Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let

$$\lambda_k \geq 0 \quad (k = 1, 2, \ldots)$$

be the Dirichlet eigenvalues for $\Omega$, that is, the eigenvalues of $-\nabla \cdot \nabla$ with zero boundary condition on $\partial \Omega$, as discussed above.

Let

$$\sum_{k=1}^{\infty} \lambda_k \psi_k(x) = \sum_{j=1}^{N} \psi_j(x)$$

be the associated eigenfunctions.

For a function $v \in L^2(\Omega)$, let $\hat{v}_k$ be the Fourier transform with respect to $-\nabla \cdot \nabla$; $\Delta_D$ (Dirichlet).

$$v(x) = \sum_{k=1}^{\infty} \hat{v}_k \psi_k(x), \quad \|v\|_2 = \sqrt{\sum_{k=1}^{\infty} |\hat{v}_k|^2}$$

$(-\Delta_D)^{1/2} = \lambda_k^{1/2} \hat{v}_k$; $(-\Delta_D)\phi_k = \sum_{k=1}^{\infty} \lambda_k \hat{v}_k \psi_k(x)$

The Laplacian $-\Delta_D$ has a positive square root -

$$(-\Delta_D)^{1/2} = \sqrt{\lambda_k} \hat{v}_k;$$

$$(-\Delta_D)\phi_k = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \hat{v}_k \psi_k(x)$$
In the usual Fourier transform in $\mathbb{R}^n$,

$$(\Delta^2 u)(x) = (-\nabla \cdot (\nabla u))^2(x) = i^2 \| \mathcal{F}(x) \|^2 = |x|^2 \mathcal{F}(x)$$

$$\Rightarrow \left( i \mathcal{F} \right)(x) = |x| \mathcal{F}(x)$$

Thus $\| \nabla u \|^2_{L^2} = \| \mathcal{F}(x) \|^2_{L^2} = \| x \mathcal{F}(x) \|^2_{L^2} = \| (i \mathcal{F}) (x) \|^2_{L^2} = \| i \mathcal{F} \|^2_{L^2}$.

So $\| \nabla u \|^2_{L^2} = \| i \mathcal{F} \|^2_{L^2} = \sum_{k=1}^{\infty} |\hat{u}_k|^2 \lambda_k$

$\| u \|^2_{H^1_0(\Omega)} = \sum_{k=1}^{\infty} (\lambda_k + 1) |\hat{u}_k|^2$

We have $u \in H^1_0(\Omega)$ iff $\sum_{k=1}^{\infty} \lambda_k |\hat{u}_k|^2 \leq 1$.

Spectral resolution:

If $P_k$ is the projection onto the span of $\psi_k$, we have

$I = \sum_{k=1}^{\infty} P_k$

For $u \in SG(-\Delta_0)$, $-\Delta_0 v = \sum_{k=1}^{\infty} \lambda_k P_k v$

For $u \in H^1_0$, $(\nabla u, \nabla v) = \sum_{k=1}^{\infty} \lambda_k |\hat{u}_k|^2$.
The max-min principle. Using the spectral representation \( \langle \nabla u, \nabla v \rangle = \sum \lambda_k \hat{u}_k \hat{v}_k \), one can deduce that

\[
\lambda_k = \inf_{\gamma < \frac{1}{k}} \sup_{\gamma \leq \frac{1}{k}, \|u\|_{W^{1,2}(\Omega)}} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2}.
\]

This principle allows one to obtain bounds on the eigenvalues \( \lambda_k \). Arrange the double sequence \( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{km} \in \mathbb{Z} \) in an increasing way: \( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{kn} \in \mathbb{Z} \). Then one shows that there exist constants \( c_1, c_2 > 0 \) such that

\[ c_1 \lambda_k \leq \lambda_k \leq c_2 \lambda_k. \]

