-\lambda}} \int \varphi(x)\,\bar{w}(x;k,\lambda)\,dx\,dk$$

$$= \int \frac{\overline{\hat{\psi}_0(k)}}{k^2-\bar{\lambda}}\,\hat{\varphi}(k,\lambda)\,dk = \left(\frac{\hat{\varphi}(k,\lambda)}{k^2-\bar{\lambda}}, \hat{\psi}_0(k)\right)$$

By the unitarity of the Fourier transform,

✓ $$(R_{\bar{\lambda}}\varphi)^{\wedge}_{0}(k) = \frac{\hat{\varphi}_1(k,\lambda)}{k^2-\bar{\lambda}}$$

5. Obtain the <u>spectral measures</u> for $-\sigma\partial\tau\partial$. ← different σ!

First, we derive a formula for the functions $\sigma(t;\varphi)$

Let R_z be the resolvent of a self-adjoint operator, and let $a<b$ be given.

$$\lim_{\varepsilon\to 0}\int_a^b \varepsilon\,\|R_{\tau\pm i\varepsilon}\varphi\|^2\,d\tau = \lim_{\varepsilon\to 0}\int_a^b \varepsilon\int_{-\infty}^{\infty} \frac{1}{t-(\tau\pm i\varepsilon)}\,d(E_t\varphi, R_{\tau\pm i\varepsilon}\varphi)\,d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{t-(\tau \pm i\varepsilon)} \, d(R_{\tau \mp i\varepsilon} E_t \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{t-(\tau \pm i\varepsilon)} \, d_t \int_{-\infty}^{\infty} \frac{1}{s-(\tau \mp i\varepsilon)} \, d_s(E_s E_t \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{t-(\tau \pm i\varepsilon)} \, d_t \int_{-\infty}^{t} \frac{1}{s-(\tau \mp i\varepsilon)} \, d_s(E_s \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \varepsilon \int_{-\infty}^{\infty} \frac{1}{|t-(\tau \pm i\varepsilon)|^2} \, d(E_t \varphi, \varphi) \, d\tau$$

$$= \lim_{\varepsilon \to 0} \int_a^b \int_{-\infty}^{\infty} \frac{\varepsilon}{(t-\tau)^2+\varepsilon^2} \, d\sigma(t;\varphi) \, d\tau$$

(!)

$$= \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} \int_a^b \frac{\varepsilon}{(t-\tau)^2+\varepsilon^2} \, d\tau \, d\sigma(t;\varphi)$$

$$= \int_{-\infty}^{\infty} \left(\pi \chi_{(a,b)} + \frac{\pi}{2} \left(\chi_{\{a\}} + \chi_{\{b\}} \right) \right) d\sigma(t;\varphi)$$

$$= \frac{\pi}{2} \Big[\big(\sigma(b+0;\varphi) + \sigma(b-0;\varphi) \big) - \big(\sigma(a+0;\varphi) - \sigma(a-0;\varphi) \big) \Big]$$

[if σ is continuous $= \pi \big(\sigma(b;\varphi) - \sigma(a;\varphi) \big)$]

av. of rt & lf. limits

(*) So $\displaystyle \lim_{\varepsilon \to 0} \int_a^b \varepsilon \| R_{\tau \pm i\varepsilon} \varphi \|^2 \, d\tau = \pi \left(\bar{\sigma}(b;\varphi) - \bar{\sigma}(a;\varphi) \right)$

Now, using the formula $(R_{\bar\lambda}\varphi)^\wedge_0(k) = \dfrac{\hat\varphi_1(k,\lambda)}{k^2-\bar\lambda}$,

we compute

$$\lim_{\varepsilon\to 0}\int_a^b \varepsilon \|R_{\tau-i\varepsilon}\varphi\|^2 d\tau = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\int_a^b \frac{\varepsilon}{(k^2-\tau)^2-\varepsilon^2}|\hat\varphi_1(k,\tau+i\varepsilon)|^2 d\tau\, dk$$

Now use the fact (needs to be proved) that $|\hat\varphi_1(k,\tau+i\varepsilon)|$ is continuous in τ and ε for $\varepsilon \geq 0$, and that

(!) $$\lim_{\varepsilon\to 0}\hat\varphi_1(k,k^2+i\varepsilon) = \hat\varphi_1(k)$$

$$= \pi\int_{-\infty}^{\infty}|\hat\varphi_1(k)|^2 \chi_{\{a\le k^2\le b\}}\, dk = \pi\int_{a\le k^2\le b}|\hat\varphi_1(k)|^2 dk$$

Since $k^2 \geq 0\ \forall k$, we may make $a=0$ and $b\geq 0$:

$$= \pi\left(\int_{-\sqrt{b}}^{-\sqrt{a}} + \int_{\sqrt{a}}^{\sqrt{b}}\right)|\hat\varphi_1(k)|^2 dk$$

$$= \pi\int_{\sqrt{a}}^{\sqrt{b}}\left(|\hat\varphi_1(k)|^2 + |\hat\varphi_1(-k)|^2\right) dk$$

by (*) p.53 $$= \pi\int_{\bar a}^{\bar b} d\sigma(\lambda;\varphi) \left(= \pi\int_a^b \sigma'(\lambda;\varphi)\,d\lambda \text{ b/c } \sigma \text{ abs. cont.}\right)$$

So with $k^2=\lambda$, we have $d\sigma(\lambda;\varphi) = \left(|\hat\varphi_1(k)|^2 + |\hat\varphi_1(-k)|^2\right)dk$.

6. Letting $a \to 0$ and $b \to \infty$, we obtain

$$\int_{-\infty}^{\infty} |\hat{\varphi}_1(k)|^2 \, dk = \int_{-\infty \text{ or } 0}^{\infty} d\sigma(\lambda; \varphi, \varphi) = (\varphi, \varphi)$$

so that $\varphi \mapsto \hat{\varphi}$ preserves the norm. In fact this map is <u>invertible</u>, as we now show.

From the result

Can take $0 \le a$

$$\int_{a \le k^2 \le b} |\hat{\varphi}_1(k)|^2 \, dk = \int_a^b d\sigma(\lambda; \varphi, \varphi) = ((E_b - E_a)\varphi, \varphi)$$

and the polarization procedure, we obtain

$$((E_b - E_a)f, \psi) = \int_{a \le k^2 \le b} \hat{f}_1(k) \overline{\hat{\psi}(k)} \, dk \quad \text{for } \psi \in C_c^\infty(\mathbb{R})$$

$$= \int_{a \le k^2 \le b} \hat{f}_1(k) \int_{-\infty}^{\infty} \overline{\psi(x)} \, w(x; k) \, dx \, dk$$

(!)

$$= \int_{-\infty}^{\infty} \overline{\psi(x)} \int_{a \le k^2 \le b} \hat{f}_1(k) \, w(x; k) \, dk \, dx$$

$$\Longrightarrow (E_b - E_a) f(x) = \int_{a \le k^2 \le b} \hat{f}_1(k) \, w(x; k) \, dk$$

Integral representation of f

Taking $a \to 0$ and $b \to \infty$, we obtain

$$\boxed{f(x) = \int_{-\infty}^{\infty} \hat{f}_1(k) \, w(x, k) \, dk}$$