Toward the mathematics of thick waveguides

We will need some background on compact operators and (limited) Sobolev-space theory. The stuff will be presented in the Hilbert-space context only.

<u>Defn</u> A linear operator $T: \mathcal{H}_1 \to \mathcal{H}_2$ is <u>compact</u> if for each bounded sequence $\{v_i\}_{i \in \mathbb{N}}$ from $\mathcal{H}_1$, the sequence $\{Tv_i\}$ has a convergent subsequence in $\mathcal{H}_2$.

[There are several insightful equivalent characterizations of compactness.]

<u>Fact</u> If T is compact, then T is bounded.

<u>Pf</u> If T is unbounded, then $\exists$ a sequence $\{v_i\}$ with $\|v_i\| = 1$ and $\|Tv_i\| \to \infty$. Such a sequence $\{Tv_i\}$ has no convergent subsequence even though $\{v_i\}$ is bounded.

<u>Fact</u> If $A: \mathcal{H}_1 \to \mathcal{H}_2$ and $B: \mathcal{H}_2 \to \mathcal{H}_3$ are bounded linear operators and either A or B is compact, then BA is compact

(Pf) In short, bounded operators preserve boundedness of sequences and convergence of sequences.

Fact If $T: \mathcal{H} \to \mathcal{H}$ is compact, then for each $\varepsilon > 0$, $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > \varepsilon\}$ is a finite set of eigenvalues of T, each possessing a finite-dimensional eigenspace.

We will prove this for self-adjoint compact operators; and we will only make use of the fact in this case.

Pf in self-adjoint case. Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be the standard resolution of the identity I for T. Since t is bounded, $\exists$ t_1 and t_2 s.th. $E_{t_1} = 0$ and $E_{t_2} = I$. Thus it suffices to prove that $E_b - E_a$ has finite-dimensional range whenever $0 \notin [a,b]$. Suppose, to the contrary, that $E_b - E_a$ has infinite-dimensional range. Put $c = \frac{a+b}{2}$. Then either $E_b - E_c$ or $E_c - E_a$ has infinite-dimensional range. Recursively, one obtains a sequence of nested intervals $[a_i, b_i] \supset [a_{i+1}, b_{i+1}]$ with $|b_i - a_i| = \frac{b_1 - a_1}{2^i}$. Set $\lambda_* = \bigcap_{i=1}^{\infty} [a_i, b_i]$ such that $E_{b_i} - E_{a_i}$ has infinite-dim'l range. One can construct recursively elements $v_i \in \mathrm{Ran}(E_{b_i} - E_{a_i})$ with $(v_i, v_j) = \delta_{ij}$. Notice that

$$(+) \qquad \|(T - \lambda_*) v_i\|^2 = \int_{a_i}^{b_i} (\lambda - \lambda_*)^2 \sigma(\lambda; v_i) \le |b_i - a_i|^2 \int_{a_i}^{b_i} \sigma(\lambda; v_i)$$

$$= |b_i - a_i|^2 \|v_i\|^2 \longrightarrow 0 \text{ as } i \to \infty.$$

The last equality holds because $v_i \in \mathrm{Ran}(E_{b_i} - E_{a_i})$.

For $i \neq j$,

$$Tv_i - Tv_j = \lambda_*(v_i - v_j) + (T-\lambda_*)v_i - (T-\lambda_*)v_j ,$$

so $\|Tv_i - Tv_j\| = \lambda_* \sqrt{2} \mp \delta_{ij}$

with $\delta_{ij} \to 0$ by (+) p. 57. Thus $\{Tv_i\}_{i=1}^{\infty}$ has no Cauchy subsequence. We conclude that the rank of $E_\lambda - E_a$ is an increasing integer-valued function that is finite at $\lambda = b$ and is therefore constant except for finite-rank jumps at a finite number of values $\lambda_i \in [a,b]$, which are eigenvalues of T with eigenspaces equal to $E_{\lambda_i^+} - E_{\lambda_i^-}$. ∎

A little on the Sobolev spaces H^1 and H^1_0.

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$.

The Sobolev space $H^1(\Omega) \subset L^2(\Omega)$ of weakly L^2-differentiable functions in Ω can be defined as the domain of the adjoint of the divergence operator $\nabla\cdot$ restricted to $[C_c^\infty(\Omega)]^n$ ($C_c^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support contained in Ω.) In other words, $f \in H^1(\Omega)$ iff $\exists\ F \in [L^2(\Omega)]^n$ such that

$$\int_\Omega f\, \nabla\cdot\Phi = -\int_\Omega F\cdot\Phi \qquad \forall\ \Phi \in [C_c^\infty(\Omega)]^n,$$

and we say that $F = \nabla f$ is the weak gradient of f. $H^1(\Omega)$ is endowed with the operator norm of ∇:

$$\|f\|_{H^1}^2 = \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 .$$

Since ∇ is the adjoint of an operator, its graph is closed, and thus $H^1(\Omega)$ is complete in the graph, or operator, norm. It is a Hilbert space with inner product inherited from its graph:

$$(f,g)_{H^1} = (\nabla f, \nabla g)_{(L^2)^n} + (f,g)_{L^2} .$$

One usually defines $H^1_0(\Omega)$ to be the <u>closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$</u>. It is also characterized as the <u>domain of the adjoint of the divergence</u> operator $\nabla\cdot$ <u>restricted to $[C^\infty(\Omega) \cap L^2(\Omega)]^n$</u>.