To do this, one considers a box \( [a, b] \times \cdots \times [c, d] \) contained in \( \Omega \) and a box \( [a', b'] \times \cdots \times [c', d'] \) that contains \( \Omega \). The \( n \times n \) Dirichlet eigenvalues of \( B_1 \) and \( B_2 \) can be found explicitly; they are

\[ \sum_{n=1}^{\infty} \left( \frac{m}{\pi (a_i - a)} \right)^2, \quad m, \ldots, m \in \mathbb{Z} \]

and

\[ \sum_{n=1}^{\infty} \left( \frac{m_i}{\pi (a_i - a)} \right)^2, \quad m_i, \ldots, m_i \in \mathbb{Z} \]

Then one uses the min-max principle to obtain the appropriate comparison.
Let \( S \) denote a cylindrical waveguide in \( \mathbb{R}^{n+1} \) with cross-section \( \Gamma \):

\[
S = \Gamma \times \mathbb{R} = (x, t)
\]

The Fourier transform for \( S \) is the tensor product of the Fourier transforms for \( \Gamma \) and \( \mathbb{R} \):

\[
u(x, t) \mapsto (F\nu)_k(\xi) = \iint_{\mathbb{R}^n} u(x, t) W_0(x, t; k, \xi) \, dx \, dt,
\]

where \( W_0 \) are the undistorted wave functions

\[
W_0(x, t; k, \xi) = \varphi_k(x) e^{i \xi \cdot t}.
\]

The partial Fourier transform with respect to \( x \) alone is

\[
\hat{u}_k(\xi) = \int_{\Gamma} u(x, t) \varphi_k(x) \, dx.
\]

The inverse Fourier transform gives the Fourier integral-sum

\[
u(x, t) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \hat{u}_k(\xi) (F\nu)_k(\xi) W_0(x, t; k, \xi) \, d\xi
\]

Notice that \( \omega_0(x, t; k, \xi) \) satisfy

\[
-\Delta W_0 = (\Box_k + \xi^2) W_0
\]
The wave functions \( \psi_0(x, z; k, \xi) \) are the generalized, or extended, eigenfunctions of \(-\Delta_D\)
in \( \Sigma \).

A major task of scattering theory is to construct the distorted wave functions \( \psi(x, z; k, \xi) \) that arise when the 
"incident field" \( \psi_0(x, z; k, \xi) \) is scattered by a defect or obstacle in the waveguide.

This obstacle is (for us) a local variation of \( \tau \) and \( \eta \) satisfying

\[
0 < \tau_- = \tau(x, z) \leq \tau_+ \leq +\infty \quad \text{for} \quad |z| \leq L
\]

\[
0 < \eta_- = \eta(x, z) \leq \eta_+ < +\infty \quad \text{for} \quad |z| \leq L
\]

\[
\tau(x, z) = \eta(x, z) = 1 \quad \text{for} \quad |z| > L.
\]

Given a fixed frequency \( \omega > 0 \), there is a finite set of wave functions \( \psi_0(x, z; k, \xi) \)
that satisfy

\[-\Delta \psi_0 = \omega^2 \psi_0 \quad \text{in} \quad \Sigma,\]

namely, those for which \( \omega^2 = \lambda_k + \xi^2 \), i.e.

\[
\begin{cases}
\lambda_k \leq \omega^2 \\
\xi^2 = \omega^2 - \lambda_k
\end{cases}
\]
Define the "propagation exponents"

\[ \xi_k(w) = \begin{cases} \sqrt{\omega^2 - \gamma_k} & \text{if } \gamma_k < \omega^2 \\ i\sqrt{\gamma_k - \omega^2} \in i\mathbb{R}^+ & \text{if } \gamma_k \geq \omega^2 \end{cases} \]

For a finite set of integers \( k \), we have

\[ \xi_k(w) > 0 \]; these are the \textbf{propagating harmonics}

because the associated wave functions

\[ \psi_0 = \psi_k(x) e^{\pm i \xi_k(w)} e^{-i \omega t} \]

are traveling waves (move to the right (+i\( \xi \)) or to the left)

The rest of the \( k \)-values, correspond to with

\[ -i \xi_k(w) > 0 \] correspond to \textbf{evanescent harmonics},

\[ \text{or exponential harmonics, because the } \psi_k \text{ functions} \]

\[ \psi_k(x) e^{\pm i \xi_k(w)} \] \textbf{are exponentially growing or decaying in } x.
Now consider $w_0(x, z; k, \xi)$ to be an incident field, where $\xi = \xi_k(\omega)$ (and $\omega < \omega^2$).

Define a scattered field $u^{sc}(x, z; k, \xi)$ such that

$$\nabla \cdot \nabla (w_0 + u^{sc}) + \rho (w_0 + u^{sc}) = 0 \quad \text{(weak sense)}$$

with

$$w_0 + u^{sc} = 0 \quad \text{on} \quad \partial S$$

radiating,

$$\text{For } z > L, \quad u^{sc}(x, z; k, \xi) = \sum_{j=1}^{\infty} \psi_j(x) e^{i \xi_j(\omega) z} \frac{c_j}{\kappa_j}$$

or outgoing,

$$\text{For } z < L, \quad u^{sc}(x, z; k, \xi) = \sum_{j=1}^{\infty} \psi_j(x) e^{-i \xi_j(\omega) z} \frac{c_j}{\kappa_j}$$

where we remember that $\omega^2 = \lambda^R + \xi^2(\omega)$.

The first two conditions declare that $w_0 + u^{sc}$ is the new field that satisfies the "distorted" waveguide, and that $u^{sc}$ is the field one adds to $w_0$ to accommodate the distortion.

The final condition is a "radiating" or "outgoing" condition, that is physically meaningful and guarantees (as we will see) quantitatively a unique solution.
Boundary values, or traces, of \( H_0 \) functions in a half-tube.

Let \( \Gamma \) be identified with \( \mathbb{T} \times \{0\} \subset S \), which is the cross-sectional part of the boundary of the half-waveguide \( \Gamma \times (0, \infty) =: S^+ \).

\[
\mathcal{R} \implies \mathcal{R}
\]

Denote by \( H_0^1(S^+) \) the subset of \( H_0^1(S) \) of \( H^1 \) functions in \( S^+ \) that vanish on the boundary of the waveguide but not on the cross-section \( \Gamma \).

\[
H_0^1(S^+) = H_0^1(S) \cap H^1(S^+)
\]

Functions in \( H_0^1(S^+) \) can be assigned boundary values, or traces on \( \Gamma \) in a natural way.

We first consider \( u \in C^\infty(S^0) \cap H_0^1(S^+) \), which has clearly defined values on \( \Gamma^0 \). We have

\[
\left( \lambda_k + 1 \right)^2 \left| \hat{u}_k(0) \right|^2 = -2 \text{Re} \int_0^\infty \sqrt{\lambda_k + 1} \hat{u}_k(z) \frac{d\hat{u}_k(z)}{dz} \, dz
\]

\[
\int_0^\infty \left( \lambda_k + 1 \right)^2 \left| \hat{u}_k(z) \right|^2 \, dz + \int_0^\infty \left| \frac{d\hat{u}_k(z)}{dz} \right|^2 \, dz
\]
Summing over \( k \), we obtain

\[
\sum_{k=1}^{\infty} \left( \gamma_k + 1 \right)^{1/2} | \tilde{u}_k(0) |^2 = \sum_{k=1}^{\infty} \int_{0}^{\infty} \left| \tilde{u}_k(z) \right|^2 \, dz + \sum_{k=1}^{\infty} \int_{0}^{\infty} \left| \frac{d \tilde{u}_k}{dz}(z) \right|^2 \, dz
\]

\[
= \| u \|_{L^2(S_+)}^2 + \int_{S_+} \| \nabla_x u \|_{L^2(S_+)}^2 \, dz \, dx + \int_{S_+} \left| \frac{\partial}{\partial z} u(x, z) \right|^2 \, dx \, dz
\]

\[
= \| u \|_{H^1(S_+)}^2
\]

Since \( C^0(S_+) \cap H^1_{00}(S_+) \) is dense in \( H^1_{00}(S_+) \),

the map \( C^0(S_+) \cap H^1_{00}(S_+) \to H^{1/2}(\mathbb{R}) : u \mapsto u|_0 \)

can be extended to all of \( H^1_{00}(S_+) \). Here, \( H^{1/2}(\mathbb{R}) \)

is the Hilbert space of functions on \( \mathbb{R} \) s.t.