The <u>interpretation</u> of $H^1_0(\Omega)$ is that it is the subspace of $H^1(\Omega)$ of functions whose boundary values (trace) on $\partial\Omega$ are zero: $u \in H^1(\Omega) : u|_{\partial\Omega} = 0$.

[There is a complete theory of boundary values of functions in Sobolev spaces, onto which we will not embark.]

<u>Theorem</u> If Ω is bounded, then the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. If $\partial\Omega$ is sufficiently smooth, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact (but we will not prove this).

[This theorem is a very special case of a vast set of compact embedding theorems for Sobolev spaces.]

The proof makes use of the <u>spectral truncations</u> v^R of a function $v \in L^2(\mathbb{R}^n)$ for $R>0$:

$$v^R = \mathcal{F}^{-1}\left[\chi_{\{|\xi|\le R\}} \mathcal{F}[v]\right].$$

First we prove two lemmas.

<u>Lemma 1</u> If $v \in H^1(\mathbb{R}^n)$, then $\|v - v^R\|_{L^2} \le \frac{1}{R}\|v\|_{H^1}$.

<u>Proof</u> $R^2\|v-v^R\|_{L^2}^2 = R^2 \int_{|\xi|>R} |\hat{v}(\xi)|^2 d\xi \le \int_{|\xi|>R} |\xi \hat{v}(\xi)|^2 d\xi$

$$\le \int |\xi \hat{v}(\xi)|^2 d\xi = \int |\nabla v(x)|^2 dx \le \|v\|_{H^1}.$$

<u>Lemma 2</u> If $v \in H^1(\mathbb{R}^n)$, then $v^R \in C^\infty$ and there is a constant C_R such that

$$|v^R(x)| < C_R \|v\|_{L^2} \quad \text{and} \quad |\nabla v^R(x)| < C_R\|v\|_{H^1}$$

for all $x \in \mathbb{R}^n$.

<u>Proof</u> $|v^R(x)| = \left| \int_{|\xi|<R} \hat{v}(\xi) e^{i\xi x} d\xi \right| \overset{\text{Hölder}}{<} C' \left[\int_{|\xi|<R} |\hat{v}(\xi)|^2 d\xi \right]^{1/2}$

$\leq C' \|\hat{v}\|_{L^2(\mathbb{R}^n)}$.

$|\nabla v^R(x)| = \left| \int_{|\xi|<R} i\xi \hat{v}(\xi) e^{i\xi x} d\xi \right| \leq C'' \left[\int_{|\xi|<R} |\xi \hat{v}(\xi)|^2 \right]^{1/2} d\xi$

$\leq C'' \|v\|_{H^1(\mathbb{R}^n)}$.

<u>Proof of Theorem p. 60</u>

Let $\{u_i\}_{i=1}^\infty$ be a sequence from $H_0^1(\Omega)$ with $\|u_i\|_{H^1} \leq M < \infty$ $\forall i$. We will demonstrate the existence of a subsequence that converges in the L^2-norm. By its definition, $H_0^1(\Omega) \subset H^1(\mathbb{R}^n)$ since $C_c^\infty(\Omega) \subset H^1(\mathbb{R}^n)$. From Lemma 1,

(*) $$\|u_i - u_i^R\|_{L^2(\mathbb{R}^n)} \leq \frac{M}{R} \quad \forall\, i \in \mathbb{N},\ \forall R > 0.$$

We infer from Lemma 2 that $\forall\, R>0$, $\{u_i^R\}_{i\in\mathbb{R}}$ is equicontinuous on $\bar{\Omega}$. Given a sequence $R_k \to \infty$, the Arzela-Ascoli Theorem and a diagonal argument produces a sequence $\{i_j\}_{j=1}^\infty$ such that $\forall k$, $u_{i_j}^{R_k}$ converges as $j \to \infty$, say $u_{i_j}^{R_k} \to u_k$, <u>uniformly</u> in Ω and $\therefore$ in $L^2(\Omega)$. $\forall k$, $\exists J$ s.th. $j > J \Rightarrow \|u_{i_j}^{R_k} - u_k\|_{L^2(\Omega)} < \frac{M}{R}$, and because of (*),

$$\|u_{i_j} - u_{i_\ell}\|_{L^2(\Omega)} \leq \frac{4M}{R} \quad \forall\ j, \ell > J.$$

Thus $\{u_{i_j}\}_{j\in\mathbb{N}}$ is Cauchy in $L^2(\Omega)$. ∎

Let us return to the wave equation in a domain Ω of $\mathbb{R}^n$:

$$\rho \frac{\partial^2}{\partial t^2} U(x,t) = \nabla \cdot \tau \nabla U(x,t) . \quad (x \in \Omega \subset \mathbb{R}^n,\ t \in \mathbb{R})$$

Assuming a solution of the <u>harmonic</u> form

$$U(x,t) = u(x)\, e^{-i\omega t},$$

one obtains the Helmholtz equation for $u(x)$:

$$\nabla \cdot \tau \nabla u + \omega^2 \rho u = 0 .$$

To motivate the definition of solutions u with τ and ρ merely assumed to be measurable, observe that <u>if</u> $\partial\Omega$ is smooth and all functions in sight are smooth (enough), then the Helmholtz equation implies $\forall\, v$ (smooth)

$$(+) \qquad 0 = \int_\Omega (\nabla \cdot \tau \nabla u + \omega^2 \rho u)\bar{v}$$

$$= -\int_\Omega \tau \nabla u \cdot \nabla \bar{v} + \omega^2 \int_\Omega \rho u \bar{v} + \int_{\partial\Omega} (\tau \partial_n u)\bar{v} .$$

In this expression, two forms appear:

$$a(u,v) := \int_\Omega \tau \nabla u \cdot \nabla \bar{v} \quad ; \quad b(u,v) = \int_\Omega \rho u \bar{v}$$

Let us consider two "converses" of this.

(1) The homogeneous "Dirichlet" boundary-condition case. If

$$\int_\Omega \tau \nabla u \cdot \nabla \bar{v} - \omega^2 \int_\Omega \rho u \bar{v} = 0 \quad \forall \; v \in C_c^\infty(\Omega),$$

then (+) implies that

$$\nabla \cdot \tau \nabla u + \omega^2 \rho u = 0 \quad \text{in } \Omega.$$

In this case, we let the forms $a(u,v)$ and $b(u,v)$ act in $H_0^1(\Omega)$, and if for some $u \in H_0^1(\Omega)$,

$$a(u,v) - \omega^2 b(u,v) = 0 \quad \forall \; v \in H_0^1(\Omega),$$

w/ eigenvalue ω^2

we call u a Dirichlet eigenfunction of (Ω, τ, ρ) ($\underline{u|_{\partial\Omega} = 0}$). It is important to observe that τ and ρ are allowed to be merely measurable and bounded.

(2) The homogeneous "Neumann" condition. If

$$\int_\Omega \tau \nabla u \cdot \nabla \bar{v} - \omega^2 \int_\Omega \rho u \bar{v} = 0 \quad \forall \; v \in C^\infty(\mathbb{R}^n),$$

then (+) implies that

$$\nabla \cdot \tau \nabla u + \omega^2 \rho u = 0 \quad \text{in } \Omega \quad \text{and} \quad \underbrace{\partial_n u = 0}_{\text{normal deriv.}} \quad \text{on } \partial\Omega.$$

In this case, we let the forms $a(u,v)$ and $b(u,v)$ act in $H^1(\Omega)$, and if for some $u \in H^1(\Omega)$,

$$a(u,v) - \omega^2 b(u,v) = 0 \quad \forall \; v \in H^1(\Omega),$$

we call u a <u>Neumann eigenfunction of</u> (Ω, τ, ρ) <u>$(\partial_n u|_{\partial\Omega} = 0)$</u> with eigenvalue ω^2. Both τ and ρ may be bounded measurable functions.

Let us now assume that τ and ρ are bounded from above and below in Ω:

$$\left.\begin{array}{l} 0 < \tau_- \leq \tau(x) \leq \tau_+ < \infty \\ 0 < \rho_- \leq \rho(x) \leq \rho_+ < \infty \end{array}\right\} \forall \; x \in \Omega$$

The form

$$\hat{a}(u,v) = a(u,v) + b(u,v)$$

$H =$ $H_0^1(\Omega)$ or $H^1(\Omega)$

is sesquilinear in $H^1(\Omega)$ and in $H_0^1(\Omega)$. Denote by H either of these spaces. Observe that $\exists \; c_\pm > 0$ s.th.

$$c_- \|u\|_H^2 \leq \tau_- \|\nabla u\|_{L^2}^2 + \rho_- \|u\|_{L^2}^2$$

$$\leq \hat{a}(u,u) = \int \tau |\nabla u|^2 + \int \rho |u|^2$$

$$\leq \tau_+ \|\nabla u\|_{L^2}^2 + \rho_+ \|u\|_{L^2}^2 \leq c_+ \|u\|_{H^1}^2 .$$

This means that $\sqrt{\hat{a}(u,u)}$ is equivalent to the Sobolev norm on H.