\[
H^{1/2}(\mathbb{R}) = \left\{ \tilde{f} \in L^2(\mathbb{R}) : \left[ (\lambda_{k+1})^{1/4} \tilde{f} \right] \in L^2 \right\},
\]

or \( \tilde{f} \in L^2 \), \( \tilde{f} \) \( \in L^2 \),

with norm

\[
\| \tilde{f} \|_{H^{1/2}(\mathbb{R})}^2 = \sum_{k=1}^{\infty} \left( \lambda_{k+1} \right)^{1/2} | \tilde{f}_k |^2.
\]
We have just seen that the trace map

\[ T : H^1_{00}(S_+) \to H^1(\mathbb{R}) \]

is bounded.

In fact, \( T \) possesses a right inverse \( E : H^1(\mathbb{R}) \to H^1_{00}(S_+) \):

\[ TEv = v \quad \forall v \in H^1(\mathbb{R}). \]

It is given, \( E \in \text{C}^0(\mathbb{R}) \), by

\[ (Ev)(z) = \frac{e^{-\frac{z}{1+i\kappa}}}{1+i\kappa} e^{-(1+i\kappa)^2 z}. \]

**Exercise**

In fact, \( Ev \) minimizes the \( H^1 \) norm over all \( v \in H^1_{00}(S_+) \) such that \( Tu = v \) \quad (Exercise)

Boundary normal derivatives

Set \( P = \mathbb{R} \times \mathbb{R} \times [-L,0] \), \( P_+ = \mathbb{R} \times \mathbb{R} \times [0,L] \),

\[ \mathcal{P} = \mathbb{R} \times (-L,L) \],

so that \( 2\mathcal{P} = \mathbb{R} \times \mathbb{R} \times [-L,L] \cup P \cup P_+ \)

(All parts of the boundary are oriented outward)
Because of the outgoing condition, $u^{sc}$ satisfies

$$\begin{align*}
2 \geq L : \quad & \frac{\partial}{\partial z} u^{sc} = \sum_{\ell=1}^{L} c_{\ell} e^{i \xi_{\ell}(w)} e^{i \xi_{\ell}(w) z} \\
2 \leq L : \quad & \frac{\partial}{\partial z} u^{sc} = \sum_{\ell=1}^{L} c_{\ell} e^{-i \xi_{\ell}(w)} e^{-i \xi_{\ell}(w) z}
\end{align*}$$

Since $|\xi_{\ell}(w)| = |\sqrt{\omega^{2} - \gamma_{k}}| \sim \sqrt{\gamma_{k}}$ as $\gamma_{k} \to \infty$,

the map

$$\begin{align*}
T : H^{\frac{1}{2}}(\mathbb{R}_{+}) & \to [H^{\frac{1}{2}}(\mathbb{R}_{+})]^* =: H^{-\frac{1}{2}}(\mathbb{R}_{+}) \\
(Tf_{\pm})_{k} &= -i \xi_{k}(w) f_{\pm,k}
\end{align*}$$

Exercise: is bounded from $H^{\frac{1}{2}}$ to its dual, called $H^{-\frac{1}{2}}$ and

the expression $\sum_{\ell=1}^{L} -i \xi_{\ell}(w) f_{\pm,k}$ is understood as

$$(Tf_{\pm})v_{\pm} = \sum_{k=1}^{\infty} -i \xi_{k}(w) f_{\pm,k} (v_{\pm})_{k}$$

for $v_{\pm} \in H^{\frac{1}{2}}(\mathbb{R}_{+})$ and $f_{\pm} \in H^{\frac{1}{2}}(\mathbb{R}_{+})$.

The map $T$ characterizes the outgoing condition.

Exercise: since $u$ outgoing

$$(Tu + \partial_{\nu}u)_{\mathbb{R}_{+}} = 0$$

for $u \in C_{0}^{\infty}(\mathbb{S}_{+}) \cap H^{1}_{loc}(\mathbb{S}_{+})$.
We now obtain the weak formulation of the scattering problem.