Since $\forall\ u \in H$, $\hat{a}(u,\cdot)$ acts as a conjugate linear functional on H and $\forall\ v \in H$,

$$\hat{a}(u,v) \le \sqrt{\hat{a}(u,u)}\,\sqrt{\hat{a}(v,v)} \le c_+ \|u\|_H \|v\|_H ,$$

the Riesz Lemma provides a linear operator $A : H \to H$ such that

Defn. of A
$$\hat{a}(u,v) = (Au,v)_H \quad \forall \quad u,v \in H ,$$

with $\|Au\|_H \le c_+ \|u\|_{H'}$, so that A is bounded by c_+ (as an operator). Also,

$$c_- \|u\|_H^2 \le \hat{a}(u,u) = (Au,u) \le \|Au\|_H \|u\|_H ,$$

which yields

$$c_- \|u\|_H \le \|Au\|_H ,$$

so that A^{-1} exists and is bounded by $\frac{1}{c_-} < \infty$.

Similarly, since $b(u,\cdot)$ acts as a conj. linear functional on H and

(+) $$|b(u,v)| = \left| \int \rho u \bar{v} \right| \le \rho_+ \left| \int u \bar{v} \right| \le \rho_+ \|u\|_{L^2} \|v\|_{L^2} \le \rho_+ \|u\|_{L^2} \|v\|_H ,$$

there exists a linear operator $B : H \to H$ such that

Defn of B

$$b(u,v) = (Bu, v)_H \qquad \forall \quad u, v \in H$$

B is the composition of three operators:

$$H \overset{\iota}{\hookrightarrow} L^2(\Omega) \xrightarrow{u \mapsto b(u,\cdot)} H^* \xrightarrow{\text{Riesz}} H .$$

The first map is the inclusion of H into L^2. We have proved that ι is compact for $H = H_0^1(\Omega)$; it is also compact for $H = H^1(\Omega)$. The second map is bounded, as shown by (†) p. 65. The third map is the Riesz isomorphism, and is thus bounded. Therefore B is compact.

The Dirichlet or Neumann eigenvalue problem

$$\nabla \cdot \tau \nabla u + \omega^2 \rho u = 0, \quad \begin{cases} u|_{\partial\Omega} = 0 & \text{(Dirichlet)} \\ \partial_n u|_{\partial\Omega} = 0 & \text{(Neumann)} \end{cases}$$

is posed in the weak sense by means of the forms a and b in $H_0^1(\Omega)$ or $H^1(\Omega)$ or equivalently, in terms of the operators A and B as follows:

E-val problem

Seek $u \in H$ such that $a(u,v) + \omega^2 b(u,v) = 0 \;\; \forall \; v \in H$, or

$$\hat{a}(u,v) + (\omega^2 - 1) b(u,v) = 0 \qquad \forall \quad v \in H,$$

or, equivalently, $Au + (\omega^2 - 1) Bu = 0$, or

$$u + (\omega^2 - 1) A^{-1} B u = 0 .$$

Since A^{-1} is bounded and B is compact, $A^{-1}B$ is compact. 'Tis a simple matter to show that A and B are both self-adjoint and that therefore $A^{-1}B$ is also self-adjoint.

Thus the spectrum of $A^{-1}B$ consists of a sequence of eigenvalues $\{\mu_i\}_{i=1}^{\infty}$ that converges to 0. Indeed 0 is not an eigenvalue because A^{-1} is injective and $(Bu, v) = 0 \ \forall \ v \Rightarrow u = 0$, which is readily checked from the definition of b and $0 < \rho_- \leq \rho \leq \rho_+ < \infty$. Let the associated eigenfunctions be $\{u_i\}_{i \in \mathbb{N}}$. These form an orthonormal Hilbert-space basis for H by virtue of the spectral theorem.

Putting $\omega_i^2 = 1 + \frac{1}{\mu_i}$,

we obtain $a(u_i, v) + \omega_i^2 b(u_i, v) = 0 \quad \forall \ v \in H$, and we can write (in the weak sense)

$$\nabla \cdot \tau \nabla u_i + \omega_i^2 \rho u_i = 0 \qquad (\omega_i \to \infty)$$

$$\text{with} \begin{cases} u_i = 0 \text{ on } \partial\Omega & \text{if } H = H_0^1(\Omega), \\ \partial_n u_i = 0 \text{ on } \partial\Omega & \text{if } H = H^1(\Omega). \end{cases}$$