Formally, the field \( U = W_0 + U^c \) satisfies

\[
\begin{cases}
\nabla \cdot \nabla u - \omega^2 p u = 0 & \text{in } \Omega \\
T u^c + \partial_n u^c = 0 & \text{on } \Gamma
\end{cases}
\]

Multiplying by \( \bar{v} \) and integrating by parts gives

\[
\int_{\Omega} \nabla u \cdot \nabla \bar{v} - \omega^2 \int_{\Omega} p u \bar{v} = \int_{\Gamma} (\partial_n u) \bar{v}
\]

\[
= \int_{\Gamma} -(Tu) \bar{v} + \int_{\Gamma} (Tu + \partial_n u) \bar{v}
\]

\[
= \int_{\Gamma} -(Tu) \bar{v} + \int_{\Gamma} (T + \partial_n) w_0 \bar{v}
\]

Now put

\[
\hat{a}(u,v) = \int_{\Omega} \nabla u \cdot \nabla \bar{v} + \int_{\Omega} p u \bar{v} + \int_{\Gamma} (Tu) \bar{v}
\]

\[
b(u,v) = \int_{\Omega} p u \bar{v}
\]

Both of these are bounded forms in \( H^1_0(\Omega) \).
The scattering problem is of the form is to seek \( u \in H^1_0(S^2) \) such that

\[
\ddot{u}(u,v) - (w^2 + 1)u(u,v) = f(u) \quad \forall \, \, v \in H^1_0(S^2),
\]

where \( f(u) := \int_{P^2} (1 + d_n w_0) v \).

**Exercise** In fact, \( f \in H^{1/2}(P^2)^* = H^{-1/2}(P^2) \).

Again, one can define an invertible operator

\[
A : H^1_0(S^2) \to H^1_0(S^2).
\]

**Exercise** such that

\[
\ddot{u}(u,v) = (A^* u, v).
\]

Also let \( f \in H^{1/2} \) be such that

\[
(A^* u, v) = f(u), \quad \text{and again,} \quad (B u, v) = b(u,v).
\]

The scattering problem becomes

\[
(A^* u, v) + (B u, v) = (f, v) \quad \forall \, \, v \in H^1_0(S^2),
\]

or

\[
u + A' B u = A'^* f
\]
6. Trapped modes

A trapped mode is a nonzero field $u \in H^1_0(\Omega)$ that satisfies

$$u + A^{-1} B u = 0,$$

i.e., $u$ is a field that exists in the guide in the absence of a source field $f$.

Define the real part of $\hat{a}$ by

$$\hat{a}_r(u,v) = \int_\Omega \nabla u \cdot \nabla \overline{v} + \int_{\partial \Omega} g u \overline{v} + \int_{\partial \Omega} (\text{Re} f) u \overline{v},$$

where $(\text{Re} f)_{g_\omega}^\pm (g_\omega) = \sum_{\omega \in \mathbb{R}} -i \xi_k(\omega) \hat{f}_{\pm, k} \overline{g_x, k}$

for $f_\pm, g_\omega \in H^{1/2}(\Omega_\pm)$.

Then $u^{t_0}$ is a trapped mode iff

$$\hat{a}(u,v) - \omega^2 b(u,v) = 0 \quad \forall \, u, v \in H^1_0$$

and

$$\hat{a}_r(u,v) - \omega^2 b(u,v) = 0 \quad \forall \, u \in H^1_0,$$

with $\xi_k(\omega) \in \mathbb{R}$.

Exercise:

$$\left\{ \begin{array}{l}
\hat{a}(u,v) = \omega^2 b(u,v) = 0 \quad \forall \, u, v \in H^1_0 \\
\hat{a}_r(u,v) = \omega^2 b(u,v) = 0 \quad \forall \, u \in H^1_0 \\
|u|_{k_\pm}^2 = 0 \quad \forall \, k \text{ with } \xi_k(\omega) \in \mathbb{R}
\end{array} \right.$$